Handbook of Differential Equations

Stationary Partial Differential Equations

Volume 6

Edited by

Michel Chipot

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Volume VI

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HANDBOOK OF DIFFERENTIAL EQUATIONS

STATIONARY PARTIAL DIFFERENTIAL EQUATIONS

Volume VI

Edited by

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North-Holland is an imprint of Elsevier
Radarweg 29, PO Box 211, 1000 AE Amsterdam, The Netherlands
Linacre House, Jordan Hill, Oxford OX2 8DP, UK

First edition 2008

Library of Congress Cataloging-in-Publication Data
A catalog record for this book is available from the Library of Congress

British Library Cataloguing in Publication Data
A catalogue record for this book is available from the British Library

ISBN: 978-0-444-53241-1
Set ISBN: 978-0-444-51743-2

For information on all North-Holland publications
visit our website at books.elsevier.com

Printed and bound in Hungary

08 09 10 11 12 10 9 8 7 6 5 4 3 2 1

Preface

This handbook is the sixth and last volume in the series devoted to stationary partial differential equations. As the preceding volumes, it is a collection of self contained, state-of-the-art surveys written by well-known experts in the field.

The topics covered by this volume include in particular domain perturbations for boundary value problems, singular solutions of semi-linear elliptic problems, positive solutions to elliptic equations on unbounded domains, symmetry of solutions, stationary compressible Navier–Stokes equation, Lotka–Volterra systems with cross-diffusion, fixed point theory for elliptic boundary value problems. I hope that these surveys will be useful for both beginners and experts and help to a wide diffusion of these recent and deep results in mathematical science.

I would like to thank all the contributors for their elegant articles. I also thank Lauren Schultz Yuhasz and Mara Vos-Sarmiento at Elsevier for the excellent editing work of this volume.

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CHAPTER 1

Domain Perturbation for Linear and Semi-Linear Boundary Value Problems

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Abstract

This is a survey on elliptic boundary value problems on varying domains and tools needed for that. Such problems arise in numerical analysis, in shape optimisation problems and in the investigation of the solution structure of nonlinear elliptic equations. The methods are also useful to obtain certain results for equations on non-smooth domains by approximation by smooth domains.

Domain independent estimates and smoothing properties are an essential tool to deal with domain perturbation problems, especially for non-linear equations. Hence we discuss such estimates extensively, together with some abstract results on linear operators.

A second major part deals with specific domain perturbation results for linear equations with various boundary conditions. We completely characterise convergence for Dirichlet boundary conditions and also give simple sufficient conditions. We then prove boundary homogenisation results for Robin boundary conditions on domains with fast oscillating boundaries, where the boundary condition changes in the limit. We finally mention some simple results on problems with Neumann boundary conditions.

The final part is concerned about non-linear problems, using the Leray-Schauder degree to prove the existence of solutions on slightly perturbed domains. We also demonstrate how to use the approximation results to get solutions to nonlinear equations on unbounded domains.

Keywords: Elliptic boundary value problem, Domain perturbation, Semilinear equations, A priori estimates, Boundary homogenisation

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1. Introduction

The purpose of this survey is to look at elliptic boundary value problems

$$\begin{aligned} \mathcal{A}_n u &= f \quad \text{in } \Omega_n, \\ \mathcal{B}_n u &= 0 \quad \text{on } \partial\Omega_n \end{aligned}$$

with all major types of boundary conditions on a sequence of open sets Ω_n in $\mathbb{R}^N$ $(N \geq 2)$. We then study conditions under which the solutions converge to a solution of a limit problem

$$\begin{aligned} \mathcal{A} u &= f \quad \text{in } \Omega, \\ \mathcal{B} u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

on some open set $\Omega \subset \mathbb{R}^N$. In the simplest case $\mathcal{A}, \mathcal{A}_n = -\Delta$ is the negative Laplace operator, and $\mathcal{B}_n, \mathcal{B}$ the Dirichlet, Robin or Neumann boundary operator, but we work with general non-selfadjoint elliptic operators in divergence form. We are interested in very singular perturbation, not necessarily of a type such that a change of variables can be applied to reduce the problem onto a fixed domain. For the theory of smooth perturbations rather complementary to ours we refer to [84] and references therein.

The main features of this exposition are the following:

- We present an L_p-theory of linear and semi-linear elliptic boundary value problems with domain perturbation in view.
- We establish domain perturbation results for linear elliptic problems with Dirichlet, Robin and Neumann boundary conditions, applicable to semi-linear problems.
- We show how to use the linear perturbation theory to deal with semi-linear problems on bounded and unbounded domains. In particular we show how to get multiple solutions for simple equations, discuss the issue of precise multiplicity and the occurrence of large solutions.
- We provide abstract perturbation theorems useful also for perturbations other than domain perturbations.
- We provide tools to prove results for linear and nonlinear equations on general domains by means of smoothing domains and operators (see Section 8).

Our aim is to build a domain perturbation theory suitable for applications to semi-linear problems, that is, problems where $f = f(x, u(x))$ is a function of x as well as the solution $u(x)$. For nonlinearities with growth, polynomial or arbitrary, we need a good theory for the linear problem in L_p for $1 < p < \infty$. Good in the context of domain perturbations means that in all estimates there is *control on domain dependence* of the constants involved. We establish such a theory in Section 2.1, where we also introduce precise assumptions on the operators. Starting from a definition of weak solutions we prove smoothing properties of the corresponding resolvent operators with control on domain dependence. The main results are Theorems 2.4.1 and 2.4.2. In particular we prove that the resolvent operators have smoothing properties independent of the domain for Dirichlet and Robin boundary conditions, but not for Neumann boundary conditions. To be able to work in a common space we consider the resolvent operator as a map acting on $L_p(\mathbb{R}^N)$, so that it becomes a pseudo-resolvent (see Section 2.5).

The smoothing properties of the resolvent operators enable us to reformulate a semi-linear boundary value problem as a fixed point problem in $L_p(\mathbb{R}^N)$. Which $p \in (1, \infty)$ we choose depends on the growth of the nonlinearity. The rule is, the faster the growth, the larger the choice of p. We also show that under suitable growth conditions, a solution in L_p is in fact in L_∞. Again, the focus is on getting control over the domain dependence of the constants involved. For a precise formulation of these results we refer to Section 3.

Let the resolvent operator corresponding to the linear problems be denoted by $R_n(\lambda)$ and $R(\lambda)$. The key to be able to pass from perturbations of the linear to perturbations of the nonlinear problem is the following property of the resolvents:

$$\text{If } f_n \rightharpoonup f \text{ weakly, then } R_n(\lambda) f_n \to R(\lambda) f \text{ strongly.}$$

Hence for all types of boundary conditions we prove such a statement. If $R(\lambda)$ is compact, it turns out that the above property is equivalent to convergence in the operator norm. An issue connected with that is also the convergence of the spectrum. We show that the above property implies the convergence of every finite part of the spectrum of the relevant differential operators. We also show here that it is sufficient to prove convergence of the resolvent in $L_p(\mathbb{R}^N)$ for some p and some λ to have them for all. These abstract results are collected in Section 4.

The most complete convergence results are known for the Dirichlet problem (Section 5). The limit problem is always a Dirichlet problem on some domain. We extensively discuss convergence in the operator norm. In particular, we look at necessary and sufficient conditions for pointwise and uniform convergence of the resolvent operators. As a corollary to the characterisation we see that convergence is independent of the operator chosen. We also give simple sufficient conditions for convergence in terms of properties of the set $\Omega_n \cap \bar{\Omega}^c$. The main source for these results is [58].

The situation is rather more complicated for Robin boundary conditions, where the type of boundary condition can change in the limit problem. In Section 6 we present three different cases. First, we look at problems with only a small perturbation of $\partial\Omega$. We can cut holes and add thin pieces outside Ω, connected to Ω only near a set of capacity zero. Second, we look at approximating domains with very fast oscillating boundary. In that case the limit problem has Dirichlet boundary conditions. Third, we deal with domains with oscillating boundary, such that the limit problem involves Robin boundary conditions with a different weight on the boundary. The second and third results are really *boundary homogenisation* results. These results are all taken from [51].

The Neumann problem is very badly behaved, and without quite severe restriction on the sequence of domains Ω_n we cannot expect the resolvent to converge in the operator norm. In particular the spectrum does not converge, as already noted in [40, page 420]. We only prove a simple convergence result fitting into the general framework established for the other boundary conditions.

After dealing with linear equations we consider semi-linear equations. A lot of this part is inspired by Dancer's paper [45] and related work. The approach is quite different since we treat linear equations first, and then use their properties to deal with nonlinear equations. The idea is to use degree theory to get solutions on a nearby domain, given a solution of the limit problem. The core of the argument is an abstract topological argument which may be useful also for other types of perturbations (see Section 9.2). We also discuss the issue of precise multiplicity of solutions and the phenomenon of large solutions. Finally, we show that the theory also applies to unbounded limit domains.

There are many other motivations to look at domain perturbation problems, so for instance variational inequalities (see [102]), numerical analysis (see [77,107,110,116–119]), potential and scattering theory (see [10,108,113,124]), control and optimisation (see [31,34,82,120]), Γ-convergence (see [24,42]) and solution structures of nonlinear elliptic equations (see [45,47,52,69]). We mention more references in the discussion on the specific boundary conditions. Some results go back a long time, see for instance [19] or [40]. The techniques are even older for the Dirichlet problem for harmonic functions with the pioneering work [93].

Finally, there are many results we do not even mention, so for instance for convergence in the L_∞-norm we refer to [8,9,14,23,26]. Furthermore, similar results can be proved for parabolic problems. The key for that are domain-independent heat kernel estimates. See for instance [7,17,47,52,59,78] and references therein. The above is only a small rather arbitrary selection of references.

2. Elliptic boundary value problems in divergence form

The purpose of this section is to give a summary of results on elliptic boundary value problems in divergence form with emphasis on estimates with control over the domain dependence.

2.1. *Weak solutions to elliptic boundary value problems*

We consider boundary value problems of the form

$$\begin{aligned} \mathcal{A}u &= f \quad \text{in } \Omega, \\ \mathcal{B}u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{2.1.1}$$

on an open subset of $\mathbb{R}^N$, not necessarily bounded or connected. Here $\mathcal{A}$ is an elliptic operator in divergence form and $\mathcal{B}$ a boundary operator to be specified later in this section. The operator $\mathcal{A}$ is of the form

$$-\operatorname{div}(A_0\nabla u + au) + b\cdot\nabla u + c_0 u \tag{2.1.2}$$

with $A_0 \in L_\infty(\Omega, \mathbb{R}^{N\times N})$, $a, b \in L_\infty(\Omega, \mathbb{R}^N)$ and $c_0 \in L_\infty(\Omega)$. Moreover, we assume that $A_0(x)$ is positive definite, uniformly with respect to $x \in \Omega$. More precisely, there exists a constant $\alpha_0 > 0$ such that

$$\alpha_0|\xi|^2 \le \xi^T A_0(x)\xi \tag{2.1.3}$$

for all $\xi \in \mathbb{R}^N$ and almost all $x \in \Omega$. We call α_0 the *ellipticity constant*.

REMARK 2.1.1. We only defined the operator $\mathcal{A}$ on Ω, but we can extend it to $\mathbb{R}^N$ by setting $a = b = 0$, $c_0 = 0$ and $A(x) := \alpha_0 I$ on Ω^c. Then the extended operator $\mathcal{A}$ also satisfies (2.1.3). In particular the ellipticity property (2.1.3) holds. Hence without loss of generality we can assume that $\mathcal{A}$ is defined on $\mathbb{R}^N$.

We further define the *co-normal derivative* associated with $\mathcal{A}$ on $\partial\Omega$ by

$$\frac{\partial u}{\partial \nu_{\mathcal{A}}} := (A_0(x)\nabla u + a(x)u) \cdot \nu,$$

where ν is the outward pointing unit normal to $\partial\Omega$. Assuming that $\partial\Omega$ is the disjoint union of Γ_1, Γ_2 and Γ_3 we define the boundary operator $\mathcal{B}$ by

$$\mathcal{B}u := \begin{cases} u|_{\Gamma_1} & \text{on } \Gamma_1 \text{ (Dirichlet b.c.)}, \\ \dfrac{\partial u}{\partial \nu_{\mathcal{A}}} & \text{on } \Gamma_2 \text{ (Neumann b.c.)}, \\ \dfrac{\partial u}{\partial \nu_{\mathcal{A}}} + b_0 u & \text{on } \Gamma_3 \text{ (Robin b.c.)} \end{cases} \tag{2.1.4}$$

with $b_0 \in L_\infty(\Gamma_3)$ nonnegative. If all functions involved are sufficiently smooth, then by the product rule

$$-v\operatorname{div}(A_0\nabla u + au) = (A_0\nabla u + au) \cdot \nabla v - \operatorname{div}(v(A_0\nabla u + au))$$

and therefore, if Ω admits the divergence theorem, then

$$\begin{aligned} -\int_\Omega & v\operatorname{div}(A_0\nabla u + au)\,dx \\ &= \int_\Omega (A_0\nabla u + au) \cdot \nabla v\,dx - \int_{\partial\Omega} (v(A_0\nabla u + au)) \cdot \nu\,d\sigma, \end{aligned}$$

where σ is the surface measure on $\partial\Omega$. Hence, if u is sufficiently smooth and $v \in C^1(\bar{\Omega})$ with $v = 0$ on Γ_1, then

$$\int_\Omega v\mathcal{A}u\,dx = \int_\Omega (A_0\nabla u + au) \cdot \nabla v + (a \cdot \nabla u + c_0 u)v\,dx + \int_{\Gamma_3} b_0 uv\,d\sigma.$$

The expression on the right-hand side defines a bilinear form. We denote by $H^1(\Omega)$ the usual Sobolev space of square integrable functions having square integrable weak partial derivatives. Moreover, $H_0^1(\Omega)$ is the closure of the set of test functions $C_c^\infty(\Omega)$ in $H^1(\Omega)$.

DEFINITION 2.1.2. For $u, v \in H^1(\Omega)$ we set

$$a_0(u, v) := \int_\Omega (A_0\nabla u + au) \cdot \nabla v + (b \cdot \nabla u + c_0 u)v\,dx.$$

The expression

$$a(u, v) := a_0(u, v) + \int_{\Gamma_3} b_0 uv\,d\sigma$$

is called the *bilinear form associated with* $(\mathcal{A}, \mathcal{B})$.

If u is a sufficiently smooth solution of (2.1.1), then

$$a(u, v) = \langle f, v\rangle := \int_{\Omega} f v\, dx \tag{2.1.5}$$

for all $v \in C^1(\bar{\Omega})$ with $v = 0$ on Γ_1. Note that (2.1.5) does not just make sense for classical solutions of (2.1.1), but for $u \in H^1(\Omega)$ as long as the boundary integral is defined. We therefore generalise the notion of solution and just require that u is in a suitable subspace V of $H^1(\Omega)$ and (2.1.5) for all $v \in V$.

ASSUMPTION 2.1.3. We require that V be a Hilbert space such that V is dense in $L_2(\Omega)$, that

$$H_0^1(\Omega) \hookrightarrow V \hookrightarrow H^1(\Omega),$$

and that

$$\left\{u \in C^1(\bar{\Omega}) \colon \operatorname{supp} u \subset \bar{\Omega} \setminus \Gamma_1, \int_{\Gamma_3} b_0 |u|^2\, d\sigma < \infty\right\} \subset V.$$

If Γ_3 is nonsmooth we replace the surface measure σ by the $(N-1)$-dimensional Hausdorff measure so that the boundary integral makes sense. We also require that

$$\|u\|_V := (\|u\|^2_{H^1(\Omega)} + \alpha_0^{-1} \|u\sqrt{b_0}\|^2_{L_2(\Gamma_3)})^{1/2} \tag{2.1.6}$$

is an equivalent norm on V.

We next consider some specific special cases.

EXAMPLE 2.1.4. (a) For a homogeneous Dirichlet problem we assume that $\Gamma_1 = \partial\Omega$ and let $V := H_0^1(\Omega)$. For the norm we can choose the usual H^1-norm, but on bounded domains we could just use the equivalent norm $\|\nabla u\|_2$. More generally, on domains Ω lying between two hyperplanes of distance D, we can work with the equivalent norm $\|\nabla u\|_2$ because of Friedrich's inequality

$$\|u\|_2 \leq D \|\nabla u\|_2 \tag{2.1.7}$$

valid for all $u \in H_0^1(\Omega)$ (see [111, Theorem II.2.D]).

(b) For a homogeneous Neumann problem we assume that $\Gamma_2 = \partial\Omega$ and let $V := H^1(\Omega)$ with the usual norm.

(c) For a homogeneous Robin problem we assume that $\Gamma_3 = \partial\Omega$. In this exposition we will always assume that Ω is a Lipschitz domain when working with Robin boundary conditions and choose $V := H^1(\Omega)$. On a bounded domain we can work with the equivalent norm

$$\|v\|_V := (\|\nabla u\|_2^2 + \|u\|^2_{L_2(\partial\Omega)})^{1/2} \tag{2.1.8}$$

(see [111, Theorem III.5.C] or [56]). It is possible to admit arbitrary domains as shown in [11,56].

We finally define what we mean by a weak solution of (2.1.1).

DEFINITION 2.1.5 (Weak solution). We say u is a weak solution of (2.1.1) if $u \in V$ and (2.1.5) holds for all $v \in V$. Moreover, we say that u is a weak solution of $\mathcal{A} = f$ in Ω if $u \in H^1_{\mathrm{loc}}(\Omega)$ such that (2.1.5) holds for all $v \in C_c^\infty(\Omega)$.

Note that a weak solution of $\mathcal{A} = f$ on Ω does not need to satisfy any boundary conditions. As we shall see, it is often easy to get a weak solution in Ω by domain approximation. The most difficult part is to verify that it satisfies boundary conditions. We next collect some properties of the form $a(\cdot\,,\cdot)$ on V. In what follows we use the norms

$$\|A\|_\infty := \Bigl(\sum_{i,j=1}^{N} \|a_{ij}\|_\infty^2\Bigr)^{1/2} \quad \text{and} \quad \|a\|_\infty := \Bigl(\sum_{i=1}^{N} \|a_i\|_\infty^2\Bigr)^{1/2}$$

for matrices $A = [a_{ij}]$ and vectors $a = (a_1, \dots, a_N)$.

PROPOSITION 2.1.6. *Suppose that $(\mathcal{A}, \mathcal{B})$ is defined as above. Then there exists $M > 0$ such that*

$$|a(u, v)| \le M\|u\|_V\|v\|_V \tag{2.1.9}$$

for all $u, v \in V$. More precisely we can set

$$M = \|A\|_\infty + \|a\|_\infty + \|b\|_\infty + \|c_0\|_\infty + \alpha_0.$$

Moreover, if we let

$$\lambda_{\mathcal{A}} := \|c_0^-\|_\infty + \frac{1}{2\alpha_0}\|a + b\|_\infty^2, \tag{2.1.10}$$

where α_0 is the ellipticity constant from (2.1.3), then

$$\frac{\alpha_0}{2}\|\nabla u\|_2^2 \le a_0(u, u) + \lambda_{\mathcal{A}}\|u\|_2^2 \tag{2.1.11}$$

for all $u \in H^1(\Omega)$. Finally, setting $\lambda_0 := \lambda_{\mathcal{A}} + \alpha_0/2$ we see that

$$\frac{\alpha_0}{2}\|u\|_V^2 \le a(u, u) + \lambda_0\|u\|_2^2$$

for all $u \in V$.

PROOF. By the Cauchy–Schwarz inequality and the definition of $a_0(\cdot\,,\cdot)$

$$\begin{aligned}|a_0(u, v)| &\le \|A\nabla u\|_2\|\nabla v\|_2 + \|au\|_2\|\nabla v\|_2 + \|b\nabla u\|_2\|v\|_2 + \|c_0u\|_2\|v\|_2\\ &\le \|A\|_\infty\|\nabla u\|_2\|\nabla v\|_2 + \|a\|_\infty\|u\|_2\|\nabla v\|_2\\ &\qquad + \|b\|_\infty\|\nabla u\|_2\|v\|_2 + \|c_0\|_\infty\|u\|_2\|v\|_2\\ &\le (\|A\|_\infty + \|a\|_\infty + \|b\|_\infty + \|c_0\|_\infty)\|u\|_V\|v\|_V\end{aligned}$$

for all $u, v \in V$. Similarly for the boundary integral

$$\int_{\Gamma_3} b_0uv\,d\sigma \le \|u\sqrt{b_0}\|_{L_2(\Gamma_3)}\|v\sqrt{b_0}\|_{L_2(\Gamma_3)} \le \alpha_0\|u\|_V\|v\|_V$$

for all $u, v \in V$, where we used (2.1.6) for the definition of the norm in V. Combining the two inequalities, (2.1.9) follows. We next prove (2.1.11). Given $u \in H^1(\Omega)$, using (2.1.3) we get

$$\begin{aligned}\alpha_0\|\nabla u\|_2^2 &\leq \int_\Omega (A\nabla u)\cdot\nabla u\,dx\\ &\leq a_0(u,u) - \int_\Omega (a+b)u\cdot\nabla u + c_0^-|u|^2\,dx\\ &\leq a_0(u,u) + \|a+b\|_\infty\|u\|_2\|\nabla u\|_2 + \|c_0^-\|_\infty\|u\|_2^2\\ &\leq a_0(u,u) + \frac{1}{2\alpha_0}\|a+b\|_\infty^2\|u\|_2^2 + \frac{\alpha_0}{2}\|\nabla u\|_2^2 + \|c_0^-\|_\infty\|u\|_2^2\end{aligned}$$

if we use the elementary inequality $xy \leq x^2/2\varepsilon + \varepsilon y^2/2$ valid for $x, y \geq 0$ and $\varepsilon > 0$ in the last step. Rearranging the inequality we get (2.1.11). If we add $\alpha_0\|u\|_2^2/2$ and the boundary integral if necessary to (2.1.11), then the final assertion follows. □

2.2. *Abstract formulation of boundary value problems*

We saw in Section 2.1 that all boundary value problems under consideration have the following structure.

ASSUMPTION 2.2.1 (Abstract elliptic problem). There exist Hilbert spaces V and H such that $V \hookrightarrow H$ and V is dense in H. Suppose there exists a bilinear form

$$a(\cdot\,,\cdot)\colon V\times V \to \mathbb{R}$$

with the following properties. There exists a constant $M > 0$ such that

$$|a(u,v)| \leq M\|u\|_V\|v\|_V \tag{2.2.1}$$

for all $u, v \in V$. Also, there exist constants $\alpha > 0$ and $\lambda_0 \geq 0$ such that

$$\alpha\|u\|_V^2 \leq a(u,u) + \lambda_0\|u\|_H^2 \tag{2.2.2}$$

for all $u \in V$.

By assumption on V and H we have

$$V \hookrightarrow H \hookrightarrow V'$$

if we identify H with its dual H' by means of the Riesz representation theorem with both embeddings being dense. In particular

$$|\langle u, v\rangle| \leq \|u\|_{V'}\|u\|_V$$

for all $u, v \in H$ and duality coincides with the inner product in H. By (2.2.1) the map $v \to a(u,v)$ is bounded and linear for every fixed $u \in V$. If we denote that functional by $Au \in V'$, then

$$a(u,v) = \langle Au, v\rangle$$

for all $u, v \in V$. The operator $A\colon V \to V'$ is linear and by (2.2.1) we have $A \in \mathcal{L}(V, V')$ with

$$\|A\|_{\mathcal{L}(V,V')} \leq M.$$

We say A is the operator induced by the form $a(\cdot\,,\cdot)$. We can also consider it as an operator on V' with domain V.

THEOREM 2.2.2. *Let $A \in \mathcal{L}(V, V')$ be defined as above. Then A is a densely defined closed operator on V' with domain V. Moreover,*

$$[\lambda_0, \infty) \subset \varrho(-A), \tag{2.2.3}$$

and

$$\|(\lambda I + A)^{-1}\|_{\mathcal{L}(V',V)} \leq \alpha^{-1} \tag{2.2.4}$$

for all $\lambda \geq \lambda_0$.

PROOF. From the Lax–Milgram theorem (see [62, Section VI.3.2.5, Theorem 7]) it follows that $(\lambda I + A)^{-1} \in \mathcal{L}(V', V)$ exists for every $\lambda \geq \lambda_0$. In particular $(\lambda_0 I + A)^{-1} \in \mathcal{L}(V', V')$ since $V \hookrightarrow V'$, so $(\lambda_0 I + A)^{-1}$ is closed on V'. As the inverse of a closed linear operator is closed we get that $\lambda_0 I + A$. Hence A is closed as an operator on V'. Since V is dense in H by assumption, V is dense in V' as well. Also from the above, (2.2.3) is true. Next let $f \in V'$ and $u \in V$ with $Au + \lambda u = f$. If $\lambda \geq \lambda_0$, then

$$\alpha\|u\|_V^2 \leq a(u, u) + \lambda\|u\|_H^2 = \langle f, u\rangle \leq \|f\|_{V'}\|u\|_V,$$

from which (2.2.4) follows by dividing by $\|u\|_V$. □

We now look at the abstract elliptic equation

$$Au + \lambda_0 u = f \quad \text{in } V', \tag{2.2.5}$$

which is equivalent to the "weak" formulation that u in V with

$$a(u, v) + \lambda_0\langle u, v\rangle = \langle f, v\rangle \tag{2.2.6}$$

for all $v \in V$. We admit $f \in V'$ in both cases, but note that for $f \in H$, the expression $\langle f, v\rangle$ is the inner product in H. The above theorem tells us that (2.2.5) has a unique solution for every $f \in V'$ whenever $\lambda \geq \lambda_0$. We now summarise the values for λ_0 for the various boundary conditions.

EXAMPLE 2.2.3. (a) If Ω is a bounded domain, or more generally if Ω is an open set lying between two parallel hyperplanes, then (2.1.7) shows that $\|\nabla u\|_2$ is an equivalent norm on $H_0^1(\Omega)$. Hence according to Proposition 2.1.6 we can choose $\lambda_0 := \lambda_{\mathcal{A}}$ in Theorem 2.2.2.

(b) In general, the Neumann problem has a zero eigenvalue. Hence by Proposition 2.1.6 we choose $\lambda_0 = \lambda_{\mathcal{A}} + \alpha_0/2$.

(c) In Example 2.1.4(c) we introduced the equivalent norm (2.1.8). If the boundary coefficient b_0 is bounded from below by a positive constant $\beta > 0$, then by Proposition 2.1.6

$$\begin{aligned} \|\nabla u\|_2^2 + \|u\|_{L_2(\partial\Omega)}^2 &\leq 2\max\left\{\frac{1}{\alpha_0}, \frac{1}{\beta}\right\}\left(\frac{\alpha_0}{2}\|\nabla u\|_2^2 + \beta\|u\|_{L_2(\partial\Omega)}^2\right) \\ &\leq 2\max\left\{\frac{1}{\alpha_0}, \frac{1}{\beta}\right\}\left(a_0(u,u) + \lambda_{\mathcal{A}}\|u\|_2^2 + \int_{\partial\Omega} b_0 u^2\, d\sigma\right). \end{aligned} \tag{2.2.7}$$

Hence in Theorem 2.2.2 we can choose $\lambda_0 := \lambda_{\mathcal{A}}$ if Ω is a bounded Lipschitz domain and $b_0 \geq \beta$ for some constant $\beta > 0$.

We frequently look at solutions in spaces other than V. To deal with such cases we look at the *maximal restriction* of the operator A to some other Banach E space with $E \hookrightarrow V'$. We let

$$D(A_E) := \{u \in V : Au \in E\}$$

with $A_E u := Au$ for all $u \in D(A_E)$ and call A_E the maximal restriction of A to E.

PROPOSITION 2.2.4. *Suppose that A_E is the maximal restriction of A to $E \hookrightarrow V'$ and that $(\lambda I + A)^{-1}(E) \subset E$ for some $\lambda \in \varrho(-A)$. Then A_E is closed and $\varrho(A) \subset \varrho(A_E)$.*

PROOF. We first prove A_E is closed. Suppose that $u_n \in D(A_E)$ with $u_n \to u$ and $Au_n \to v$ in E. As $E \hookrightarrow V'$ convergence is also in V'. Because A is closed in V' with domain E we conclude that $u \in V$ and $Au = v$. We know that $u, v \in E$, so $v \in D(A_E)$ and $A_E u = v$, proving that A_E is closed. By assumption $(\lambda I + A)^{-1}(E) \subset E$. Because $\lambda I + A_E$ is closed, also its inverse is a closed operator on E. Hence by the closed graph theorem $(\lambda I + A_E)^{-1} \in \mathcal{L}(E)$. In particular, the above argument shows that $\varrho(A) \subset \varrho(A_E)$. □

As a special case in the above proposition we can set $E := H$. Sometimes it is useful to prove properties of the operator A via the associated semigroup it generates on H. A proof of the following proposition can be found in [64, §XVII.6, Proposition 3].

PROPOSITION 2.2.5. *Under the above assumption, $-A_H$ generates a strongly continuous analytic semigroup e^{-tA_H} on H. Moreover,*

$$\|(\lambda I + A)^{-1}\|_{\mathcal{L}(H)} \leq \frac{1}{\lambda - \lambda_0}$$

for all $\lambda > \lambda_0$.

PROOF. We only prove the resolvent estimate. If $u = (\lambda I + A)^{-1} f$ for some $\lambda > \lambda_0$ and $f \in H$, then

$$(\lambda - \lambda_0)\|u\|_H^2 \leq a(u,u) + \lambda\|u\|_H^2 = \langle f, u\rangle \leq \|f\|_H \|u\|_H.$$

If we rearrange the inequality, then the resolvent estimate follows. □

2.3. *Formally adjoint problems*

When working with non-selfadjoint problems it is often necessary to consider the adjoint problem. Suppose that $a(\cdot\,,\cdot)$ is a bilinear form on a Hilbert space $V \hookrightarrow H$ as in the previous section. We define a new bilinear form

$$a^\sharp\colon V \times V \to \mathbb{R}$$

by setting

$$a^\sharp(u, v) := a(v, u) \tag{2.3.1}$$

for all $u, v \in V$. If $a(\cdot\,,\cdot)$ satisfies (2.2.1) and (2.2.2), then clearly $a^\sharp(\cdot\,,\cdot)$ has the same properties with the same constants M, α and λ_0. We denote the operator induced on V by $A^\sharp$. It is given by

$$a^\sharp(u, v) = \langle A^\sharp u, v\rangle$$

for all $u, v \in V$. We now relate $A^\sharp$ to the dual A' of A.

PROPOSITION 2.3.1. *Suppose $a(\cdot\,,\cdot)$ is a bilinear form satisfying Assumption 2.2.1 and $a^\sharp(\cdot\,,\cdot)$ and $A^\sharp$ as defined above. Then $A' = A^\sharp \in \mathcal{L}(V, V')$. Moreover, $A'_H = A^\sharp_H \in \mathcal{L}(H, H)$ is the adjoint of the maximal restriction A_H. Finally, if $a(\cdot\,,\cdot)$ is a symmetric form, then A_H is self-adjoint.*

PROOF. Since every Hilbert space is reflexive $V'' = V$ and so $A' \in \mathcal{L}(V'', V') = \mathcal{L}(V, V')$. Now by definition of A and $A^\sharp$ we have

$$\langle Au, v\rangle = a(u, v) = a^\sharp(v, u) = \langle A^\sharp v, u\rangle$$

for all $u, v \in V$. Next look at the maximal restriction A_H. From the above

$$\langle A_H u, v\rangle = \langle u, A^\sharp_H v\rangle$$

for all $u \in D(A_H)$ and all $v \in D(A^\sharp_H)$, so $A'_H = A^\sharp$. Finally, since $A' = A^\sharp = A$ if A is a symmetric form, the maximal restrictions A_H and $A^\sharp_H$ are the same, so A_H is self-adjoint. □

Let us now look at boundary value problems $(\mathcal{A}, \mathcal{B})$ given by (2.1.2) and (2.1.4) satisfying the assumptions made in Section 2.1. Let $a(\cdot\,,\cdot)$ be the form associated with $(\mathcal{A}, \mathcal{B})$ as in Definition 2.1.2. Then $a^\sharp(\cdot\,,\cdot)$ is the form associated with the *formally adjoint boundary value problem* $(\mathcal{A}^\sharp, \mathcal{B}^\sharp)$ given by

$$\mathcal{A}^\sharp = -\operatorname{div}(A_0^T(x)\nabla u + b(x)u) + a(x)\cdot\nabla u + c_0 u \tag{2.3.2}$$

Table 2.1. Constants in (2.4.1) for Dirichlet problems

Condition on Ω	Value of d	Value of λ_0	Value of c_a
$N \geq 3$, any Ω	N	$\lambda_{\mathcal{A}}$	$c(N)/\alpha_0$
$N = 2$, any Ω	any $d \in (2, \infty)$	$\lambda_{\mathcal{A}} + \alpha_0/2$	$c(d)/\alpha_0$
$N = 2$, Ω between parallel hyperplanes of distance D	any $d \in (2, \infty)$	$\lambda_{\mathcal{A}}$	$c(d)(1 + D^2)/\alpha_0$

and

$$\mathcal{B}^\sharp u := \begin{cases} u|_{\Gamma_1} & \text{on } \Gamma_1 \text{ (Dirichlet b.c.)}, \\ \dfrac{\partial u}{\partial \nu_{\mathcal{A}^\sharp}} & \text{on } \Gamma_2 \text{ (Neumann b.c.)}, \\ \dfrac{\partial u}{\partial \nu_{\mathcal{A}^\sharp}} + b_0 u & \text{on } \Gamma_3 \text{ (Robin b.c.)}, \end{cases} \tag{2.3.3}$$

where

$$\frac{\partial u}{\partial \nu_{\mathcal{A}^\sharp}} := (A_0^T(x)\nabla u + b(x)u) \cdot \nu.$$

Note that $(\mathcal{A}^\sharp, \mathcal{B}^\sharp)$ has the same structure as $(\mathcal{A}, \mathcal{B})$ with A_0 replaced by its transposed A_0^T and the roles of a and b interchanged. If A and $A^\sharp$ are the corresponding operators induced on V and H, then all the assertions of Proposition 2.3.1 apply.

2.4. *Global a priori estimates for weak solutions*

In our treatment of domain perturbation problems, global L_p-L_q-estimates for weak solutions to (2.1.1) play an essential role, especially in the nonlinear case with f depending on u. We provide a simple test to obtain such estimates and apply them to the three boundary conditions. The estimates are only based on an embedding theorem for the space V of weak solutions introduced in Section 2.1.

As it turns out, the key to control the domain dependence of L_p-estimates for solutions of (2.1.1) is the constant in a Sobolev-type inequality. More precisely, let $a(\cdot\,,\cdot)$ be the form associated with the boundary value problem $(\mathcal{A}, \mathcal{B})$ as given in Assumption 2.1.2. We require that there are constants $d > 2$, $c_a > 0$ and $\lambda_0 \geq 0$ such that

$$\|u\|^2_{2d/(d-2)} \leq c_a(a(u, u) + \lambda_0\|u\|_2^2) \tag{2.4.1}$$

for all $u \in V$, where V is the space of weak solutions associated with $(\mathcal{A}, \mathcal{B})$. We always require that V satisfies Assumption 2.1.3. Note that $d = N$ is the smallest possible d because of the optimality of the usual Sobolev inequality in $H_0^1(\Omega)$. To obtain control over domain dependence of the constants d, c_a and λ_0, we need to choose d larger in some cases. Explicit values for the three boundary conditions are listed in Tables 2.1–2.3. In these tables, the constant $c(\cdot)$ only depends on its argument, α_0 is the ellipticity constant from (2.1.3), and $\lambda_{\mathcal{A}}$ is given by (2.1.10). Proofs of (2.4.1) for the various cases are given

Table 2.2. Constants in (2.4.1) for Robin problems if $b_0 \geq \beta$ for some $\beta > 0$

Condition on Ω	Value of d	Value of λ_0	Value of c_a
$N \geq 2$, $\lvert\Omega\rvert < \infty$	$2N$	$\lambda_{\mathcal{A}}$	$c(N)(1 + \lvert\Omega\rvert^{1/N}) \max\{\frac{1}{\alpha_0}, \frac{1}{\beta}\}$
$N \geq 2$, $\lvert\Omega\rvert \leq \infty$	$2N$	$\lambda_{\mathcal{A}} + \alpha_0/2$	$c(N) \max\{\frac{1}{\alpha_0}, \frac{1}{\beta}\}$

Table 2.3. Constants in (2.4.1) for Neumann problems

Condition on Ω	Value of d	Value of λ_0	Value of c_a
$N \geq 3$, cone condition	N	$\lambda_{\mathcal{A}} + \alpha_0/2$	$c(d, \text{cone})/\alpha_0$
$N = 2$, cone condition	any $d \in (2, \infty)$	$\lambda_{\mathcal{A}} + \alpha_0/2$	$c(d, \text{cone})/\alpha_0$
$N \geq 2$, special class of Ω	$d > 2$ depending on Ω	$\lambda_{\mathcal{A}} + \alpha_0/2$	$c(d, \text{class})/\alpha_0$
No condition on Ω	"$d = \infty$" no smoothing	N/A	N/A

in Sections 2.4.2 and 2.4.3. It turns out that there are domain-independent estimates and smoothing properties for Dirichlet and Robin boundary conditions, but not for Neumann boundary conditions.

THEOREM 2.4.1. *Suppose that $\Omega \subset \mathbb{R}^N$ is an open set and $u \in V$ is a weak solution of* (2.1.1). *If* (2.4.1) *holds, then there exists a constant $C > 0$ only depending on d and $p \geq 2$ such that*

$$\|u\|_{dp/(d-2p)} \leq c_a C(\|f\|_p + \lambda_0 \|u\|_p) \tag{2.4.2}$$

if $p \in [2, d/2)$, and

$$\|u\|_\infty \leq c_a C(\|f\|_p + \lambda_0 \|u\|_p) + \|u\|_p \tag{2.4.3}$$

or

$$\|u\|_\infty \leq c_a C(\|f\|_p + \lambda_0 \|u\|_p) + \|u\|_{2d/(d-2)} \tag{2.4.4}$$

if $p > d/2$. Moreover, if $\lambda_0 = 0$ or $u \in L_p(\Omega)$ (if $\lvert\Omega\rvert < \infty$ for instance), then the above estimates are valid for $p \in [2d/(d+2), 2)$ as well.

PROOF. Let A be the operator induced by the form $a(\cdot\,,\cdot)$ on V. Given $u \in V$ we set $u_q := |u|^{q-2}u$ for $q \geq 2$. Assuming that (2.4.1) holds, it follows from [57, Proposition 5.5] that

$$\|u\|^q_{dq/(d-2)} \leq c_a \frac{q}{2}(\langle Au, u_q\rangle + (q-1)\lambda_0 \langle u, u_q\rangle)$$

for all $q \geq 2$ and $u \in V$ for which the expression on the right-hand side is finite. Then we apply [57, Theorem 4.5] to get the estimates. Compared to that reference, we have replaced the term $\|u\|_p$ by $\|u\|_{2d/(d-2)}$ in (2.4.4). We can do this by replacing q_0 in equation (4.11) in the proof of [57, Theorem 4.5] by $q_0 := 1 + \frac{2d}{p'(d-2)}$ and then complete the proof in a similar way. Finally, the limitation that $p \geq 2$ comes from proving that $u \in L_p(\Omega)$ first if $\lambda_0 \neq 0$. If $\lambda_0 = 0$, or if we know already that $u \in L_p(\Omega)$, then this is not necessary (see also [61, Theorem 2.5]), and we can admit $p \in [2d/(d+2), 2)$. □

The above theorem tells us that for $p \geq 2$ and $\lambda \geq \lambda_{\mathcal{A}}$

$$(\lambda I + A)^{-1} \colon L_p(\Omega) \cap L_2(\Omega) \to L_{m(p)}(\Omega)$$

if we set

$$m(p) := \begin{cases} \dfrac{dp}{d-2p} & \text{if } p \in (1, d/2), \\ \infty & \text{if } p > d/2 \end{cases} \tag{2.4.5}$$

and A is the operator associated with the problem (5.1.2) as constructed in Section 2.2.

We next want to derive domain-independent bounds for the norm of the resolvent operator by constructing an operator in $L_p(\Omega)$ for $p \in (1, \infty)$. Let A_2 denote the maximal restriction of A to $L_2(\Omega)$. By Proposition 2.2.5, the operator $-A_2$ is the generator of a strongly continuous analytic semigroup on $L_2(\Omega)$. Moreover, still assuming that (2.4.1) holds, e^{-tA_2} has an kernel satisfying pointwise Gaussian estimates and therefore interpolates to $L_p(\Omega)$ for all $p \in (1, \infty)$ (see [55]). Denote by $-A_p$ its infinitesimal generator. The dual semigroup is a strongly continuous analytic semigroup on $L_{p'}(\Omega)$ and its generator is A'_p. Let $A^\sharp_p$ be the corresponding operators associated with the formally adjoint problem. Since $(A^\sharp_2)' = A_2$ by Proposition 2.3.1 we get $(A^\sharp_p)' = A_{p'}$. We denote the exponent conjugate to p by p', that is,

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

From [55, Theorem 5.1]

$$\|e^{-tA_p}\|_{\mathcal{L}(L_p)} \leq e^{\omega_p t}, \qquad \omega_p := \max\{p-1, p'-1\}\lambda_0 \tag{2.4.6}$$

for all $t > 0$ and $p \in (1, \infty)$.

Solutions of the abstract equation $(\lambda I + A_p)u = f$ with $f \in L_p(\Omega)$ are called generalised solutions of the corresponding elliptic problem in $L_p(\Omega)$. If $1 < p < 2N/(N+2)$ they are not weak solutions in general, but solutions in an even weaker sense. Note that we defined A_p by means of semigroup theory to be able to easily get a definition for domains with unbounded measure, because for such domains we cannot expect $L_p(\Omega) \subset V'$ for $p > 2$.

THEOREM 2.4.2. *Let A_p as defined above and $p \in (1, \infty)$. Then $(\omega_p, \infty) \subset \varrho(-A_p)$ with $\omega_p := \max\{p-1, p'-1\}\lambda_0$, and*

$$\|(\lambda I + A_p)^{-1}\|_{\mathcal{L}(L_p)} \leq \frac{1}{\lambda - \omega_p} \tag{2.4.7}$$

for all $\lambda > \omega_p$. Furthermore, for every $p > 1$, $p \neq N/2$, there exists a constant $C > 0$ only depending on d, p and c_a and λ, λ_0 such that

$$\|(\lambda I + A_p)^{-1}\|_{\mathcal{L}(L_p, L_{m(p)})} \leq C \tag{2.4.8}$$

for all $\lambda > \lambda_0$. If Ω is bounded, then $(\lambda I + A)^{-1} \colon L_p(\Omega) \to L_q(\Omega)$ is compact for all $q \in [p, m(p))$. Finally, if $A^\sharp_p$ is the operator associated with the formally adjoint problem, then $A'_p = A^\sharp_{p'}$.

PROOF. As $-A_p$ generates a strongly continuous semigroup, (2.4.6) implies that $(\omega_p, \infty) \subset \varrho(-A_p)$ and that

$$(\lambda I + A_p)^{-1} = \int_0^\infty e^{-tA_p} e^{-\lambda t}\, dt$$

for all $\lambda > \omega_p$ (see [125, Section IX.4]). Hence (2.4.7) follows if we take into account (2.4.6). Now for $f \in L_2(\Omega) \cap L_p(\Omega)$ we have $u := (\lambda I + A_p)^{-1} \in L_p(\Omega)$. Hence, if $2d/(d+2) \leq p < d/2$, then by (2.4.2)

$$\|u\|_{dp/(d-2p)} \leq c_a C(\|f + \lambda u\|_p + \lambda_0 \|u\|_p) \leq c_a C \left(1 + \frac{\lambda + \lambda_0}{\lambda - \omega_p}\right) \|f\|_p.$$

A similar estimate is obtained by using (2.4.3) if $p > d/2$. Now (2.4.8) follows since $L_2(\Omega) \cap L_p(\Omega)$ is dense in $L_p(\Omega)$ if we choose C appropriately. If $1 < p < 2d/(d+2)$, then we use a duality argument. In that case $q := m(p)' > 2d/(d-2)$ and a simple calculation reveals that $p = m(q)'$. Because

$$(\lambda I + A_q^\sharp)^{-1} \in \mathcal{L}(L_q(\mathbb{R}^N), L_{m(q)}(\mathbb{R}^N)),$$

by duality

$$((\lambda I + A_q^\sharp)^{-1})' = (\lambda I + (A_q^\sharp)')^{-1} = (\lambda I + A_p)^{-1} \in \mathcal{L}(L_p(\mathbb{R}^N), L_{m(p)}(\mathbb{R}^N))$$

with equal norm. Compactness of the resolvent on $L_p(\Omega)$ for $1 < p < \infty$ follows from [57, Section 7]. Now compactness as an operator from $L_p(\Omega)$ to $L_q(\Omega)$ for $q \in [p, m(p))$ follows from a compactness property of the Riesz–Thorin interpolation theorem (see [94]). □

REMARK 2.4.3. Note that the above theorem is not optimal, but it is sufficient for our purposes. In particular the condition $\lambda > \omega_p$ could be improved by various means. If A has compact resolvent, then the spectrum of A_p is independent of p because the above smoothing properties of the resolvent operator show that every eigenfunction is in $L^\infty(\Omega)$. Also if $p = N$, then the spectrum is independent of p by [95] because of Gaussian bounds for heat kernels (see [55]).

2.4.1. *Sobolev inequalities associated with Dirichlet problems*

If $N \geq 3$ there exists a constant $c(N)$ only depending on the dimension N such that

$$\|u\|_{2N/(N-2)} \leq c(N) \|\nabla u\|_2 \tag{2.4.9}$$

for all $u \in H^1(\mathbb{R}^N)$ (see [76, Theorem 7.10]). If $N = 2$, then for every $q \in [2, \infty)$ there exists a constant c_q only depending on q such that

$$\|u\|_q \leq c_q \|u\|_{H^1(\mathbb{R}^N)}$$

for all $u \in H^1(\mathbb{R}^2)$ (see [96, Theorem 8.5]). If $q \in (2, \infty)$ and $d := 2q/(q-2)$, then $q = 2d/(d-2)$. Hence for every $d > 2$ there exists $c_d > 0$ only depending on d such that

$$\|u\|_{2d/(d-2)} \leq c_d \|u\|_{H^1(\mathbb{R}^N)}. \tag{2.4.10}$$

If Ω is lying between two parallel hyperplanes of distance D, then using (2.1.7) we conclude that

$$\|u\|_{2d/(d-2)} \leq c_d \|u\|_{H^1(\mathbb{R}^N)} \leq c_d \sqrt{1+D^2} \|\nabla u\|_2 \tag{2.4.11}$$

for all $u \in H^1_0(\Omega)$. Combining the above with the basic inequalities in Proposition 2.1.6 we can summarise the constants appearing in (2.4.1) in Table 2.1.

The L_∞-estimates for Dirichlet problems are very well known, see for instance [76, Chapter 8]. The estimates for $p < N/2$ are not as widely known, and sometimes stated with additional assumptions on the structure of the operators, see [38, Appendix to Chapter 3], [112, Theorem 4.2] or without proof in [45, Lemma 1]. A complete proof is contained in [57].

2.4.2. *Maz'ya's inequality and Robin problems*

It may be surprising that solutions of the elliptic problem with Robin boundary conditions

$$\begin{aligned} \mathcal{A}u &= f \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu_{\mathcal{A}}} + b_0 u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{2.4.12}$$

with $b_0 \geq \beta$ for some constant $\beta > 0$ satisfy domain-independent estimates similar to the ones for problems with Dirichlet boundary conditions. The estimates were first established in [56]. In the present discussion we will only work with bounded Lipschitz domains $\Omega \subset \mathbb{R}^N$, but note that the result could be generalised to arbitrary domains. We refer to [11,56] for details.

The weak solutions of the Robin problem on a Lipschitz domain are in $H^1(\Omega)$ as discussed in Example 2.1.4(c). In the usual Sobolev inequality $\|u\|_{2N/(N-2)} \leq c\|u\|_{H^1}$, the constant c depends on the shape of the domain as examples of domains with a cusp show (see [2, Theorem 5.35]). Hence only a weaker statement can be true. The key is an inequality due to Maz'ya from [100] (see [101, Section 3.6]) stating that

$$\|u\|_{N/(N-1)} \leq c(N)(\|\nabla u\|_1 + \|u\|_{L_1(\partial\Omega)})$$

for all $u \in W^1_1(\Omega) \cap C(\bar{\Omega})$, where $c(N)$ is the isoperimetric constant depending only on $N \geq 2$. Substituting u^2 into the above inequality we get

$$\begin{aligned} \|u\|^2_{2N/(N-1)} &\leq c(N)(2\|u\nabla u\|_1 + \|u\|^2_{L_2(\partial\Omega)}) \\ &\leq c(N)(\|u\|^2_{H^1} + \|u\|^2_{L_2(\partial\Omega)}). \end{aligned} \tag{2.4.13}$$

By the density of $H^1(\Omega) \cap C(\bar{\Omega})$ in $H^1(\Omega)$ the inequality is valid for all $u \in H^1(\Omega)$. If Ω has finite measure, then similarly

$$\|u\|^2_{2N/(N-1)} \leq c(N)(1+|\Omega|)^{1/N}(\|\nabla u\|^2_2 + \|u\|^2_{L_2(\partial\Omega)}) \tag{2.4.14}$$

for all $u \in H^1(\Omega)$ with a constant $c(N)$ different from the original one, but only depending on N. Combining (2.2.7) with (2.4.13) or (2.4.14) we therefore get (2.4.1) with $d = 2N$ as displayed in Table 2.2.

2.4.3. *Sobolev inequalities associated with Neumann problems*
The smoothing properties for Dirichlet and Robin problems we established in the previous sections were based on the validity of a Sobolev-type inequality for functions in the space of weak solutions with a constant independent of the shape of the domain. The space of weak solutions for the Neumann problem

$$\begin{aligned} \mathcal{A}u &= f \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu_{\mathcal{A}}} &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

is $H^1(\Omega)$. In the case of the Robin problem we could use an equivalent norm on $H^1(\Omega)$ involving a boundary integral. As pointed out in Section 2.4.2, the boundary integral is of higher order than the H^1-norm if the domain is bad. Hence for the Neumann problem the constant c in the Sobolev inequality

$$\|u\|_{2N/(N-2)} \leq c\|u\|_{H^1} \tag{2.4.15}$$

depends on the shape and not just the measure of Ω. There are no easy necessary and sufficient conditions for the inequality to be true. A sufficient condition is that Ω satisfies an (interior) cone condition, that is, there exists an open cone $C \subset \mathbb{R}^N$ with vertex at zero such that for every $x \in \partial\Omega$ there is an orthogonal transformation T such that $x+T(C) \subset \Omega$ (see [2, Definition 4.3]). The constant c in (2.4.15) depends on the length and the angle of the cone C (see [2, Lemma 5.12]). A cone condition is however not necessary for getting a Sobolev inequality uniformly with respect to a family of domains. Shrinking holes of fixed shape to a point is sufficient (see [53, Section 2]). Alternatively an extension property is also sufficient. This includes domains with fractal boundary (quasi-disks) as shown in [101, Section 1.5.1]. We could replace the cone C by a standard polynomial cusp and get an inequality of the form

$$\|u\|_{2d/(d-2)} \leq c\|u\|_{H^1} \tag{2.4.16}$$

with $d \geq N$ depending on the sharpness of the cusp (see [2, Theorem 5.35] or [101, Section 4.4]). For general domains there is no such $d > 2$, and there are no smoothing properties of the resolvent operator. This corresponds to the degenerate case "$d = \infty$" because $2d/(d-2) \to 2$ as $d \to \infty$. Combining Proposition (2.1.11) with the above we get (2.4.1) with constants as displayed in Table 2.3.

2.5. *The pseudo-resolvent associated with boundary value problems*

When dealing with varying domains we want to embed our problem into a fixed large space. In this section we want to explain how to do that. We define the inclusion $i_\Omega\colon L_p(\Omega) \to L_p(\mathbb{R}^N)$ to be the trivial extension

$$i_\Omega(u) := \begin{cases} u & \text{on } \Omega, \\ 0 & \text{on } \Omega^c. \end{cases} \tag{2.5.1}$$

We also sometimes write $\tilde{u} := i_\Omega(u)$ for the trivial extension. The above extension operator also acts as an operator

$$i_\Omega\colon H_0^1(\Omega) \to H^1(\mathbb{R}^N).$$

Indeed, by definition $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$. We can identify $C_c^\infty(\Omega)$ with the set $\{u \in C_c^\infty(\mathbb{R}^N)\colon \operatorname{supp} u \subset \Omega\}$, and view $H_0^1(\Omega)$ as the closure of $C_c^\infty(\Omega)$ in $H^1(\mathbb{R}^N)$.

Furthermore, we let $r_\Omega\colon L_p(\mathbb{R}^N) \to L_p(\Omega)$ be the restriction

$$r_\Omega(u) := u|_\Omega. \tag{2.5.2}$$

If $f \in H^{-1}(\mathbb{R}^N)$, then we restrict the functional to the closed subspace $H_0^1(\Omega)$ of $H^1(\mathbb{R}^N)$ to get an element of $H^{-1}(\Omega)$. More formally we define the restriction operator $r_\Omega\colon H^{-1}(\mathbb{R}^N) \to H^{-1}(\Omega)$ by

$$r_\Omega(f) := f|_{H_0^1(\Omega)}. \tag{2.5.3}$$

On the subspace $L_2(\mathbb{R}^N)$ of $H^{-1}(\mathbb{R}^N)$, the two definitions coincide. We next prove that the operators i_Ω and r_Ω are dual to each other.

LEMMA 2.5.1. *Let $\Omega \subset \mathbb{R}^N$ be open. Let i_Ω and r_Ω as defined above and $1 < p < \infty$. Then*

$$i_\Omega \in \mathcal{L}(L_p(\Omega), L_p(\mathbb{R}^N)) \cap \mathcal{L}(H_0^1(\Omega), H^1(\mathbb{R}^N))$$

and $\|i_\Omega\| = \|r_\Omega\| = 1$. Moreover

$$i'_\Omega = r_\Omega \in \mathcal{L}(L_{p'}(\mathbb{R}^N), L_{p'}(\Omega)) \cap \mathcal{L}(H^{-1}(\mathbb{R}^N), H^{-1}(\Omega))$$

and $r'_\Omega = i_\Omega$, where p' is the conjugate exponent to p.

PROOF. The first assertion follows directly from the definition of the operators. If $f \in H^{-1}(\mathbb{R}^N)$ or $L_{p'}(\mathbb{R}^N)$, then by definition of i_Ω and r_Ω

$$\langle f, i_\Omega(u)\rangle = \langle r_\Omega(f), u\rangle$$

for all $u \in C_c^\infty(\Omega)$. By density of $C_c^\infty(\Omega)$ in $H_0^1(\Omega)$ and $L_p(\Omega)$ we get $i'_\Omega = r_\Omega$. By the reflexivity of the spaces involved we therefore also have $r'_\Omega = i_\Omega$. □

Let $(\mathcal{A}_n, \mathcal{B}_n)$ and $(\mathcal{A}, \mathcal{B})$ be elliptic boundary value problems on open sets Ω_n and Ω, respectively. Suppose that A_n and A are the corresponding operators induced as discussed in Section 2.2. We can then embed the problems in $\mathbb{R}^N$ as follows.

DEFINITION 2.5.2. Let A_n, A defined as above. We set

$$R_n(\lambda) := i_{\Omega_n}(\lambda I + A_n)^{-1} r_{\Omega_n} \quad \text{and} \quad R(\lambda) := i_\Omega(\lambda I + A)^{-1} r_\Omega$$

whenever the inverse operators exist. Similarly we define $R_n^\sharp(\lambda)$ and $R^\sharp(\lambda)$ for the formally adjoint problem.

The family of operators $R(\lambda)$ and $R_n(\lambda)$ form a pseudo-resolvent as defined for instance in [125, Section VIII.4]. In particular they satisfy the resolvent equation

$$R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu)$$

for all $\lambda, \mu \in \varrho(-A)$. Using Lemma 2.5.1 and the last assertion in Theorem 2.4.2 we get the following properties of the pseudo-resolvent.

LEMMA 2.5.3. *If $\lambda \in \varrho(A)$, then $R(\lambda)' = R^\sharp(\lambda)$.*

REMARK 2.5.4. From the above it also follows that we can replace $(\lambda + A)^{-1}$ by $R(\lambda)$ in (2.4.7) and (2.4.8) with all constants being the same.

3. Semi-linear elliptic problems

The purpose of this section is to formulate semi-linear boundary value problems as a fixed point equation in $L_p(\mathbb{R}^N)$. Then we derive an L_∞-estimate for solutions in terms of an L_p-norm, provided the nonlinearity satisfies a growth condition.

3.1. *Abstract formulation of semi-linear problems*

We now want to look at properties of weak solutions of the semi-linear boundary value problem

$$\begin{aligned} \mathcal{A}u &= f(x, u(x)) && \text{in } \Omega, \\ \mathcal{B}u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.1.1}$$

where $(\mathcal{A}, \mathcal{B})$ are as discussed in Section 2.1. We use the smoothing properties from Section 2.4 to show that under suitable growth conditions on f, the boundary value problem (3.1.1) can be viewed as a fixed point equation in $L_p(\mathbb{R}^N)$ for a some range of $p \in (1, \infty)$.

We assume that $V \subset H^1(\Omega)$ is the space of weak solutions for the boundary conditions under consideration as introduced in Section 2.1 and discussed in detail for different boundary conditions in Section 2.4. We also assume that $f \colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a function with properties to be specified. Given $u \colon \Omega \to \mathbb{R}$ we define the *superposition operator* $F(u)$ by

$$F(u)(x) := f(x, u(x)) \tag{3.1.2}$$

for all $x \in \Omega$, provided that $F(u) \in V'$. We call $u \in V$ a weak solution of (3.1.1) if

$$a(u, v) = \langle F(u), v\rangle$$

for all $v \in V$. Here $a(\cdot\,,\cdot)$ is the form associated with $(\mathcal{A}, \mathcal{B})$ as in Definition 2.1.2. If A is the operator induced by $(\mathcal{A}, \mathcal{B})$ we can rewrite (3.1.1) as

$$Au = F(u). \tag{3.1.3}$$

To be able to show that $F(u) \in V'$ we need to make some assumptions on f.

ASSUMPTION 3.1.1. Suppose that $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, that is, $f(\cdot\,,\xi)\colon \Omega \to \mathbb{R}$ is measurable for all $\xi \in \mathbb{R}$, and $f(x,\cdot) \in C(\mathbb{R})$ for almost all $x \in \Omega$. Further suppose that there exist a function $g \in L_1(\Omega) \cap L_\infty(\Omega)$ and constants $1 \le \gamma < \infty$ and $c \ge 0$ such that

$$|f(x,\xi)| \le g(x) + c|\xi|^\gamma \tag{3.1.4}$$

for all $(x,\xi) \in \Omega \times \mathbb{R}$.

The above growth conditions on f lead to the following mapping properties of the superposition operator.

LEMMA 3.1.2. *Suppose that f satisfies Assumption 3.1.1 with $1 \le \gamma \le p$. Then the corresponding superposition operator F is in $C(L_p(\Omega), L_{p/\gamma}(\Omega))$. Moreover,*

$$\|F(u)\|_{p/\gamma} \le \|g\|_1 + \|g\|_\infty + c\|u\|_p^\gamma$$

for all $u \in L_p(\Omega)$.

PROOF. By [6, Theorem 3.1] we have

$$\|F(u)\|_{p/\gamma} \le \|g\|_{p/\gamma} + c\|u\|_p^\gamma \le \|g\|_1^{\gamma/p}\|g\|_\infty^{1-\gamma/p} + c\|u\|_p.$$

Then use Young's inequality to get

$$\|g\|_1^{\gamma/p}\|g\|_\infty^{1-\gamma/p} \le \frac{\gamma}{p}\|g\|_1 + \left(1 - \frac{\gamma}{p}\right)\|g\|_\infty \le \|g\|_1 + \|g\|_\infty.$$

Continuity is proved in [6, Theorem 3.7]. □

Note that [6, Theorem 3.1] shows that the growth conditions are necessary and sufficient for F to map $L_p(\Omega)$ into $L_{p/\gamma}(\Omega)$. In particular, if F maps $L_p(\Omega)$ into itself and $p \in (1,\infty)$, then $\gamma = 1$, that is, f grows at most linearly.

From now on we assume that (2.4.1) is true for all $v \in V$. Then by Theorem 2.4.2

$$(\lambda I + A)^{-1} \in \mathcal{L}(L_p(\Omega)) \cap \mathcal{L}(L_p(\Omega), L_{m(p)}(\Omega))$$

for all $\lambda \in \varrho(-A)$ with $m(p)$ given by (2.4.5). If we fix $\lambda \in \varrho(-A)$, then we can rewrite (3.1.3) as $Au + \lambda u = F(u) + \lambda u$ and hence in form of the fixed point equation

$$u = (\lambda I + A)^{-1}(F(u) + \lambda u).$$

To be able to consider this as an equation in $L_p(\Omega)$ we need that the right-hand side is in $L_p(\Omega)$ if $u \in L_p(\Omega)$. Taking into account Lemma 3.1.2 and the smoothing property of the resolvent we need that $m(p/\gamma) \ge p$. When looking at boundedness of weak solutions and convergence properties with respect to the domain it is necessary to require $m(p/\gamma) > p$, or equivalently,

$$1 \le \gamma < 1 + \frac{2p}{d}. \tag{3.1.5}$$

Because every weak solution of (3.1.1) lies in $L_{2d/(d-2)}(\Omega)$ we can assume that $p \ge 2d/(d-2)$. Then automatically $1 \le \gamma \le p$ as required in Lemma 3.1.2.

PROPOSITION 3.1.3. *Suppose that* $(\mathcal{A}, \mathcal{B})$ *is such that* (2.4.1) *holds for some* $d > 2$. *Moreover, let* $2d/(d-2) \le p < \infty$ *such that* Assumption 3.1.1 *holds with* γ *satisfying* (3.1.5). *Fix* $\lambda \in \varrho(-A)$ *and set*

$$G(u) := (\lambda I + A)^{-1}(F(u) + \lambda u).$$

Then $G \in C(L_p(\Omega), L_p(\Omega))$ *and*

$$\begin{aligned}\|G(u)\|_p \le \|R(\lambda)\|_{\mathcal{L}(L_{p/\gamma}, L_p)}(\|g\|_1 + \|g\|_\infty + c\|u\|_p^\gamma) \\ + |\lambda|\|R(\lambda)\|_{\mathcal{L}(L_p)}\|u\|_p \end{aligned} \tag{3.1.6}$$

for all $u \in L_p(\Omega)$. *Furthermore, if* Ω *is bounded, then* G *is compact, that is,* G *maps bounded sets of* $L_p(\Omega)$ *onto relatively compact sets of* $L_p(\Omega)$. *Finally,* $u \in L_p(\Omega) \cap V$ *is a weak solution of* (3.1.1) *if and only if* u *is a fixed point of*

$$u = G(u)$$

in $L_p(\Omega)$.

PROOF. By Lemma 3.1.2, $F \in C(L_p(\Omega), L_{p/\gamma}(\Omega))$ is bounded with

$$\|F(u)\|_{p/\gamma} \le \|g\|_1 + \|g\|_\infty + c\|u\|_p^\gamma,$$

so F is bounded. Clearly (3.1.5) implies $m(p/\gamma) > p$, so Theorem 2.4.2 shows that $R(\lambda) \in \mathcal{L}(L_{p/\gamma}(\Omega), L_p(\Omega)) \cap \mathcal{L}(L_p(\Omega))$ with the operator being compact if Ω is bounded. Hence G is continuous as claimed and compact if Ω is bounded. From the definition of G

$$\|G(u)\|_p \le \|R(\lambda)\|_{\mathcal{L}(L_{p/\gamma}, L_p)}\|F(u)\|_{p/\gamma} + |\lambda|\|R(\lambda)\|_{\mathcal{L}(L_p)}\|u\|_p.$$

Combining it with the estimate of $\|F(u)\|_{p/\gamma}$ from above we obtain (3.1.6). The last assertion is evident from the definition of G. □

REMARK 3.1.4. Because every weak solution lies in $L_{2d/(d-2)}(\Omega)$ the above condition is automatically satisfied if $\gamma < (d+2)/(d-2)$, that is, the growth is subcritical for the exponent d.

3.2. *Boundedness of weak solutions*

We apply results from Section 2.4 to show that weak solutions of (2.1.1) are in L_∞ if they are in $L_p(\mathbb{R}^N)$ and the nonlinearity satisfies a growth condition.

THEOREM 3.2.1. *Suppose that* $(\mathcal{A}, \mathcal{B})$ *is such that* (2.4.1) *holds for some* $d > 2$ *and* $\lambda_0 \ge 0$. *Moreover, let* $p \ge 2d/(d-2)$ *such that* Assumption 3.1.1 *holds with* γ *satisfying* (3.1.5). *Suppose that* $u \in V \cap L_p(\Omega)$ *is a weak solution of* (3.1.1). *If* $\lambda_0 = 0$, *then* $u \in L_\infty(\Omega)$ *and there exists an increasing function* $q\colon [0,\infty) \to [0,\infty)$ *such that*

$$\|u\|_\infty \le q(\|u\|_p).$$

That function only depends on γ, p, *an upper bound for* $\|g\|_1 + \|g\|_\infty$ *and* c *from* Assumption 3.1.1 *and the constants* c_a, C *from* Theorem 2.4.1.

If $\lambda_0 > 0$ *and in addition* $u \in L_{p/\gamma}(\Omega)$, *then* $u \in L_\infty(\Omega)$ *and*

$$\|u\|_\infty \le q(\|u\|_p, \|u\|_{p/\gamma})$$

with the function q *also depending on* λ_0.

PROOF. Suppose that $u \in L_p(\Omega)$ is a solution of (3.1.1) with p and γ satisfying (3.1.5). We set

$$p_{k+1} := m(p_k/\gamma) \quad \text{and} \quad p_0 := p$$

and note that $p_0/\gamma = p/\gamma \ge 2d/(d+2)$. Using (2.4.2) with $\lambda_0 = 0$ and Lemma 3.1.2

$$\|u\|_{p_{k+1}} \le c_a C \|F(u)\|_{p_k/\gamma} \le c_a C(\|g\|_1 + \|g\|_\infty + \|u\|_{p_k}^\gamma) \tag{3.2.1}$$

as long as $p_k < \gamma d/2$. From (3.1.5) the sequence (p_k) is increasing. Hence, again using (3.1.5) we get

$$\begin{aligned} p_{k+1} - p_k &= \frac{dp_k}{d\gamma - 2p_k} - p_k = \left(\frac{d}{d\gamma - 2p_k} - 1\right) p_k \\ &\ge \left(\frac{d}{d\gamma - 2p} - 1\right) p > 0 \end{aligned}$$

as long as $p_k/\gamma < d/2$. Therefore we can choose $m \in \mathbb{N}$ such that $p_m < \gamma d/2 < p_{m+1}$. Then by (2.4.2) and Lemma 3.1.2

$$\begin{aligned} \|u\|_\infty &\le c_a C \|F(u)\|_{p_{m+1}/\gamma} + \|u\|_{p_{m+1}} \\ &\le c_a C(\|g\|_1 + \|g\|_\infty + \|u\|_{p_{m+1}}^\gamma) + \|u\|_{p_{m+1}} \end{aligned} \tag{3.2.2}$$

and we are done. We now obtain the L_∞-bound for u by applying (3.2.1) inductively to $k = 0, \dots, m$, and finally using (3.2.2). It is now obvious how to define the function q having the required properties. If $\lambda_0 > 0$, then (3.2.1) has to be replaced by

$$\begin{aligned} \|u\|_{p_{k+1}} &\le c_a C(\|F(u)\|_{p_k/\gamma} + \lambda_0 \|u\|_{p_k/\gamma}) \\ &\le c_a C(\|g\|_1 + \|g\|_\infty + \|u\|_{p_k}^\gamma + \lambda_0 \|u\|_{p_k/\gamma}). \end{aligned}$$

Now the assertion follows in a similar manner as in the case $\lambda_0 = 0$. □

REMARK 3.2.2. On domains with finite measure we often work with (3.1.4), where g is a constant. In that case dependence on $\|g\|_1 = g|\Omega|$ means dependence on the measure of the domain and the magnitude of g. Moreover, if Ω has finite measure, then the condition $u \in L_{p/\gamma}(\Omega)$ is automatically satisfied because $p/\gamma \le p$.

4. Abstract results on linear operators

Many convergence properties of the resolvents reduce to an abstract perturbation theorem. We collect these results here. The first is a characterisation of convergence in the operator norm if the limit is compact. Then we discuss a spectral mapping theorem and how to apply it to get continuity of the spectrum and the corresponding projections. Finally we use an interpolation argument to extend convergence in $L_p(\mathbb{R}^N)$ for some p to all p.

4.1. *Convergence in the operator norm*

The aim of this section is to prove a characterisation of convergence in the operator norm useful in the context of domain perturbations. Recall that a sequence of operators (T_n) on a Banach space E is called *strongly convergent* if $T_n f \to Tf$ for all $f \in E$.

PROPOSITION 4.1.1. *Suppose that E, F are Banach spaces, E is reflexive and that $T_n, T \in \mathcal{L}(E, F)$. Then the following assertions are equivalent.*

(1) *T is compact and $T_n \to T$ in $\mathcal{L}(E, F)$;*
(2) *$T_n f_n \to Tf$ in F whenever $f_n \rightharpoonup f$ weakly in E;*
(3) *$T_n \to T$ strongly and $T_n f_n \to 0$ in F whenever $f_n \rightharpoonup 0$ weakly in E.*

PROOF. We first prove that (1) implies (2). Assuming that $f_n \rightharpoonup f$ weakly in E we have

$$\|T_n f_n - Tf\|_F \leq \|T_n - T\|_{\mathcal{L}(E,F)}\|f_n\|_E + \|T(f_n - f)\|_F.$$

The first term on the right-hand side converges to zero because $T_n \to T$ in $\mathcal{L}(E, F)$ by assumption and weakly convergent sequences are bounded. By compactness of T and since $f_n - f \rightharpoonup 0$ weakly in E, also the second term converges to zero, proving (2).

Clearly (2) implies (3) so it remains to prove that (3) implies (1). We start by showing that T is compact. Because E is reflexive we only need to show that $Tf_n \to 0$ in F whenever $f_n \rightharpoonup 0$ weakly in E (see [39, Proposition VI.3.3]). Assume now that $f_n \rightharpoonup 0$ weakly in E. Clearly (2) in particular shows that $T_n \to T$ strongly, so $T_k f_n \to Tf_n$ as $k \to \infty$ for every fixed $n \in \mathbb{N}$. Hence for every $n \in \mathbb{N}$ there exists $k_n \geq n$ such that $\|T_{k_n} f_n - Tf_n\|_F \leq 1/n$, and thus

$$\begin{aligned}\|Tf_n\|_F &\leq \|Tf_n - T_{k_n} f_n\|_F + \|T_{k_n}(f_n - f_{k_n})\|_F + \|T_{k_n} f_{k_n}\|_F \\ &\leq \frac{1}{n} + \|T_{k_n}(f_n - f_{k_n})\|_F + \|T_{k_n} f_{k_n}\|_F.\end{aligned}$$

By assumption $\|T_{k_n} f_{k_n}\|_F \to 0$ as $n \to \infty$ since $f_n \rightharpoonup 0$ and likewise $\|T_{k_n}(f_n - f_{k_n})\|_F \to 0$ as $n \to \infty$ because $f_n - f_{k_n} \rightharpoonup 0$. Hence the right-hand side of the above inequality converges to zero as $n \to \infty$, so $Tf_n \to 0$ in F and thus T is compact.

To prove that T_n converges in $\mathcal{L}(E, F)$, we assume to the contrary that this is not the case and derive a contradiction. Then there exist $\varepsilon > 0$ and $f_n \in E$ with $\|f_n\| = 1$ such that $\varepsilon \leq \|T_n f_n - Tf_n\|_F$ for all $n \in \mathbb{N}$. As bounded sets in a reflexive space are weakly sequentially compact there exists a subsequence (f_{n_k}) such that $f_{n_k} \rightharpoonup f$ weakly in E. Therefore

$$\begin{aligned}0 < \varepsilon &\leq \|T_{n_k} f_{n_k} - Tf_{n_k}\|_F \\ &\leq \|T_{n_k}(f_{n_k} - f)\|_F + \|T_{n_k} f - Tf\|_F + \|T(f - f_{n_k})\|_F. \qquad (4.1.1)\end{aligned}$$

The first term converges to zero by assumption as $f_{n_k} - f \rightharpoonup 0$ weakly in E. The second term converges to zero as $T_n \to T$ strongly, and the last term converges to zero as T is compact and $f - f_{n_k} \rightharpoonup 0$ weakly in E. However, this contradicts (4.1.1), showing that T_n must converge in $\mathcal{L}(E, F)$. Hence (1) holds, completing the proof of the proposition. □

Note that we do not require the T_n to be compact. Hence the sequence of operators is not necessarily collectively compact as in [5].

4.2. *A spectral mapping theorem*

When looking at domain perturbation problems we embedded the problems involved into one single space by making use of inclusions and restrictions as introduced in Section 2.5. The purpose of this section is to show that we can still apply the standard perturbation theory of linear operators to show continuity of the spectrum and the corresponding projections.

Suppose that E, F are Banach spaces, and that A is a closed densely defined operator on F with domain $D(A)$. Moreover, suppose that there exist $i \in \mathcal{L}(F, E)$ and $r \in \mathcal{L}(E, F)$ such that $ri = I_F$. For $\lambda \in \varrho(A)$ we consider the pseudo-resolvent

$$R(\lambda) := i(\lambda I - A)^{-1} r.$$

We then have the following spectral mapping theorem. A pseudo-resolvent is a family of linear operators satisfying the resolvent equation

$$R(\lambda) - R(\mu) = (\mu - \lambda) R(\lambda) R(\mu)$$

for λ, μ in some open subset of $\mathbb{C}$.

PROPOSITION 4.2.1. *Suppose that $\lambda \in \varrho(A)$, and that $\mu \neq \lambda$. Then $\mu \in \varrho(A)$ if and only if $(\mu - \lambda)^{-1} \in \varrho(R(\lambda))$. If that is the case, then*

$$R(\mu) = \frac{1}{\mu - \lambda} R(\lambda) \left(\frac{1}{\mu - \lambda} I_E - R(\lambda) \right)^{-1}. \tag{4.2.1}$$

PROOF. Replacing A by $\lambda I_F - A$ we can assume without loss of generality that $\lambda = 0$ and thus $A^{-1} \in \mathcal{L}(F)$. Now $0 \neq \mu \in \varrho(A)$ if and only if $1/\mu \in \varrho(A^{-1})$ (see [92, Theorem III.6.15]), so we only need to prove that $1/\mu \in \varrho(R(0))$ if and only if $1/\mu \in \varrho(A^{-1})$. To do so we first split the equation

$$\frac{1}{\mu} u - R(0) u = f \tag{4.2.2}$$

into an equivalent system of equations. Observe that $P := ir$ is a projection. If we set $E_1 := P(E)$ and $E_2 := (I - P)(E)$, then $E = E_1 \oplus E_2$. By construction, the image of $R(0)$ is in E_1. As $r = rP$ we have $PR(0) = R(0)P$. Setting $u_1 := Pu$ and $u_2 := (I_E - P)u$, equation (4.2.2) is equivalent to the system

$$\left(\frac{1}{\mu} I_E - R(0) \right) u_1 = Pf, \tag{4.2.3}$$

$$\frac{1}{\mu} u_2 = (I - P) f. \tag{4.2.4}$$

Assume now that $\mu \in \varrho(A^{-1})$, and fix $f \in E$ arbitrary. It follows that $u_1 := i(\mu^{-1} I_F - A^{-1})^{-1} Pf$ is the unique solution of (4.2.3), and $u_2 := \mu(I - P) f$ is the unique solution of (4.2.4). Hence $u := u_1 + u_2$ is the unique solution of (4.2.2) and the map $f \to (u_1, u_2)$ is continuous, showing that $1/\mu \in \varrho(R(0))$.

Next assume that $1/\mu \in \varrho(R(0))$, and that $g \in F$ is arbitrary. Set $f := i(g)$ and note that $Pf = f$ in that case. By assumption (4.2.3) has a unique solution u_1. As $(I-P)f = 0$ the solution of (4.2.4) is zero. Hence $r(u_1)$ is the unique solution of $(\mu^{-1}I - A^{-1})u = g$, showing that $1/\mu \in \varrho(A^{-1})$. We finally prove identity (4.2.1), provided $\lambda, \mu \in \varrho(A)$. By the resolvent equation

$$(\lambda I - A)^{-1} = (\mu I - A)^{-1}(I_F - (\lambda - \mu)(\lambda I - A)^{-1}).$$

Using that $ri = I_F$ this yields

$$\begin{aligned} R(\lambda) &= i(\mu I_F - A)^{-1}(I_F r - (\lambda - \mu) ri(\lambda I - A)^{-1} r) \\ &= i(\mu I_F - A)^{-1} r(I_E - (\lambda - \mu) i(\lambda I - A)^{-1} r) \\ &= R(\mu)(I_E - (\lambda - \mu)R(\lambda)) \\ &= (\lambda - \mu)R(\mu)\left(\frac{1}{\lambda - \mu} I_E - R(\lambda)\right). \end{aligned}$$

As we know that $(\lambda - \mu)^{-1} \in \varrho(R(\lambda))$, identity (4.2.1) follows by rearranging the above equation. □

4.3. *Convergence properties of resolvent and spectrum*

We consider a situation similar to the one in Section 4.2, but with a sequence of closed operators A_n defined on Banach spaces F_n with domains $D(A_n)$. Moreover suppose that there exist a Banach space E and operators $i_n \in \mathcal{L}(F_n, E)$ and $r \in \mathcal{L}(E, F_n)$ such that $r_n i_n = I_{F_n}$. We also deal with a limit problem involving a closed densely defined operator A on a Banach space F. For $\lambda \in \varrho(A_n) \cap \varrho(A)$ we consider the *pseudo-resolvents*

$$R_n(\lambda) := i_n(\lambda I - A_n)^{-1} r_n \quad \text{and} \quad R(\lambda) := i(\lambda I - A)^{-1} r$$

similarly as in the concrete case of boundary value problems in Section 2.5. We then have the following theorem about convergence of the pseudo-resolvents.

THEOREM 4.3.1. *Suppose that $R_n(\lambda) \to R(\lambda)$ in $\mathcal{L}(E)$ for some $\lambda \in \mathbb{C}$. Then, for every $\mu \in \varrho(A)$ we have $\mu \in \varrho(A_n)$ for $n \in \mathbb{N}$ large enough, and $R_n(\mu) \to R(\mu)$ in $\mathcal{L}(E)$.*

PROOF. Suppose that $R_n(\lambda) \to R(\lambda)$ in $\mathcal{L}(E)$ for some $\lambda \in \mathbb{C}$, and that $\mu \in \varrho(A)$. By Proposition 4.2.1 we have $(\mu - \lambda)^{-1} \in \varrho(-R(\lambda))$ and so [92, Theorem IV.2.25] implies that $(\mu - \lambda)^{-1} \in \varrho(-R_n(\lambda))$ if only n is large enough. Applying Proposition 4.2.1 again we see that $\mu \in \varrho(A_n)$ if n is large enough. Using (4.2.1) we get

$$\begin{aligned} \lim_{n\to\infty} R_n(\mu) &= \lim_{n\to\infty} \frac{1}{\mu - \lambda} R_n(\lambda)\left(\frac{1}{\mu - \lambda} I_E - R_n(\lambda)\right)^{-1} \\ &= \frac{1}{\mu - \lambda} R(\lambda)\left(\frac{1}{\mu - \lambda} I_E - R(\lambda)\right)^{-1} = R(\mu) \end{aligned}$$

in $L(E)$. Here we use that the map $T \mapsto (\alpha I - T)^{-1}$ is continuous as a map from $\mathcal{L}(E)$ into itself if $\alpha \in \varrho(T)$ (see [121, Theorem IV.1.5]). This completes the proof of the theorem. □

From the above we get the upper semi-continuity of separated parts of the spectrum and in particular the continuity of every finite system of eigenvalues. Recall that a spectral set is a subset of the spectrum which is open and closed in the spectrum. To every spectral set we can consider the corresponding spectral projection (see [92, Section III.6.4]). The following properties of the spectral projections immediately follow from [92, Theorem IV.3.16] and Proposition 4.2.1.

COROLLARY 4.3.2. *Suppose that $R_n(\lambda) \to R(\lambda)$ in $L(E)$ for some $\lambda \in \mathbb{C}$, that $\Sigma \subset \sigma(-A_\Omega) \subset \mathbb{C}$ is a compact spectral set, and that Γ is a rectifiable closed simple curve enclosing Σ, separating it from the rest of the spectrum. Then, for n sufficiently large, $\sigma(A_n)$ is separated by Γ into a compact spectral set Σ_n and the rest of the spectrum. Denote by P and P_n the corresponding spectral projections. Then the dimension of the images of P and P_n are the same, and P_n converges to P in norm.*

REMARK 4.3.3. As a consequence of the above corollary we get the continuity of every finite system of eigenvalues (counting multiplicity) and of the corresponding spectral projection. In particular, we get the continuity of an isolated eigenvalue of simple algebraic multiplicity and its eigenvector when normalised suitably (see [92, Section IV.3.5] for these facts on perturbation theory).

In all cases of domain perturbation we look at, we have that the resolvents $R_n(\lambda)$ act on $L_p(\mathbb{R}^N)$ for all $p \in (1, \infty)$ with image in $L_{m(p)}(\mathbb{R}^N)$ with $m(p)$ given by (2.4.5).

THEOREM 4.3.4. *Suppose that for every $p \in (1, \infty)$ there exists $M, \lambda > 0$ such that*

$$\|R_n(\lambda)\|_{\mathcal{L}(L_p)} + \|R_n(\lambda)\|_{\mathcal{L}(L_p, L_{m(p)})} \leq M \tag{4.3.1}$$

for all $n \in \mathbb{N}$. If $R(\lambda)$ is compact on $L_p(\Omega)$ for some $p \in (1, \infty)$ and $\lambda \in \varrho(A)$, then it is compact for all $p \in (1, \infty)$ and all $\lambda \in \varrho(A)$. Moreover, the following assertions are equivalent:

(1) *There exist $p_0 \in (1, \infty)$ and $\lambda > 0$ such that $R_n(\lambda) f_n \to R(\lambda) f$ in $L_{p_0}(\mathbb{R}^N)$ whenever $f_n \rightharpoonup f$ weakly in $L_p(\mathbb{R}^N)$.*
(2) *There exist $p_0 \in (1, \infty)$ and $\lambda > 0$ such that $R_n(\lambda) \to R(\lambda)$ in $\mathcal{L}(L_{p_0}(\mathbb{R}^N))$.*
(3) *For every $\lambda \in \varrho(A)$ and $p \in (1, \infty)$ we have $R_n(\lambda) f_n \to R(\lambda) f$ in $L_q(\mathbb{R}^N)$ for all $q \in [p, m(p))$, whenever $f_n \rightharpoonup f$ weakly in $L_p(\mathbb{R}^N)$.*
(4) *For every $\lambda \in \varrho(A)$ and $p \in (1, \infty)$*
$$R_n(\lambda) \to R(\lambda) \quad \text{in } \mathcal{L}(L_p(\mathbb{R}^N), L_q(\mathbb{R}^N))$$
for all $q \in [p, m(p))$.

Assertions (2) *and* (4) *are equivalent without the compactness of $R(\lambda)$.*

PROOF. The equivalence of (1) and (2) follow directly from Proposition 4.1.1. We show that (2) implies (4). Note that the argument does not make use of the compactness of $R(\lambda)$. Together with Theorem 4.3.1 it follows from (2) that $R_n(\lambda) \to R(\lambda)$ in $\mathcal{L}(L_p(\mathbb{R}^N))$ for all $\lambda \in \varrho(A)$. Fix $p \in (1, \infty)$ and then $p_1 \in (1, \infty)$ such that either $p_0 < p < p_1$ or $p_0 > p > p_0$. Choose $\lambda \in \varrho(A)$ such that (4.3.1) holds for $p = p_1$. Then by the Riesz–Thorin interpolation theorem

$$\begin{aligned}\|R_n(\lambda) - R(\lambda)\|_{\mathcal{L}(L_p)} &\leq \|R_n(\lambda) - R(\lambda)\|^{1-\theta}_{\mathcal{L}(L_{p_0})} \|R_n(\lambda) - R(\lambda)\|^{\theta}_{\mathcal{L}(L_{p_1})} \\ &\leq (2M)^{\theta} \|R_n(\lambda) - R(\lambda)\|^{1-\theta}_{\mathcal{L}(L_{p_0})}\end{aligned}$$

if we choose $\theta \in (0, 1)$ such that

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

(see [22, Theorem 1.1.1]). Hence $R_n(\lambda) \to R(\lambda)$ in $\mathcal{L}(L_p(\mathbb{R}^N))$. If $p < q < m(p)$, then again by (4.3.1) and the Riesz–Thorin interpolation theorem

$$\begin{aligned}\|R_n(\lambda) - R(\lambda)\|_{\mathcal{L}(L_p, L_q)} &\leq \|R_n(\lambda) - R(\lambda)\|^{1-\theta}_{\mathcal{L}(L_p)} \|R_n(\lambda) - R(\lambda)\|^{\theta}_{\mathcal{L}(L_p, L_{m(p)})} \\ &\leq (2M)^{\theta} \|R_n(\lambda) - R(\lambda)\|^{1-\theta}_{\mathcal{L}(L_p)}\end{aligned}$$

with $\theta \in (0, 1)$ such that

$$\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{m(p)}.$$

Hence $R_n(\lambda) \to R(\lambda)$ in $\mathcal{L}(L_p(\mathbb{R}^N), L_q(\mathbb{R}^N))$ for all $q \in [p, m(p))$. Recall that we had fixed λ for this argument. Let $\mu \in \varrho(A)$ be arbitrary with $\lambda \neq \mu$. Since the map $T \mapsto (\alpha I - T)^{-1}$ is continuous as a map from $\mathcal{L}(E)$ into itself if $\alpha \in \varrho(T)$ (see [121, Theorem IV.1.5])

$$S_n := \frac{1}{\mu - \lambda}\left(\frac{1}{\mu - \lambda} I_E - R_n(\lambda)\right)^{-1} \to S := \frac{1}{\mu - \lambda}\left(\frac{1}{\mu - \lambda} I_E - R(\lambda)\right)^{-1}$$

in $\mathcal{L}(L_p(\mathbb{R}^N))$. Hence, using the identity (4.2.1),

$$\begin{aligned}\|R_n(\mu) - R(\mu)\|_{\mathcal{L}(L_p, L_q)} &= \|R_n(\lambda) S_n - R(\lambda) S\|_{\mathcal{L}(L_p, L_q)} \\ &\leq \|R_n(\lambda)\|_{\mathcal{L}(L_p, L_q)} \|S_n - S\|_{\mathcal{L}(L_p)} + \|R_n(\lambda) - R(\lambda)\|_{\mathcal{L}(L_p, L_q)} \|S\|_{\mathcal{L}(L_p)}\end{aligned}$$

for all $n \in \mathbb{N}$. Using what we have proved already, we get $R_n(\mu) \to R(\mu)$ in $\mathcal{L}(L_p(\mathbb{R}^N), L_q(\mathbb{R}^N))$ for all $q \in [p, m(p))$. Since all operators $R(\lambda)$ interpolate, a compactness property of the Riesz–Thorin interpolation theorem shows that $R(\lambda)$ is compact as an operator in $\mathcal{L}(L_p(\mathbb{R}^N))$ and $\mathcal{L}(L_p(\mathbb{R}^N), L_q(\mathbb{R}^N))$ for all $p \in (1, \infty)$ and $q \in [p, m(p))$ if $R(\lambda)$ is compact (see [94]). Now the equivalence of (3) and (4) follows from Proposition 4.1.1. □

5. Perturbations for linear Dirichlet problems

The most complete results on domain perturbation are for problems with Dirichlet boundary conditions. After stating the main assumptions we will give a complete characterisation of convergence of solutions for the Dirichlet problem on a domain Ω_n to a solution of the corresponding problem on Ω.

Theorem 5.2.4 is the main theorem on strong convergence and Theorem 5.2.6 the main result on convergence in the operator norm. Section 5.3 is then concerned with *necessary conditions* and Section 5.4 with *sufficient conditions* for convergence.

5.1. *Assumptions and preliminary results*

Given open sets $\Omega_n \subset \mathbb{R}^N$ ($N \geq 2$) we ask under what conditions the solutions of

$$\begin{aligned} \mathcal{A}_n u + \lambda u &= f_n \quad \text{in } \Omega_n, \\ u &= 0 \quad \text{on } \partial\Omega_n \end{aligned} \tag{5.1.1}$$

converge to a solution of the corresponding problem

$$\begin{aligned} \mathcal{A} u + \lambda u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{5.1.2}$$

on a limit domain Ω as $n \to \infty$. Most of the results in this section are taken from [58], but here we allow perturbations of $\mathcal{A}$ as well. We make the following basic assumptions on the operators $\mathcal{A}_n$ below.

ASSUMPTION 5.1.1. We let $\mathcal{A}_n$ be operators of the form

$$-\operatorname{div}(A_{0n}(x)\nabla u + a_n(x)u) + b_n(x)\cdot\nabla u + c_{0n}u \tag{5.1.3}$$

with $A_{0n} \in L_\infty(\mathbb{R}^N, \mathbb{R}^{N\times N})$, $a_n, b_n \in L_\infty(\mathbb{R}^N, \mathbb{R}^N)$ and $c_{0n} \in L_\infty(\mathbb{R}^N)$. Moreover, assume that the ellipticity constant $\alpha_0 > 0$ can be chosen uniformly with respect to $n \in \mathbb{N}$, and that

$$\sup_{n\in\mathbb{N}}\{\|A_{0n}\|_\infty, \|a_n\|_\infty, \|b_n\|_\infty, \|c_{0n}\|_\infty\} < \infty. \tag{5.1.4}$$

We also assume that $\mathcal{A}$ is an operator of the form (2.1.2) and that

$$\lim_{n\to\infty} A_{0n} = A_0, \qquad \lim_{n\to\infty} a_n = a, \qquad \lim_{n\to\infty} b_n = b, \qquad \lim_{n\to\infty} c_{0n} = c_0 \tag{5.1.5}$$

almost everywhere in $\mathbb{R}^N$.

The condition that $\mathcal{A}_n, \mathcal{A}$ be defined on $\mathbb{R}^N$ is no restriction since by Remark 2.1.1 we can always extend them to $\mathbb{R}^N$. The bilinear forms associated with the boundary value problem are given by

$$a_n(u, v) := \int_\Omega (A_{0n}\nabla u + a_n u) \cdot \nabla v + (b_n \cdot \nabla u + c_{0n} u)v\, dx \tag{5.1.6}$$

for all $u, v \in H_0^1(\Omega_n)$ and by

$$a(u, v) := \int_\Omega (A_0\nabla u + au) \cdot \nabla v + (b \cdot \nabla u + c_0 u)v\, dx.$$

for all $u, v \in H_0^1(\Omega)$. Applying Proposition 2.1.6 we get the following properties.

PROPOSITION 5.1.2. *Suppose that Assumption 5.1.1 is satisfied. Then there exists $M > 0$ such that*

$$|a_n(u, v)| \le M\|u\|_{H_0^1}\|v\|_{H_0^1} \tag{5.1.7}$$

for all $u, v \in H_0^1(\mathbb{R}^N)$ and all $n \in \mathbb{N}$. Moreover,

$$\frac{\alpha_0}{2}\|\nabla u\|_2^2 \le a_n(u, u) + \lambda\|u\|_2^2$$

for all $\lambda \in \mathbb{R}$ with

$$\lambda \ge \lambda_{\mathcal{A}} := \sup_{n\in\mathbb{N}}\left(\|c_{0n}^-\|_\infty + \frac{1}{2\alpha_0}\|a_n + b_n\|_\infty\right), \tag{5.1.8}$$

and

$$\frac{\alpha_0}{2}\|u\|_{H^1}^2 \le a_n(u, u) + \lambda\|u\|_2^2$$

for all $\lambda \in \mathbb{R}$ with

$$\lambda \ge \lambda_0 := \lambda_{\mathcal{A}} + \frac{\alpha_0}{2} \tag{5.1.9}$$

for all $u \in H_0^1(\mathbb{R}^N)$ and all $n \in \mathbb{N}$. Similar inequalities hold for $a(\cdot\,,\cdot)$ with the same constants. Finally,

$$\lim_{n\to\infty} a_n(u_n, v_n) = a(u, v) \tag{5.1.10}$$

if $u_n \rightharpoonup u$ weakly and $v_n \to v$ strongly in $H^1(\mathbb{R}^N)$ or vice versa.

PROOF. The first properties follow from Proposition 2.1.6. For the last, note the following fact. If c_n is bounded in $L_\infty(\mathbb{R}^N)$ with $c_n \to c$ pointwise and $w_n \to w$ in $L_2(\mathbb{R}^N)$, then $c_n w_n \to cw$ in $L_2(\mathbb{R}^N)$ as well. Indeed,

$$\begin{aligned}\|c_n w_n - cw\|_2 &\le \|c_n(w_n - w)\|_2 + \|(c_n - c)w\|_2\\ &\le \|c_n\|_\infty\|w_n - w\|_2 + \|(c_n - c)w\|_2,\end{aligned}$$

where the first term on the right-hand side converges to zero because $\|c_n\|_\infty$ is bounded and $w_n \to w$ in $L_2(\mathbb{R}^N)$. The second term converges to zero by the dominated convergence theorem. Hence under the given assumptions, every term in (5.1.6) is the L_2 inner product of a strongly and a weakly converging sequence and therefore (5.1.10) follows. □

Depending on the domains we can use $\|\nabla u\|_2$ as a norm on $H_0^1(\Omega)$, for instance if the measure of Ω_n is uniformly bounded, or if all Ω_n are contained between two parallel hyperplanes. Since we do not want to restrict ourselves to such a situation we will generally work with $\lambda \geq \lambda_0$ as given in (5.1.9). From the results in Section 2.2 we construct operators

$$A_n \in \mathcal{L}(H_0^1(\Omega_n), H^{-1}(\Omega_n)) \quad \text{and} \quad A \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega)),$$

where by definition

$$H^{-1}(\Omega) = (H_0^1(\Omega))'.$$

Generally, the right-hand side f of (5.1.2) is in $H^{-1}(\Omega)$, so the linear functional f is defined on the closed subspace $H_0^1(\Omega)$ of $H^1(\mathbb{R}^N)$. By the Hahn–Banach theorem (see [125, Theorem 6.5.1]) there exists an extension $\tilde{f}$ of f with $\|\tilde{f}\|_{H^{-1}(\mathbb{R}^N)} = \|f\|_{H^{-1}(\Omega)}$. Hence we can assume without loss of generality that $f \in H^{-1}(\mathbb{R}^N)$.

Suppose that $R_n(\lambda)$, $R(\lambda)$ are given as in Definition 2.5.2. From Theorem 2.2.2 we conclude that

$$[\lambda_0, \infty) \subset \varrho(-A_n) \cap \varrho(-A),$$

and also the uniform estimate

$$\|R_n(\lambda)\|_{\mathcal{L}(H^{-1}, H_0^1)} \leq \frac{2}{\alpha_0} \tag{5.1.11}$$

for all $\lambda \geq \lambda_0$ and all $n \in \mathbb{N}$.

We summarise the results of this section in the following proposition. It is a uniform a priori estimate for weak solutions of (5.1.1) and (5.1.2).

PROPOSITION 5.1.3. *If $\lambda \geq \lambda_0$, then*

$$\|R_n(\lambda)\|_{\mathcal{L}(H^{-1}(\mathbb{R}^N), H^1(\mathbb{R}^N))} \leq \frac{2}{\alpha_0}$$

for all $n \in \mathbb{N}$. A similar estimate with the same constant hold for $R(\lambda)$.

PROOF. The claim follows by combining Lemma 2.5.3, Proposition 2.1.6, Theorem 2.2.2 and Proposition 5.1.2. □

5.2. *The main convergence result*

In this section we summarise the main convergence results for Dirichlet problems. The bulk of the proof will be given in Section 5.5.

When proving that the solutions of (5.1.1) converge to a solution of (5.1.2), the following two conditions appear very naturally.

ASSUMPTION 5.2.1. Suppose that $\Omega_n, \Omega \subset \mathbb{R}^N$. The weak limit points of every sequence $u_n \in H_0^1(\Omega_n)$ lie in $H_0^1(\Omega)$.

ASSUMPTION 5.2.2. Suppose that $\Omega_n, \Omega \subset \mathbb{R}^N$ are open sets and for every $u \in H_0^1(\Omega)$ there exists $u_n \in H_0^1(\Omega_n)$ such that $u_n \to u$ in $H^1(\mathbb{R}^N)$.

If the above conditions are satisfied it is often said that $\Omega_n \to \Omega$ in the sense of Mosco as this is equivalent to $H_0^1(\Omega_n) \to H_0^1(\Omega)$ as subspaces of $H^1(\mathbb{R}^N)$ in the sense of Mosco [102, Section 1]. The conditions also appear in a more disguised form in [116], and explicitly in [119]. A discussion in terms of capacity appears in [25].

DEFINITION 5.2.3 (Mosco convergence). We say $\Omega_n \to \Omega$ *in the sense of Mosco*, if the open sets $\Omega_n, \Omega \subset \mathbb{R}^N$ satisfy Assumption 5.2.1 and Assumption 5.2.2.

For the formulation of the main convergence result for Dirichlet problems we use the notation and framework introduced in Section 5.1. In particular, $R_n(\lambda) f$ and $R(\lambda) f$ are the weak solutions of (5.1.1) and (5.1.2) extended to $\mathbb{R}^N$ by zero. Also recall that we can choose $f_n, f \in H_0^1(\mathbb{R}^N)$ without loss of generality by extending the functionals by means of the Hahn–Banach Theorem if necessary.

THEOREM 5.2.4. *If $\lambda \geq \lambda_0$, then the following assertions are equivalent.*

(1) $\Omega_n \to \Omega$ *in the sense of Mosco;*
(2) $R_n(\lambda) f_n \rightharpoonup R(\lambda) f$ *weakly in* $H^1(\mathbb{R}^N)$ *whenever* $f_n \rightharpoonup f$ *weakly in* $H^{-1}(\mathbb{R}^N)$*;*
(3) $R_n(\lambda) f_n \to R(\lambda) f$ *in* $H^1(\mathbb{R}^N)$ *whenever* $f_n \to f$ *in* $H^{-1}(\mathbb{R}^N)$*;*
(4) $R_n(\lambda) f \rightharpoonup R(\lambda) f$ *weakly in* $H^1(\mathbb{R}^N)$ *for* f *in a dense subset of* $H^{-1}(\mathbb{R}^N)$.

The property that $R_n(\lambda) f \to R(\lambda) f$, at least in the case of the Laplace operator, is often called the γ-convergence of the solutions (see for instance [27]). Note that in particular, the above theorem implies that convergence is independent of the operator under consideration, a result that has been proved for a restricted class of operators in [15].

The above theorem does not say anything about convergence in the operator norm, it is only a theorem on the strong convergence of the resolvent operators. Strong convergence does not imply the convergence of the eigenvalues to the corresponding eigenvalues of the limit problem. However, according to Corollary 4.3.2 we get convergence of every finite part of the spectrum if the pseudo-resolvents converge in the operator norm.

If we assume that there is a bounded open set B such that $\Omega_n, \Omega \subset B$ for all $n \in \mathbb{N}$, then we get convergence in the operator norm.

COROLLARY 5.2.5. *Suppose that $\Omega_n \to \Omega$ in the sense of Mosco. Moreover, suppose that there exists a bounded open set B such that $\Omega_n, \Omega \subset B$ for all $n \in \mathbb{N}$. Finally let $\lambda \in \varrho(-A)$. Then $\lambda \in \varrho(-A_n)$ for n large enough, and $R_n(\lambda) \to R(\lambda)$ in $\mathcal{L}(H^{-1}(\mathbb{R}^N), L_q(\mathbb{R}^N))$ for all $q \in [1, 2d/(d-2))$.*

PROOF. If $\lambda \geq \lambda_0$, then from Theorem 5.2.4 we have that $u_n := R_n(\lambda) f_n \rightharpoonup u := R(\lambda) f$ weakly in $H^1(\mathbb{R}^N)$ whenever $f_n \rightharpoonup f$ weakly in $H^{-1}(\Omega)$. Since $u_n \in H_0^1(B)$ for all $n \in \mathbb{N}$ and B is bounded, Rellich's Theorem implies that $u_n \to u$ in $L_q(\mathbb{R}^N)$ for all $q \in [1, 2d/(d-2))$. Hence $R_n(\lambda) \to R(\lambda)$ in $\mathcal{L}(H^{-1}(\mathbb{R}^N), L_q(\mathbb{R}^N))$ for all $q \in [1, 2d/(d-2))$ by Proposition 4.1.1. The remaining assertions follow from Theorem 4.3.1. □

We now want to look at the situation where only Ω is bounded, but not necessarily Ω_n. We then get necessary and sufficient conditions for convergence in the operator norm. We denote by $\lambda_1(U)$ the *spectral bound* of $-\Delta$ on the open set U with Dirichlet boundary conditions. It is given by the variational formula

$$\lambda_1(U) = \inf_{\substack{u\in H_0^1(U)\\ u\neq 0}} \frac{\|\nabla u\|^2_{L_2(U)}}{\|u\|^2_{L_2(U)}}. \tag{5.2.1}$$

For convenience we set

$$\lambda_1(\emptyset) := \infty.$$

We then have the following characterisation of convergence in the operator norm.

THEOREM 5.2.6. *Suppose that Ω is bounded and that $\Omega_n \to \Omega$ in the sense of Mosco. Then the following assertions are equivalent.*

(1) *There exists $\lambda > 0$ such that $R_n(\lambda) \to R(\lambda)$ in $\mathcal{L}(H^{-1}(\mathbb{R}^N), L_2(\mathbb{R}^N))$.*
(2) *For every $\lambda \in \varrho(-A)$ we have $\lambda \in \varrho(-A_n)$ for n large enough, and $R_n(\lambda) \to R(\lambda)$ in $\mathcal{L}(L_p(\mathbb{R}^N), L_q(\mathbb{R}^N))$ for all $q \in [p, m(p))$ and all $p \in (1,\infty)$, where $m(p)$ is defined by (2.4.5) with $d = N$.*
(3) *There exists an open set B with $\bar{\Omega} \subset B$ such that $\lambda_1(\Omega_n \cap \bar{B}^c) \to \infty$ as $n \to \infty$.*

PROOF. We know that $R(\lambda) \in \mathcal{L}(H^{-1}(\mathbb{R}^N), H^1(\mathbb{R}^N))$. Since Ω is bounded, Rellich's Theorem implies that $R(\lambda) \in \mathcal{L}(H^{-1}(\mathbb{R}^N), L_2(\mathbb{R}^N))$ is compact. Hence by Proposition 4.1.1, assertion (1) is equivalent to the following statement:

(1') For some λ large enough $R_n(\lambda) f_n \to R(\lambda) f$ in $L_2(\mathbb{R}^N)$ whenever $f_n \rightharpoonup f$ weakly in $H^{-1}(\mathbb{R}^N)$.

Note that the above implies that $R_n(\lambda) f_n \to R(\lambda) f$ in $L_2(\mathbb{R}^N)$ whenever $f_n \rightharpoonup f$ weakly in $L_2(\mathbb{R}^N)$. From Theorem 2.4.2 we have uniform a priori estimates for $\lambda > 0$ large enough, and therefore Theorem 4.3.4 shows that (1') is equivalent to (2). Hence it remains to show that (1') is equivalent to (3).

Suppose that (1') is true, but not (3). Then there exists a bounded open set B containing $\bar{\Omega}$ such that $\lambda_1(\Omega_n \setminus \bar{B}^c) \not\to \infty$. Hence for every $k \in \mathbb{N}$ there exist $n_k > k$ and $\varphi_{n_k} \in C_c^\infty(\Omega_{n_k} \setminus \bar{B}^c)$ and $c > 0$ such that $\|\varphi_{n_k}\|_2 = 1$ and

$$0 \le \lambda_{n_k} \le \|\nabla \varphi_{n_k}\|_2^2 \le c$$

for all $k \in \mathbb{N}$. We define functionals $f_{n_k} \in H^{-1}(\mathbb{R}^N)$ by

$$\langle f_{n_k}, v\rangle := a_{n_k}(\varphi_{n_k}, v) + \lambda \langle \varphi_{n_k}, v\rangle$$

for all $v \in H^1(\mathbb{R}^N)$. By (5.1.7) and the choice of φ_{n_k} we have

$$\|f_{n_k}\|_{H^{-1}} \le (M + \lambda)\|\varphi_{n_k}\|_{H^1} \le (M + \lambda)\sqrt{1 + c^2}$$

for all $k \in \mathbb{N}$. This means that (f_{n_k}) is a bounded sequence in $H^{-1}(\mathbb{R}^N)$, and therefore has a subsequence converging weakly in $H^{-1}(\mathbb{R}^N)$ to some $f \in H^{-1}(\mathbb{R}^N)$. We denote that subsequence again by (f_{n_k}). By definition of f_{n_k} we have $\varphi_{n_k} = R_{n_k}(\lambda) f_{n_k}$ and by assumption (1')

$$\varphi_{n_k} = R_{n_k}(\lambda) f_{n_k} \to R(\lambda) f$$

in $L_2(\mathbb{R}^N)$. Since $\operatorname{supp}(\varphi_{n_k}) \cap \bar{\Omega} = \emptyset$, the definition of f_{n_k} implies that $f|_{H^{-1}(\Omega)} = 0$ and so $\varphi_{n_k} \to 0$ in $L_2(\mathbb{R}^N)$. However, this is impossible since we chose φ_{n_k} such that $\|\varphi_{n_k}\|_2 = 1$ for all $k \in \mathbb{N}$. Hence we have a contradiction, so (1') implies (3).

We finally prove that (3) implies (1'). Suppose that $f_n \rightharpoonup f$ weakly in $H^{-1}(\mathbb{R}^N)$. Then by Theorem 5.2.4

$$u_n := R_n(\lambda) f_n \rightharpoonup u := R(\lambda) f$$

weakly in $H^1(\mathbb{R}^N)$. Let B be an open set as in (3) and choose an open bounded set U with $\bar{B} \subset U$. Then by Rellich's theorem $u_n \to u$ in $L_2(U)$. Hence it remains to show that $u_n \to 0$ in $L_2(\mathbb{R}^N \setminus U)$. We choose a cutoff function $\psi \in C^\infty(\mathbb{R}^N)$ such that $0 \le \psi \le 1$, $\psi = 0$ on $\bar{B}$ and $\psi = 1$ on $\mathbb{R}^N \setminus U$. Then, $\psi u_n \in H^1(\Omega \cap \bar{B}^c)$, and setting $\lambda_n := \lambda_1(\Omega \setminus \bar{B})$ we get

$$\begin{aligned} \lambda_n \|u_n\|^2_{L_2(\mathbb{R}^N \setminus U)} &\le \lambda_n \|\psi u_n\|^2_{L_2(\mathbb{R}^N \setminus \bar{B})} \\ &\le \|\nabla(\psi u_n)\|^2_{L_2(\mathbb{R}^N \setminus \bar{B})} = \|\nabla(\psi u_n)\|_2^2 \end{aligned} \tag{5.2.2}$$

for all $n \in \mathbb{N}$. Since (f_n) is bounded in $H^{-1}(\mathbb{R}^N)$, the sequence (u_n) is bounded in $H^1(\mathbb{R}^N)$. Hence

$$\|\nabla(\psi u_n)\|_2^2 \le (\|\psi\|_\infty^2 + \|\psi\|_\infty^2)\|u_n\|_{H^1}^2$$

is bounded. Because $\lambda_n \to \infty$ by assumption, (5.2.2) implies that $u_n \to 0$ in $L_2(\mathbb{R}^N \setminus U)$. Hence $u_n \to u$ in $L_2(\mathbb{R}^N)$ as claimed. This completes the proof of the theorem. □

REMARK 5.2.7. (a) Note that condition (3) in the above theorem is always satisfied if $\lambda_1(\Omega_n \setminus \bar{\Omega}^c) \to \infty$ as $n \to \infty$. Indeed, from the monotonicity of the first eigenvalue of the Dirichlet problem with respect to the domain

$$\lambda_1(\Omega_n \setminus \bar{\Omega}^c) \le \lambda_1(\Omega_n \setminus \bar{B}^c)$$

for every bounded set B with $\Omega \subset B$. The monotonicity is a consequence of the variational formula (5.2.1).

(b) Note that (3) is also satisfied if $|\Omega_n \cap \bar{\Omega}^c| \to 0$, whether or not Ω_n is bounded. To see this let B_n be a ball of the same volume as $\Omega_n \cap \bar{\Omega}^c$. As the measure goes to zero $\lambda_1(B_n) \to \infty$, and by the isoperimetric inequality for the first eigenvalue of the Dirichlet problem (see [20,83]) we get

$$\lambda_1(\Omega_n \cap \bar{\Omega}^c) \ge c\lambda_1(B_n) \to \infty.$$

EXAMPLE 5.2.8. We give a situation, where we get convergence in the operator norm, but Ω_n has unbounded measure for all $n \in \mathbb{N}$. We can take a disk and attach an infinite strip. We then let the width of the strip tend to zero. Then by Friedrich's inequality (2.1.7)

$$\lambda_1(\Omega_n \setminus \bar{\Omega}) \ge \frac{1}{D^2} \to \infty$$

if the width D of the strip goes to zero. The situation is depicted in Figure 5.1.

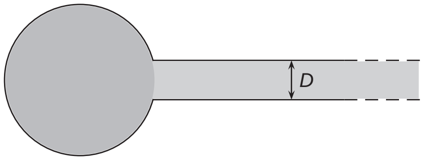

Fig. 5.1. Disc with an infinite strip attached.

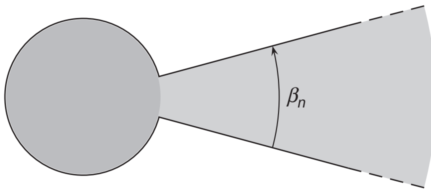

Fig. 5.2. Disc with an infinite cone attached.

In contrast, if we attach a cone of angle β_n rather than a strip, then convergence is not in the operator norm if $\beta_n \to 0$ as shown in Figure 5.2. The fact that these domains converge in the sense of Mosco follows from Theorem 5.4.5 below. Note that $\lambda_1(\Omega_n) = 0$ for all $n \in \mathbb{N}$, and therefore $\lambda_1(\Omega_n) \not\to \lambda_1(\Omega)$. This means that there is no convergence of the spectrum. More examples are given in [58, Section 8]. Examples where just part of the spectrum converges can be found in [107].

5.3. *Necessary conditions for convergence*

In this section we collect some necessary conditions for convergence in the sense of Mosco. For a convergence result such as the one in Theorem 5.2.4 we clearly need that the support of the limit function is in $\bar{\Omega}$. We give a simple characterisation of such a requirement in terms of the spectral bound of the Laplacian on bounded sets outside the limit set $\bar{\Omega}$.

THEOREM 5.3.1. *For open sets $\Omega_n, \Omega \subset \mathbb{R}^N$ the following assertions are equivalent.*

(1) *The weak limit points of every sequence $u_n \in H_0^1(\Omega_n)$, $n \in \mathbb{N}$, in $H^1(\mathbb{R}^N)$ have support in $\overline{\Omega}$;*

(2) *For every open bounded set B with $\bar{B} \subset\subset \mathbb{R}^N \setminus \overline{\Omega}$*
$$\lim_{n\to\infty} \lambda_1(\Omega_n \cap B) = \infty; \tag{5.3.1}$$

(3) *There exists an open covering $\mathcal{O}$ of $\mathbb{R}^N \setminus \overline{\Omega}$ such that (5.3.1) holds for all $B \in \mathcal{O}$.*

PROOF. Suppose that (1) holds and let B be a bounded open set with $\bar{B} \subset \mathbb{R}^N \setminus \overline{\Omega}$. Set $\lambda_n := \lambda_1(\Omega_n \cap B)$. Then, by the variational characterisation (5.2.1) of the spectral bound, for every $n \in \mathbb{N}$ there exists $v_n \in C_c^\infty(\Omega_n \cap B)$ with

$$(\lambda_n + 1)\|v_n\|_2^2 \geq \|\nabla v_n\|_2^2 = 1. \tag{5.3.2}$$

Since B is bounded (2.1.7) implies that (v_n) is bounded in $H_0^1(B)$. Hence there exists a subsequence (v_{n_k}) converging weakly to some v in $H_0^1(B)$. By assumption $\operatorname{supp}(v) \subset \bar{B} \subset\subset \mathbb{R}^N \setminus \overline{\Omega}$, and so (1) implies that $v = 0$. As B is bounded Rellich's Theorem shows that $\|v_{n_k}\|_2 \to 0$. Hence, (5.3.2) can only be true if $\lambda_{n_k} - 1 \to \infty$, implying that $\lambda_{n_k} \to \infty$ as $k \to \infty$. The above arguments apply to every weakly convergent subsequence of (v_n) and therefore (1) implies (2).

Clearly (2) implies (3) and so it remains to prove that (3) implies (1). Suppose that $u_n \in H_0^1(\Omega_n)$, and that $u_{n_k} \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$ as $k \to \infty$. Let $\mathcal{O}$ be an open covering of $\mathbb{R}^N \setminus \overline{\Omega}$ with the properties stated in (3). Fix $B \subset \mathcal{O}$ and let $\varphi \in C_c^\infty(B)$. Then $\varphi u_n \in H_0^1(\Omega_n \cap \overline{\Omega}^c \cap B)$. The map $u_n \to u_n\varphi$ is a bounded linear map from $H^1(\mathbb{R}^N)$ to $H_0^1(B)$ and therefore is weakly continuous. Hence, if $u_{n_k} \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$, then $\varphi u_{n_k} \rightharpoonup \varphi u$ weakly in $H_0^1(B)$. As B is bounded, Rellich's theorem implies that $\varphi u_{n_k} \to \varphi u$ in $L_2(\mathbb{R}^N)$. Now by (5.3.1) and the boundedness of the sequence $(\|\nabla(\varphi u_{n_k})\|_2)$

$$\|\varphi u\|_2^2 = \lim_{k\to\infty} \|\varphi u_{n_k}\|_2^2 \leq \lim_{k\to\infty} \frac{\|\nabla(\varphi u_{n_k})\|_2^2}{\lambda_1(\Omega_{n_k} \cap B)} = 0.$$

Hence $\varphi u = 0$ almost everywhere for all $\varphi \in C_c^\infty(B)$, so $u = 0$ almost everywhere in B. As $\mathcal{O}$ is a covering of $\mathbb{R}^N \setminus \overline{\Omega}$ it follows that $\operatorname{supp} u \subset \overline{\Omega}$ as claimed. □

REMARK 5.3.2. The above condition does not imply Assumption 5.2.1. The reason is that a function $u \in H^1(\mathbb{R}^N)$ with $\operatorname{supp}(u) \subset \bar{\Omega}$ does not need to be in $H_0^1(\Omega)$. We discuss conditions for that in the next subsection.

We next give a characterisation of Assumption 5.2.2 in terms of capacity. A related result appears in [108, Proposition 4.1] and a proof is given in [77, page 75] or [119, page 24]. Our exposition follows [58, Section 7]. Recall that the capacity (or more precisely (1, 2)-capacity) of a compact set $E \subset \mathbb{R}^N$ is given by

$$\operatorname{cap}(E) := \inf\{\|u\|_{H^1}^2 : u \in H_0^1(\mathbb{R}^N) \text{ and } u \geq 1 \text{ in a neighbourhood of } E\}$$

(see [80, Section 2.35]). We could also define capacity with respect to an open set U and define for $E \subset U$ compact $E \subset \mathbb{R}^N$ given by

$$\operatorname{cap}_U(E) := \inf\{\|u\|_{H^1}^2 : u \in H_0^1(U) \text{ and } u \geq 1 \text{ in a neighbourhood of } E\}.$$

It turns out that $\operatorname{cap}_U(E) = 0$ if and only if $\operatorname{cap}(E) = 0$. Moreover, we can work with $u \in C_c^\infty(\mathbb{R}^N)$ and $u \in C_c^\infty(U)$, respectively rather than the Sobolev spaces.

PROPOSITION 5.3.3. *Let $\Omega_n, \Omega \subset \mathbb{R}^N$ be open sets. Then the following conditions are equivalent.*

(1) *Assumption 5.2.2;*

(2) *For every open set $B \subset \mathbb{R}^N$ and every $\varphi \in C_c^\infty(\Omega \cap B)$ there exists $\varphi_n \in C_c^\infty(\Omega_n \cap B)$ such that $\varphi_n \to \varphi$ in $H_0^1(\Omega \cap B)$;*

(3) *For every compact set $K \subset \Omega$*
$$\lim_{n\to\infty} \operatorname{cap}(K \cap \Omega_n^c) = 0.$$

PROOF. We prove that (1) implies (3). Fix a compact set $K \subset \Omega$ and let $\varphi \in C_c^\infty(\Omega)$ with $0 \le \varphi \le 1$ and $\varphi = 1$ in a neighbourhood of K. By assumption there exists $u_n \in H_0^1(\Omega)$ such that $u_n \to \varphi$ in $H^1(\mathbb{R}^N)$. As $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$ there exists $\varphi_n \in C_c^\infty(\Omega)$ such that $\|u_n - \varphi_n\|_{H^1} < 1/n$. Then

$$\|\varphi_n - \varphi\|_{H^1} \le \|\varphi_n - u_n\|_{H^1} + \|u_n - \varphi\|_{H^1} \le \frac{1}{n} + \|u_n - \varphi\|_{H^1} \to 0.$$

Now set $\psi_n := \varphi - \varphi_n$. Then by construction $\psi_n \in H^1(\mathbb{R}^N)$ and $\psi_n = 1$ in a neighbourhood of $K \cap \Omega_n^c$. Hence by definition of capacity

$$\operatorname{cap}(K \cap \Omega_n^c) \le \|\psi_n\|_{H^1}^2 = \|\varphi_n - \varphi\|_{H^1}^2 \to 0$$

as claimed.

We next prove that (3) implies (2). We fix an open set $B \subset \mathbb{R}^N$. Clearly we only need to consider the case where $\Omega \cap B \ne \emptyset$. Let $\varphi \in C_c^\infty(\Omega \cap B)$. By definition of capacity there exists $\psi_n \in C_c(\Omega \cap B)$ such that $\psi_n = 1$ on $\operatorname{supp}\varphi \cap \Omega_n^c$ and such that

$$\|\psi_n\|_{H^1}^2 \le \operatorname{cap}(\operatorname{supp}\varphi \cap \Omega_n^c) + \frac{1}{n}$$

for all $n \in \mathbb{N}$. Hence by assumption $\psi_n \to 0$ in $H^1(\mathbb{R}^N)$. We now set $\varphi_n := (1 - \psi_n)\varphi$. Then by construction $\varphi_n \in C_c^\infty(\Omega \cap B)$ and

$$\begin{aligned}\|\varphi_n - \varphi\|_{H^1} = \|\varphi\psi_n\|_{H^1} &\le \|\varphi\psi_n\|_2 + \|\psi_n \nabla\varphi + \varphi\nabla\psi_n\|_2 \\ &\le (\|\varphi\|_\infty + \|\nabla\varphi\|_\infty)\|\psi_n\|_2 + \|\varphi\|_\infty \|\nabla\psi_n\|_2 \\ &\le 2(\|\varphi\|_\infty + \|\nabla\varphi\|_\infty)\|\psi_n\|_{H^1} \to 0.\end{aligned}$$

Hence we have found $\varphi_n \in C_c^\infty(\Omega \cap B)$ with $\varphi_n \to \varphi$ in $H^1(\mathbb{R}^N)$, proving (2).

We finally prove that (2) implies (1). For given $u \in H_0^1(\Omega)$ there exists $\varphi_k \in C_c^\infty(\Omega)$ such that $\varphi_n \to u$ in $H^1(\Omega)$. Now by (2) there exists $\varphi_{k,n} \in C_c^\infty(\Omega_n)$ such for every fixed $k \in \mathbb{N}$ we have $\varphi_{k,n} \to \varphi_k$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$. Hence for every $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that $\|\varphi_{k,n} - \varphi_k\|_{H^1} < 1/k$ for all $n > n_k$. We can also arrange that $n_k < n_{k+1}$ for all $k \in \mathbb{N}$. Now we set $u_n := \varphi_{k,n}$ whenever $n_k < n \le n_{k+1}$. Then $u_n \in H_0^1(\Omega_n)$ and our aim is to show that $u_n \to u$ in $H^1(\mathbb{R}^N)$. To do so fix $\varepsilon > 0$. As $\varphi_k \to u$ in $H^1(\mathbb{R}^N)$ there exists $k_0 \in \mathbb{N}$ such that $1/k + \|\varphi_k - u\|_{H^1} < \varepsilon$ for all $k > k_0$. Given $n > n_{k_0+1}$ there exists $k > k_0$ such that $n_k < n \le n_{k+1}$ and so by construction $u_n = \varphi_{k,n}$ and

$$\|u_n - u\|_{H^1} \le \|\varphi_{k,n} - \varphi_k\|_{H^1} + \|\varphi_k - u\|_{H^1} \le 1/k + \|\varphi_k - u\|_{H^1} < \varepsilon.$$

This shows that $\|u_n - u\|_{H^1} < \varepsilon$ for all $n > n_{k_0+1}$. As $\varepsilon > 0$ was arbitrary we conclude that $u_n \to u$ in $H^1(\mathbb{R}^N)$ as claimed. □

5.4. *Sufficient conditions for convergence*

In this section we collect some simple sufficient conditions for $\Omega_n \to \Omega$ in the sense of Mosco. First we look at approximations of an open set Ω by open sets from the inside.

PROPOSITION 5.4.1. *Suppose that* $\Omega_n, \Omega \subset \mathbb{R}^N$ *are open sets. If supposing that* $\Omega_n \subset \Omega_{n+1} \subset \Omega$ *for all* $n \in \mathbb{N}$, *and* $\Omega = \bigcup_{n\in\mathbb{N}} \Omega_n$, *then* $\Omega_n \to \Omega$ *in the sense of Mosco.*

PROOF. Since $H_0^1(\Omega_n) \subset H_0^1(\Omega)$ for all $n \in \mathbb{N}$, Assumption 5.2.1 is clearly satisfied. Suppose that $u \in H_0^1(\Omega)$. If $\varphi \in C_c^\infty(\Omega)$, then by assumption there exists $n_0 \in \mathbb{N}$ such that $\operatorname{supp}(\varphi) \subset \Omega_n$ for all $n \geq n_0$. We now choose $\phi_n \in C_c^\infty(\Omega_n)$ arbitrary for $1 \leq n \leq n_0$ and $\varphi_n := \varphi$ for all $n > n_0$. Then clearly $\varphi_n \to \varphi$ in $H^1(\mathbb{R}^N)$ and Assumption 5.2.2 follows from Proposition 5.3.3. □

For approximations from the outside we need a weak regularity condition on the boundary of Ω. We define

$$H_0^1(\bar{\Omega}) := \{u \in H^1(\mathbb{R}^N) : u = 0 \text{ almost everywhere on } \bar{\Omega}^c\}$$

We make the following definition.

DEFINITION 5.4.2. We say the open set $\Omega \subset \mathbb{R}^N$ is *stable* if $H_0^1(\Omega) = H_0^1(\bar{\Omega})$.

The above notion of stability is the same as the stability of the Dirichlet problem for harmonic functions on Ω as introduced in Keldyš [93]. An excellent discussion of bounded stable sets is given in [79]. A discussion on the connections between stability of the Dirichlet problem for harmonic functions and the Poisson problem by more elementary means is presented in [9].

PROPOSITION 5.4.3. *An open set* $\Omega \subset \mathbb{R}^N$ *is stable if one of the following conditions is satisfied:*

(1) Ω *has the segment property except possibly on a set of capacity zero;*
(2) *for all* $x \in \partial\Omega$ *except possibly a set of capacity zero*
$$\liminf_{r\to 0} \frac{\operatorname{cap}(\bar{\Omega}^c \cap B(x,r))}{\operatorname{cap}(\Omega^c \cap B(x,r))} > 0,$$
where $B(x,r)$ *is the ball of radius* r *centred at* x.

The last condition is necessary and sufficient for the stability of Ω.

PROOF. For a proof of (1) we refer to [77, p. 77/78], [119, Section 3.2] or [124, Satz 4.8]), and for (2) to [1, Theorem 11.4.1]. □

More characterisations of stability are in [79, Theorem 11.9]. Note that, if Ω is Lipschitz (or even smoother), then Ω satisfies the segment condition and Ω is therefore stable. According to [67, Theorem V.4.4], the segment condition is equivalent to $\partial\Omega$ to be continuous.

PROPOSITION 5.4.4. *Suppose that* $\Omega \subset \Omega_{n+1} \subset \Omega_n$ *for all* $n \in \mathbb{N}$, *and that* $\bigcap_{n\in\mathbb{N}} \Omega_n \subset \bar{\Omega}$. *If* Ω *is stable, then* $\Omega_n \to \Omega$ *in the sense of Mosco.*

PROOF. Since $H_0^1(\Omega) \subset H_0^1(\Omega_n)$ for all $n \in \mathbb{N}$, Assumption 5.2.2 is clearly satisfied. Suppose now that (u_n) is a sequence in $H_0^1(\Omega_n)$. Since $\bigcap_{n\in\mathbb{N}} \Omega_n \subset \bar{\Omega}$ it follows that every weak limit point u of that sequence has support in $\overline{\Omega}$. Hence by the stability of Ω

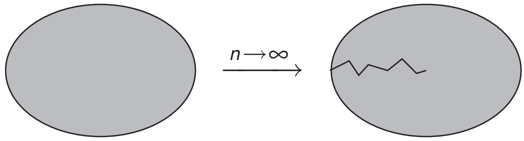

Fig. 5.3. Cracking domain.

we get that $u \in H_0^1(\Omega)$ as required in Assumption 5.2.1. Hence $\Omega_n \to \Omega$ in the sense of Mosco. □

THEOREM 5.4.5. *Suppose that $\Omega_n, \Omega \subset \mathbb{R}^N$ are open (not necessarily bounded) sets, and that Ω is stable. Then $\Omega_n \to \Omega$ in the sense of Mosco if and only if the following two conditions are satisfied.*

(1) $\operatorname{cap}(K \cap \Omega_n^c) \to 0$ *as $n \to \infty$ for all compact sets $K \subset \Omega$;*

(2) *There exists an open covering $\mathcal{O}$ of $\mathbb{R}^N \setminus \overline{\Omega}$ such that $\lambda_1(U \cap \Omega_n) \to \infty$ as $n \to \infty$ for all $U \in \mathcal{O}$;*

PROOF. The assertion follows from Proposition 5.3.3 and Theorem 5.3.1, together with the definition of stability. □

Note that the requirement that Ω be stable is too much in certain cases. An example is a cracking domain as in Figure 5.3. It is sufficient to require a condition on

$$\Gamma := \bigcap_{n\in\mathbb{N}} \left(\overline{\bigcup_{k\geq n} (\Omega_k \cap \partial\Omega)} \right) \subset \partial\Omega. \tag{5.4.1}$$

For instance for the cracking domain we consider, the set Γ consists of the end point of the crack. As that set is of capacity zero, we get convergence in the sense of Mosco. Also if $\Gamma \subset \partial\Omega$ satisfies a segment condition except at a set of capacity zero, we also get convergence of Ω_n. A discussion of this condition is given in [58, Section 7] or [119]. Examples of cracking domains also appear in [124].

5.5. *Proof of the main convergence result*

To prove Theorem 5.2.4 we will proceed as follows. First we prove that (1) implies (2) and that (2) implies (3). We then observe that Assumption 5.2.1 follows from (2), whereas Assumption 5.2.2 follows from (3). Hence we could try to prove that (3) implies (2) to get from (3) back to (1). However, this does not seem to be possible. Instead we prove that (3) implies a statement similar to (2), but for the formally adjoint problem. That still implies Assumption 5.2.2, so (3) implies (1). We then prove the equivalence to (4) separately by using the uniform estimate on the norm of $R_n(\lambda)$ and the density.

We start by proving that (1) implies (2).

PROPOSITION 5.5.1. *Suppose $\Omega_n, \Omega \subset \mathbb{R}^N$ are open sets with $\Omega_n \to \Omega$ in the sense of Mosco. If $\lambda \geq \lambda_0$, then $R_n(\lambda) f_n \rightharpoonup R(\lambda) f$ weakly in $H^1(\mathbb{R}^N)$ whenever $f_n \rightharpoonup f$ weakly in $H^{-1}(\Omega)$.*

PROOF. Let $f_n \rightharpoonup f$ weakly in $H^{-1}(\mathbb{R}^N)$. Set $u_n := R_n(\lambda) f_n$. By Proposition 5.1.3 we have

$$\|u_n\|_{H^1} \leq \frac{2}{\alpha_0} \|f_n\|_{H^{-1}}$$

for all $n \in \mathbb{N}$. Since (f_n) is weakly convergent and therefore bounded, the sequence (u_n) is bounded in $H^1(\mathbb{R}^N)$. Hence it has a weakly convergent subsequence (u_{n_k}) with limit v. By Assumption 5.2.1 we have $v \in H_0^1(\Omega)$. Given $\varphi \in C_c^\infty(\Omega)$ Assumption 5.2.2 implies that there exist $\varphi_n \in H_0^1(\Omega_n)$ with $\varphi_n \to \varphi$ in $H^1(\mathbb{R}^N)$. Because u_n is a weak solution of (5.1.1) we get

$$a_n(u_n, \varphi_n) = \langle f_n, \varphi_n \rangle$$

for all $n \in \mathbb{N}$. Since $u_n \rightharpoonup v$ weakly and $\varphi_n \to \varphi$ strongly we can use (5.1.10) to pass to the limit in the above identity. Hence

$$a_n(v, \varphi) = \langle f, \varphi \rangle$$

for all $\varphi \in C_c^\infty(\Omega)$, showing that v is a weak solution of (5.1.2). Since (5.1.2) has a unique solution we conclude that $v = R(\lambda) f$ and that the whole sequence converges. □

Next we prove that (2) implies (3).

PROPOSITION 5.5.2. *Suppose $\Omega_n, \Omega \subset \mathbb{R}^N$ are open sets and that $\lambda \geq \lambda_0$. Moreover, suppose $f_n \to f$ in $H^{-1}(\mathbb{R}^N)$ with $R_n(\lambda) f_n \rightharpoonup R(\lambda) f$ weakly in $H^1(\mathbb{R}^N)$. Then $R_n(\lambda) f_n \to R(\lambda) f$ in $H^1(\mathbb{R}^N)$.*

PROOF. Assume that $f_n \to f$ in $H^{-1}(\mathbb{R}^N)$, so that $u_n := R_n(\lambda) f_n \rightharpoonup u := R(\lambda) f$ weakly in $H^1(\mathbb{R}^N)$. Hence because in every instance a strongly and a weakly convergent sequence is paired,

$$\lim_{n\to\infty} (a_n(u_n, u_n) + \lambda \|u_n\|_2^2) = \lim_{n\to\infty} \langle f_n, u_n \rangle = \langle f, u \rangle = a(u, u) + \lambda \|u\|_2^2,$$

and also

$$\lim_{n\to\infty} (a_n(u_n, u) + \lambda \langle u_n, u \rangle) = \lim_{n\to\infty} (a_n(u, u_n) + \lambda \langle u, u_n \rangle) = a(u, u) + \lambda \|u\|_2^2.$$

Therefore

$$\begin{aligned} a_n(u_n - u, u_n - u) + \lambda \|u_n - u\|_2^2 &= a_n(u_n, u_n) + \lambda \|u_n\|_2^2 \\ &- (a_n(u_n, u) + \lambda \langle u_n, u \rangle) - (a_n(u, u_n) + \lambda \langle u, u_n \rangle) + a(u, u) + \lambda \|u\|_2^2 \to 0. \end{aligned}$$

From Proposition 5.1.2 we get

$$\frac{2}{\alpha_0} \|u_n - u\|_{H^1}^2 \leq a_n(u_n - u, u_n - u) + \lambda \|u_n - u\|_2^2 \to 0,$$

showing that $u_n \to u$ strongly in $H^1(\mathbb{R}^N)$. □

As remarked earlier we cannot prove directly the converse of the above proposition, but we can prove the corresponding weak convergence property for the formally adjoint problem we introduced in Section 2.3.

PROPOSITION 5.5.3. *Suppose $\Omega_n, \Omega \subset \mathbb{R}^N$ are open sets and that $\lambda \geq \lambda_0$. Suppose that $R_n(\lambda) f_n \to R(\lambda) f$ in $H^1(\mathbb{R}^N)$ whenever $f_n \to f$ in $H^{-1}(\mathbb{R}^N)$. Then $R_n^\sharp(\lambda) f_n \rightharpoonup R^\sharp(\lambda)$ weakly in $H^1(\mathbb{R}^N)$ whenever $f_n \rightharpoonup f$ weakly in $H^{-1}(\mathbb{R}^N)$.*

PROOF. Assume that $f_n \rightharpoonup f$ weakly in $H^{-1}(\mathbb{R}^N)$ and fix $g \in H^{-1}(\mathbb{R}^N)$. Then by (3) $R_n(\lambda)g \to R(\lambda)g$ and so by Lemma 2.5.3

$$\langle g, R_n^\sharp(\lambda) f_n\rangle = \langle R_n(\lambda)g, f_n\rangle \to \langle R(\lambda)g, f\rangle = \langle g, R^\sharp(\lambda) f\rangle,$$

completing the proof of the proposition. □

We now prove that the weak convergence property (2) implies Assumption 5.2.2 and and the strong convergence property (3) implies Assumption 5.2.1.

LEMMA 5.5.4. *Suppose $\Omega_n, \Omega \subset \mathbb{R}^N$ are open sets and that $\lambda \geq \lambda_0$. Suppose that $R_n(\lambda) f_n \rightharpoonup R(\lambda) f$ in $H^1(\mathbb{R}^N)$ whenever $f_n \rightharpoonup f$ in $H^{-1}(\mathbb{R}^N)$. Then Assumption 5.2.1 holds.*

PROOF. Let $u_n \in H_0^1(\Omega_n)$ and define functionals $f_n \in H^{-1}(\mathbb{R}^N)$ by

$$\langle f_n, v\rangle := a_n(u_n, v) + \lambda\langle u_n, v\rangle$$

for all $v \in H^1(\mathbb{R}^N)$. By (5.1.7)

$$\|f_n\|_{H^{-1}} \leq (M+\lambda)\|u_n\|_{H^1}$$

for all $n \in \mathbb{N}$. Now assume that $u_{n_k} \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$. We need to show that $u \in H_0^1(\Omega)$. Since (u_{n_k}) is bounded, the above shows that (f_{n_k}) is bounded in $H^{-1}(\mathbb{R}^N)$. Because every bounded sequence in a Hilbert space has a weakly convergent subsequence there exists a further subsequence, denoted again by (f_{n_k}), with $f_{n_k} \rightharpoonup f$. Now by assumption and the definition of f_n

$$u_{n_k} := R_{n_k}(\lambda) f_{n_k} \rightharpoonup R(\lambda) f = u.$$

Since $u \in H_0^1(\Omega)$ Assumption 5.2.1 follows. □

LEMMA 5.5.5. *Suppose $\Omega_n, \Omega \subset \mathbb{R}^N$ are open sets and that $\lambda \geq \lambda_0$. Suppose that $R_n(\lambda) f_n \to R(\lambda) f$ in $H^1(\mathbb{R}^N)$ whenever $f_n \to f$ in $H^{-1}(\mathbb{R}^N)$. Then Assumption 5.2.2 holds.*

PROOF. Fix $\varphi \in H_0^1(\Omega)$ and define $f \in H^{-1}(\mathbb{R}^N)$ by

$$\langle f, v\rangle := a(\varphi, v) + \lambda\langle \varphi, v\rangle$$

for all $v \in H^1(\mathbb{R}^N)$. Then by (3) and the definition of f

$$\varphi_n := R_n(\lambda) f \to R(\lambda) f = \varphi$$

in $H^1(\mathbb{R}^N)$. Since $\varphi_n \in H_0^1(\Omega_n)$, Assumption 5.2.2 follows. □

We finally need to prove that (4) is equivalent to the other assertions. Clearly (2) implies (4), so we only need to prove that (4) implies (3). The proof is an abstract argument just using a uniform bound on the norms of $R_n(\lambda)$.

LEMMA 5.5.6. *Suppose $\Omega_n, \Omega \subset \mathbb{R}^N$ are open sets and that $\lambda \geq \lambda_0$. Suppose that $R_n(\lambda) f \rightharpoonup R(\lambda) f$ for f in a dense subset of $H^{-1}(\mathbb{R}^N)$, then $R_n(\lambda) f_n \to R(\lambda) f$ in $H^1(\mathbb{R}^N)$ whenever $f_n \to f$ in $H^{-1}(\mathbb{R}^N)$.*

PROOF. Let V be the dense set of $H^{-1}(\mathbb{R}^N)$ for which $R_n(\lambda) g \rightharpoonup R(\lambda) g$ weakly in $H^1(\mathbb{R}^N)$ for all $g \in V$. Then by Proposition 5.5.2 convergence is actually in $H^1(\mathbb{R}^N)$, so $R_n(\lambda) \to R(\lambda)$ strongly on the dense subset V. Since the norms of $\|R_n(\lambda)\|$ is uniformly bounded by Proposition 5.1.3, we have strong convergence on $H^1(\mathbb{R}^N)$. Also because of the uniform bound and the strong convergence, $R_n(\lambda) f_n \to R(\lambda) f$ in $H^1(\mathbb{R}^N)$ whenever $f_n \to f$ in $H^{-1}(\Omega)$. □

6. Varying domains and Robin boundary conditions

An interesting feature of Robin problems is, that for a sequence of domains, the boundary condition can change in the limit. We consider three different cases.

6.1. *Summary of results*

Given open sets $\Omega_n \subset \mathbb{R}^N$ ($N \geq 2$) we consider convergence of solutions of the Robin problems

$$\begin{aligned} \mathcal{A}_n u + \lambda u &= f_n && \text{in } \Omega_n, \\ \frac{\partial}{\partial \nu_{\mathcal{A}}} u + b_{0n} u &= 0 && \text{on } \partial\Omega_n \end{aligned} \tag{6.1.1}$$

to a solution of a limit problem

$$\begin{aligned} \mathcal{A}u + \lambda u &= f && \text{in } \Omega, \\ \mathcal{B}u &= 0 && \text{on } \partial\Omega \end{aligned} \tag{6.1.2}$$

on a domain Ω as $n \to \infty$. On the operators $\mathcal{A}_n$ we make the same assumptions as in the case of the Dirichlet problem. We also need a positivity assumption on the boundary coefficient b_0.

ASSUMPTION 6.1.1. Suppose that $\mathcal{A}_n, \mathcal{A}$ satisfy Assumption 5.1.1. Moreover, let $b_{0n} \geq \beta$ for some constant $\beta > 0$.

The above conditions allow us to make use of the domain-independent a priori estimates proved in Section 2.4.2.

The situation is not as simple as in the case of Dirichlet boundary conditions, where the limit problem satisfies Dirichlet boundary conditions as well. Here, the boundary

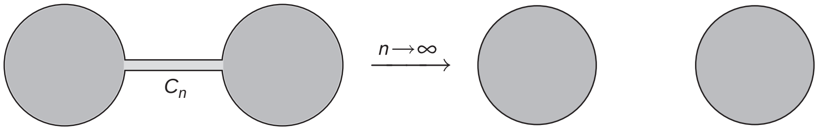

Fig. 6.1. Dumbbell like domain converging to two circles.

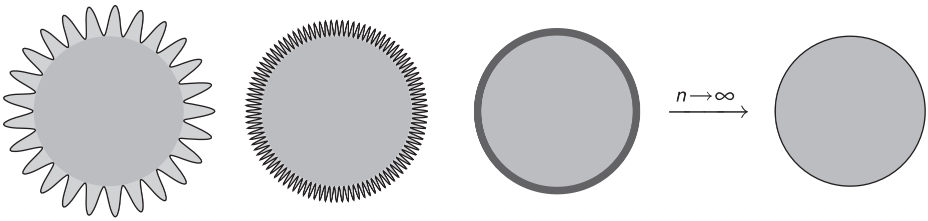

Fig. 6.2. Domains with fast oscillating boundaries approaching a disc.

conditions of the limit problem depends on how the domains Ω_n approach Ω. We consider the following three cases:

(1) The boundary $\partial\Omega$ is only modified in the neighbourhood of a very small set, namely a set of capacity zero. A prototype of such an approximation is a dumbbell with a handle C_n shrinking to a line. The limit set consists of two disconnected sets as shown in Figure 6.1. It is also possible to cut small holes and shrink them to a set of capacity zero. The limit problem is then a Robin problem with the same boundary conditions as the approximating problems. How much boundary we add is irrelevant. A precise statement is in Theorem 6.3.3.

(2) The boundaries of the approaching domains are wildly oscillating. If the oscillations, which do not necessarily need to be periodic, are very fast, then the limit problem turns out to be a Dirichlet problem. See Figure 6.2 for an example. A magnification of the boundary of the last domain shown in the sequence is displayed in Figure 6.3. The precise result is stated in Theorem 6.4.3.

(3) The boundaries of the approaching domains oscillate moderately, not necessarily in a periodic fashion. The limit domain can then be a Robin problem with a different coefficient b_0 in the boundary conditions. Again, an example is as shown in Figure 6.2, with boundary not oscillating quite as fast. The precise result is stated in Theorem 6.5.1.

To see that such phenomena are to be expected, look at the model of heat conduction. The boundary conditions describe a partially insulated boundary, where the loss of heat is proportional to the temperature at the boundary. If the boundary becomes longer the loss is bigger. If the additional boundary is only connected to the main body by a small set as for instance the handle in the case of the dumbbell, then its influence on the temperature inside the major parts of the body is negligible. This corresponds to case one. In the second case, the oscillating boundary will act like a radiator and cool the body better and better, the longer the boundary gets. As the length of the boundary goes to infinity, the

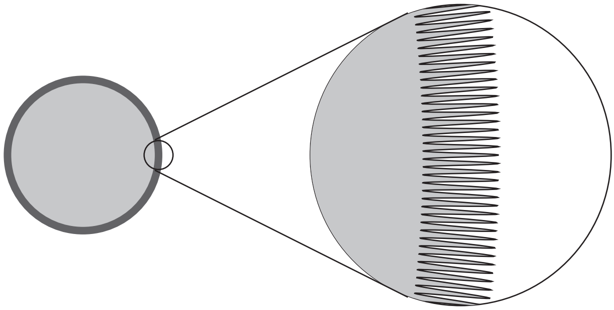

Fig. 6.3. Enlarged portion of a very fast oscillating boundary.

cooling becomes perfect and we get Dirichlet boundary conditions. In the third case the boundary oscillates, but its surface area does not go to infinity and therefore we just get a better cooling, meaning that we have Robin boundary conditions with a possibly larger different boundary coefficient. It is worth noting that the cooling can only get better, not worse.

The first case shows that in a way, the Robin problem behaves very similar to the Dirichlet problem, where we get convergence in the operator norm and therefore convergence of the spectrum. This is in sharp contrast to the Neumann problem. The last two phenomena are *boundary homogenisation results*, where we get the *effective boundary conditions* in the limit. Our exposition follows [51]. In parts we make stronger assumptions to avoid lengthy technical proofs. For periodic oscillations, using very different techniques, other boundary homogenisation results complementing ours are proved in [21,36,71,109] with very different methods. Problems with small holes (obstacles) were considered in [105,106,122,123]. There are applications to problems with nonlinear boundary conditions in [16].

According to Table 2.2 we can set $\lambda_0 = \lambda_{\mathcal{A}}$ given by (5.1.8). Therefore, by Theorem 2.2.2, the Robin problem (6.1.1) is uniquely solvable for all $\lambda \geq \lambda_{\mathcal{A}}$ and all $f_n \in L_p(\Omega_n)$ if $p \geq 2N/(N+1)$. Let $i_{\Omega_n}(f)$ be the extension of $f \in L_p(\Omega)$ by zero outside Ω_n and $r_{\Omega_n}(f)$ the restriction of $f \in L_p(\mathbb{R}^N)$ to Ω_n. Finally let A_n be the operator induced by the form $a_n(\cdot\,,\cdot)$ induced by (6.1.1). We let $R_n(\lambda)$ and $R(\lambda)$ be the pseudo-resolvents associated with the problems $(\mathcal{A}_n, \mathcal{B}_n)$ and the limit problem $(\mathcal{A}, \mathcal{B})$ as in Definition 2.5.2. In all three cases we show that, if $p \in (1, \infty)$, then

$$R_n(\lambda) \to R(\lambda)$$

in $\mathcal{L}(L_p(\mathbb{R}^N), L_q(\mathbb{R}^N))$ for all $q \in [1, m(p))$, where $m(p)$ is given by

$$m(p) := \begin{cases} Np(N-p)^{-1} & \text{if } p \in (1, N), \\ \infty & \text{if } p > N. \end{cases} \tag{6.1.3}$$

Since $R(\lambda)$ is compact, Theorem 4.3.4 shows that the above is equivalent to

$$\lim_{n\to\infty} R_n(\lambda) f_n = R(\lambda) f$$

in $L_q(\mathbb{R}^N)$ for all $q \in [1, m(p))$, whenever $f_n \rightharpoonup f$ weakly in $L_p(\mathbb{R}^N)$. The latter property is essential for dealing with semi-linear problems.

After proving some preliminary results we devote to each of the three cases a separate section, where we give a precise statement and a proof of the convergence theorems.

6.2. *Preliminary results*

To prove the three results mentioned in the previous section we make use of some preliminary results. They are about convergence to a solution of the limit problem on Ω without consideration of boundary conditions.

Given open sets Ω_n we work with functions in $u_n \in H^1(\Omega_n)$ and look at their convergence properties in an open set Ω. We do not assume that Ω_n has an extension property. Even if it has, then its norm does not need to be uniformly bounded with respect to $n \in \mathbb{N}$. Instead we extend u_n and ∇u_n by zero outside Ω_n and consider convergence in $L_2(\mathbb{R}^N)$. We denote these functions by $\tilde{u}_n$ and $\widetilde{\nabla u}_n$. The argument used is very similar, but slightly more complicated than the one given on the proof of Proposition 5.5.1. The complication arises because in the present case u_n cannot be considered as an element of $H^1(\mathbb{R}^N)$.

PROPOSITION 6.2.1. *Suppose that $\Omega_n, \Omega \subset \mathbb{R}^N$ satisfy Assumption 5.2.2 and $\mathcal{A}_n$ Assumption 5.1.1. Furthermore, suppose that $u_n \in H^1(\Omega_n)$ are weak solutions of $\mathcal{A}_n u_n = f_n$ with $f_n \in L_2(\mathbb{R}^N)$ and $f_n \rightharpoonup f$ weakly in $L_2(\mathbb{R}^N)$. If $\|u_n\|_{H^1}$ is uniformly bounded, then there exists a subsequence (u_{n_k}) and $u \in H^1(\Omega)$ such that $\tilde{u}_{n_k} \rightharpoonup u$ and $\widetilde{\nabla u}_{n_k} \rightharpoonup \nabla u$ weakly in $L_2(\Omega)$ and $L_2(\Omega, \mathbb{R}^N)$, respectively. Moreover, u is a weak solution of $\mathcal{A}u = f$ in Ω.*

PROOF. By assumption $\|u_n\|_{H^1}$ is uniformly bounded, and so the functions u_n and ∇u_n, extended by zero outside Ω_n are bounded sequences in $L_2(\mathbb{R}^N)$. Hence there exists a subsequence such that $u_{n_k} \rightharpoonup u$ weakly in $L_2(\Omega)$ and $\nabla u_{n_k} \rightharpoonup v$ weakly in $L_2(\Omega, \mathbb{R}^N)$. Renumbering the subsequence we assume that (u_n) and (∇u_n) converge. Fix now $\varphi \in C_c^\infty(\Omega)$. By assumption and Proposition 5.3.3 there exists $\varphi_n \in C_c^\infty(\Omega_n \cap \Omega)$ such that $\varphi_n \to \varphi$ in $H^1(\Omega)$. As $u_n, \varphi_n \in H^1(\Omega_n)$

$$\int_\Omega \nabla u_n \varphi_n \, dx = \int_{\Omega_n} \nabla u_n \varphi_n \, dx = -\int_{\Omega_n} u_n \nabla \varphi_n \, dx = -\int_\Omega u_n \nabla \varphi_n \, dx$$

for all $n \in \mathbb{N}$. Because $\tilde{u}_n, \widetilde{\nabla u}_n$ converge weakly and $\varphi_n, \nabla \varphi_n$ strongly we can pass to the limit and get

$$\int_\Omega v\varphi \, dx = -\int_\Omega u \nabla \varphi \, dx.$$

As $\varphi \in C_c^\infty(\Omega)$ was arbitrary, this means that $v = \nabla u$ is the weak gradient of u in Ω, so $u \in H^1(\Omega)$. Next we observe that, as in the proof of (5.1.10), we have $a_n(u_n, \varphi_n) \to a(u, \varphi)$. We know that $a_n(u_n, \varphi_n) = \langle f_n, \varphi_n \rangle$, and passing to the limit $a(u, \varphi) = \langle f, \varphi \rangle$. Hence $\mathcal{A}u = f$ as claimed. □

The above is a very weak result. In particular, there are no assumptions on Ω_n outside Ω. If we add some more assumptions we get strong convergence.

PROPOSITION 6.2.2. *In addition to the assumptions of Proposition 6.2.1, suppose that (u_n) is bounded in $L_r(\mathbb{R}^N)$ for some $r > 2$. Moreover, suppose that for every $\varepsilon > 0$ there exists a compact set $K \subset \Omega$ and $n_0 \in \mathbb{N}$ such that $K \subset \Omega_n$ and $|(\Omega_n \cup \Omega) \setminus K| < \varepsilon$ for all $n > n_0$. Then there exists a subsequence $(\tilde{u}_{n_k})$ such that $\tilde{u}_{n_k} \to u$ in $L_q(\mathbb{R}^N)$ for all $q \in [2, r)$, and $u = 0$ almost everywhere in Ω^c.*

PROOF. Proposition 6.2.1 guarantees that a subsequence of $(\tilde{u}_n)$ converges weakly in $L_2(\mathbb{R}^N)$ to a solution of $\mathcal{A}u = f$. Denote that subsequence again by $(\tilde{u}_n)$. We show that convergence takes place in the L_2-norm. Because $(\tilde{u}_n)$ is also bounded in $L_r(\mathbb{R}^N)$ we can also assume that the subsequence converges weakly in $L_r(\mathbb{R}^N)$, or weakly* if $r = \infty$. Hence $u \in L_r(\mathbb{R}^N)$.

Fix $\varepsilon > 0$ and a compact set $K \subset \Omega$ and $n_1 \in \mathbb{N}$ such that $K \subset \Omega_n$ and $|(\Omega_n \cup \Omega) \setminus K| < \varepsilon/2$ for all $n > n_1$. Then choose an open set $U \subset K$ with $|(\Omega_n \cup \Omega) \setminus U| < \varepsilon$ for all $n > n_1$. By assumption $(\tilde{u}_n)$ is bounded in $H^1(K)$ and so Rellich's theorem implies that $\tilde{u}_n \to u$ in $L_2(U)$. Hence there exists $n_2 \in \mathbb{N}$ such that $\|\tilde{u}_n - u\|_{L_2(U)} < \varepsilon$ for all $n > n_2$. Using that $u_n \in L_r(\mathbb{R}^N)$ we get by Hölder's inequality

$$\begin{aligned}\|\tilde{u}_n - u\|_2 &= \|\tilde{u}_n - u\|_{L_2(U)} + \|\tilde{u}_n - u\|_{L_2((\Omega_n \cup \Omega)\setminus U)} \\ &< \varepsilon + \big|(\Omega_n \cup \Omega) \setminus U\big|^{1/2-1/r}(\|\tilde{u}_n\|_r + \|u\|_r) < \varepsilon + \varepsilon^{1/2-1/r}(\|\tilde{u}_n\|_r + \|u\|_r)\end{aligned}$$

for all $n > n_0 := \max\{n_1, n_2\}$. By the uniform bound on $\|u_n\|_r$ we conclude that $\tilde{u}_n \to u$ in $L_2(\mathbb{R}^N)$. Since $\Omega^c \subset U^c$ the above argument also shows that

$$\|u_n\|_{L_2(\Omega^c)} \le \|u_n\|_{L_2((\Omega_n \cup \Omega)\setminus U)} < \varepsilon^{1/2-1/r}\|u_n\|_r$$

for all $n > n_0$. Hence $\tilde{u}_n \to 0$ in $L_2(\Omega^c)$ and thus $u = 0$ on Ω^c almost everywhere. By the uniform bound on $\|u_n\|_r$ and interpolation

$$\|\tilde{u}_n - u\|_q \le \|\tilde{u}_n - u\|_2^{\theta}\|\tilde{u}_n - u\|_r^{1-\theta} \to 0$$

for $q \in [2, r)$, where $\theta = \frac{2(r-q)}{q(r-2)}$. Hence $\tilde{u}_n \to u$ in $L_q(\mathbb{R}^N)$ for all $q \in [2, \infty)$. □

We now use the a priori estimates for the solutions of the Robin problem to verify the boundedness assumptions made in the previous propositions.

COROLLARY 6.2.3. *Suppose Ω_n, Ω are bounded Lipschitz domains $p > N$ and $f_n \rightharpoonup f$ in $L_p(\mathbb{R}^N)$. Let u_n be the weak solution of (6.1.1). If $\lambda \ge \lambda_0$, then there exists a constant M independent of $n \in \mathbb{N}$ such that*

$$\|u_n\|_{V_n} + \|u_n\|_\infty \le M. \tag{6.2.1}$$

Moreover, if $\Omega \subset \Omega_n$ for all $n \in \mathbb{N}$ and $|\Omega_n \setminus \Omega| \to 0$, then there exists a subsequence (u_{n_k}) of (u_n) converging to a weak solution of

$$\mathcal{A}u + \lambda u = f$$

in $L_q(\mathbb{R}^N)$ for all $q \in [2, \infty)$. Finally, $u = 0$ on $\bar{\Omega}^c$ almost everywhere.

PROOF. By Theorem 2.4.1 with constants from Table 2.2 there exists a constant C only depending on N, p and an upper bound for $|\Omega_n|$ such that

$$\|u_n\|_\infty \le C \max\left\{\frac{1}{\alpha_0}, \frac{1}{\beta}\right\} \|f_n\|_p. \tag{6.2.2}$$

Similarly, using the norm

$$\|u_n\|_{V_n} := \left(\|\nabla u_n\|_2 + \int_{\partial\Omega_n} u_n^2\, d\sigma\right)^{1/2}$$

we get from Theorem 2.2.2 that

$$\|u_n\|_{V_n} \le C_1 \|f_n\|_{V_n'} \le C_2 \|f_n\|_p$$

for all $n \in \mathbb{N}$ with constants C_1, C_2 independent of $n \in \mathbb{N}$. Using that weakly convergent sequences in $L_p(\mathbb{R}^N)$ are bounded we get the existence of a constant M independent of $n \in \mathbb{N}$ such that (6.2.1) holds for all $n \in \mathbb{N}$. The second part follows from Proposition 6.2.2 because Assumption 5.2.2 and all other assumptions are clearly satisfied. □

6.3. *Small modifications of the original boundary*

Without further mentioning we use the notation and setup from Section 6.1 and suppose $(\mathcal{A}_n, \mathcal{B}_n)$ satisfy Assumption 6.1.1. We look at a situation where the original boundary remains largely unperturbed. How much boundary we add outside the domain or inside as holes is almost irrelevant. What we mean by small modifications of $\partial\Omega$ we specify as follows.

ASSUMPTION 6.3.1. Suppose that $\Omega_n \subset \mathbb{R}^N$ are bounded open sets satisfying a Lipschitz condition. Let Ω be an open set and $K \subset \bar{\Omega}$ be a compact set of capacity zero such that for every neighbourhood U of K there exists $n_0 \in \mathbb{N}$ such that

$$\bar{\Omega} \cap \left(\overline{\Omega_n \cap (\Omega \cup U)^c}\right) = \emptyset \quad \text{and} \quad \Omega \subset \Omega_n \cup U \tag{6.3.1}$$

for all $n > n_0$. Moreover, assume that

$$\lim_{n\to\infty} |\Omega_n \cap \Omega^c| = 0.$$

Note that the first condition in (6.3.1) means that U allows us to separate $\bar{\Omega}$ from $\overline{\Omega_n \cap (\Omega \cup U)^c}$ which is the part of $\bar{\Omega}_n$ outside $\bar{\Omega}$ as shown in Figure 6.4. Note that the above assumption also allows us to cut holes in Ω shrinking to a set of capacity zero as $n \to \infty$. If (6.3.1) holds, then also

$$\partial\Omega \cap U^c \subset \partial\Omega_n. \tag{6.3.2}$$

This means that $\partial\Omega$ is contained in $\partial\Omega_n$ except for a very small set. To see this let $x \in \partial\Omega \cap U^c$. If W is a small enough neighbourhood of x, then in particular $W \cap \Omega \subset W \cap \Omega_n$ by (6.3.1) and so $W \cap \Omega_n \ne \emptyset$ for every neighbourhood of x. Moreover,

$$\partial\Omega \cap \Omega_n \cap U^c \subset \Omega^c \cap \Omega_n \cap U^c \subset \overline{\Omega_n \cap (\Omega \cup U)^c}$$

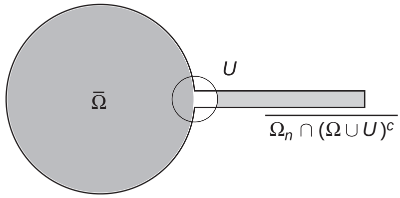

Fig. 6.4. Separation of the part of Ω_n from $\bar{\Omega}$ by U.

and because $\partial\Omega \subset \bar{\Omega}$ (6.3.1) implies that $\partial\Omega \cap \Omega_n \cap U^c = \emptyset$. Hence (6.3.2) follows. We next use the above to construct a sequence of cutoff functions.

LEMMA 6.3.2. *Suppose Ω_n, Ω satisfy Assumption 6.3.1. Then there exists a sequence $\psi_n \in C^\infty(\bar{\Omega}) \cap C^\infty(\bar{\Omega}_n)$ such that $0 \le \psi_n \le 1$, $\psi_n \to 0$ on $\Omega_n \cap \bar{\Omega}^c$ and $\psi_n \to 1$ in $H^1(\Omega)$.*

PROOF. Fix a sequence U_k of bounded open sets such that $K \subset \bar{U}_{k+1} \subset U_k$ for all $k \in \mathbb{N}$, that $\bigcap U_k = K$ and that $\operatorname{cap}(\bar{U}_k) \to 0$ converges to zero. We can make such a choice because K is a compact set with $\operatorname{cap}(K) = 0$. By (6.3.1), for every $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that

$$\bar{\Omega} \cap \overline{(\Omega_n \cap (\Omega \cup U_k)^c)} = \emptyset \quad \text{and} \quad \Omega \subset \Omega_n \cup U_k$$

for all $n > n_k$. Let now $\theta_k \in C_c^\infty(U_{k+1})$ be such that $0 \le \theta_k \le 1$ and $\theta_k = 1$ on $\bar{U}_k$. Since $\operatorname{cap}(\bar{U}_k)$ converges to zero we can make that choice such that $\theta_k \to 0$ in $H^1(\mathbb{R}^N)$. For $n_k < n \le n_{k+1}$ we now set

$$\psi_n(x) := \begin{cases} 1 - \theta_k(x) & \text{if } x \in \bar{\Omega} \cup \bar{U}_k, \\ 0 & \text{if } x \in (\bar{\Omega} \cup \bar{U}_k)^c. \end{cases}$$

By choice of n_k and θ_k the function ψ_n is well defined with the required properties. □

We now prove the main theorem of this section.

THEOREM 6.3.3. *Suppose Ω_n are bounded Lipschitz domains satisfying Assumption 6.3.1 and that $b_{0n} \rightharpoonup b_0$ weakly in $L_r(\partial\Omega)$ for some $r \in (1, \infty)$ or weakly* in $L_\infty(\partial\Omega)$. If $1 < p < \infty$ and $\lambda \in \varrho(-A)$, then for n large enough $\lambda \in \varrho(-A_n)$ and $R_n(\lambda) \to R(\lambda)$ in $\mathcal{L}(L_p(\mathbb{R}^N), L_q(\mathbb{R}^N))$ for all $q \in [1, m(p))$. Here $m(p)$ is given by (6.1.3) and $R(\lambda)$ is the resolvent operator associated with the problem*

$$\begin{aligned} \mathcal{A}u + \lambda u &= f && \text{in } \Omega, \\ \frac{\partial}{\partial \nu_{\mathcal{A}}} u + b_0 u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{6.3.3}$$

PROOF. We only need to consider $p > N$ and $\lambda > \lambda_{\mathcal{A}}$ with $\lambda_{\mathcal{A}}$ defined by (5.1.8). Given that $f_n \rightharpoonup f$ weakly in $L_p(\mathbb{R}^N)$, we show that $R_n(\lambda) f_n \to R(\lambda) f$ in $L_p(\mathbb{R}^N)$. Since $R(\lambda)$ is compact Theorem 4.3.4 and the uniform a priori estimates from Theorem 2.4.1 in conjunction with Table 2.2 complete the proof.

Let $p > N$, $\lambda > \lambda_{\mathcal{A}}$ and suppose that $f_n \rightharpoonup f$ weakly in $L_p(\mathbb{R}^N)$. Set $u_n := R(\lambda)f_n$ and $u := R(\lambda)f$. Then Corollary 6.2.3 implies the uniform bound (6.2.1). Also note that Assumption 6.3.1 implies Assumption 5.2.2 and hence Proposition 6.2.2 guarantees that a subsequence of $u_n := R_n(\lambda)f_n$ converges to some $u \in H^1(\Omega)$ in $L_q(\mathbb{R}^N)$ for all $q \in (1, \infty)$. Moreover, $u = 0$ in $\mathbb{R}^N \setminus \Omega$ almost everywhere and u satisfies $\mathcal{A}u + \lambda u = f$ in Ω. Since (6.3.3) has a unique solution, u is the only possible accumulation point for the sequence (u_n), and therefore the whole sequence converges. To simplify notation we therefore assume that u_n converges.

We assumed that Ω_n are Lipschitz domains and that for every open set U containing K the set boundary $\partial\Omega$ is contained in $\partial\Omega_n \cup U$ for n large enough. Hence $\partial\Omega$ satisfies a Lipschitz condition except possibly at $K \cap \partial\Omega$. Because the latter set has capacity zero the restrictions of $C_c^\infty(\mathbb{R}^n)$ to Ω are dense in $H^1(\Omega)$. Fix $\varphi \in C_c^\infty(\mathbb{R}^N)$ and set $\varphi_n := \psi_n\varphi$ with ψ_n the cutoff functions from Lemma 6.3.2. Then by construction $\varphi_n \in H^1(\Omega_n) \cap H^1(\Omega)$ with $\varphi_n \to \varphi$ in $H^1(\Omega)$. In particular

$$a_n(u_n, \varphi_n) + \lambda\langle u_n, \varphi_n\rangle = \langle f_n, \varphi_n\rangle$$

for all $n \in \mathbb{N}$. Clearly $\langle u_n, \varphi_n\rangle \to \langle u, \varphi\rangle$ and $\langle f_n, \varphi_n\rangle \to \langle f, \varphi\rangle$, so we only have to deal with $a_n(u_n, \varphi_n)$. An argument similar to that used in the proof of Proposition 5.1.2 shows that

$$\lim_{n\to\infty} a_{0n}(u_n, \varphi_n), = a_0(u, \varphi_n)$$

where $a_{0n}(\cdot\,,\cdot)$ is the form corresponding to $(\mathcal{A}_n, \mathcal{B}_n)$ excluding the boundary integral as in Definition 2.1.2. We only need to show that

$$\lim_{n\to\infty} \int_{\partial\Omega_n} b_{0n} u_n \varphi_n \, d\sigma = \lim_{n\to\infty} \int_{\partial\Omega} b_{0n} u_n \varphi_n \, d\sigma = \int_{\partial\Omega} b_0 u \varphi \, d\sigma \tag{6.3.4}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^N)$. The first equality is because of the properties of the cutoff functions ψ_n, so we prove the second one. Fix an open set U such that $K \cap \partial\Omega \subset U$. Because $\partial\Omega \cap U^c$ is Lipschitz, the trace operator from $H^1(\Omega)$ into $L_2(\partial\Omega \cap U^c)$ is compact (see [104, Théorème 2.6.2]). As $u_n \rightharpoonup u$ weakly in $H^1(\Omega)$ the corresponding traces converge in $L_2(\partial\Omega \cap U^c)$ and by the uniform bound (6.2.1) and interpolation in $L_s(\partial\Omega \cap U^c)$ for all $s \in [1, \infty)$. By construction, for big enough n, we have $\varphi_n = \varphi$ on $\partial\Omega \cap U^c$. Hence by the assumptions on b_{0n}

$$\lim_{n\to\infty} \int_{\partial\Omega\cap U^c} b_{0n} u_n \varphi_n \, d\sigma = \lim_{n\to\infty} \int_{\partial\Omega\cap U^c} b_{0n} u_n \varphi \, d\sigma = \int_{\partial\Omega\cap U^c} b_0 u \varphi \, d\sigma.$$

The assumptions on b_{0n} also imply that (b_{0n}) is bounded in $L_r(\partial\Omega)$, and therefore

$$\left|\int_{\partial\Omega\cap U} b_{0n} u_n \varphi \, d\sigma\right| \le \|u_n\varphi_n\|_\infty \|b_{0n}\|_{L_r(\partial\Omega)} \sigma(\partial\Omega \cap U).$$

Since we can choose the measure $\sigma(\partial\Omega \cap U)$ to be arbitrarily small and $\|u_n\varphi_n\|_\infty$ is uniformly bounded (6.3.4) follows, completing the proof of the theorem. □

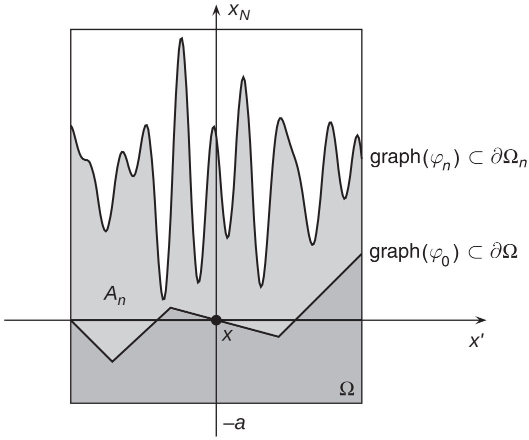

Fig. 6.5. $\partial\Omega$, $\partial\Omega_n$ are locally graphs with respect to the same coordinate system.

6.4. *Boundary homogenisation: Limit is a Dirichlet problem*

In this section we look at a sequence of problems of the form (5.1.1), where $\partial\Omega_n$ is different to $\partial\Omega$ on large parts of the domain. We assume that $\partial\Omega_n$ and $\partial\Omega$ are, at least after a change of coordinates, the graph of a function with respect to the same coordinate system for all $n \in \mathbb{N}$ as illustrated in Figure 6.5. We make this more precise as follows.

ASSUMPTION 6.4.1. Let Ω, Ω_n be domains in $\mathbb{R}^N$ such that $\Omega \subset \Omega_n$ and

$$\lim_{n\to\infty} |\Omega_n \setminus \Omega| = 0.$$

We further assume that for every $x \in \partial\Omega$ there exists a coordinate system with x at the centre and a cylinder $Z = B \times (-a, \infty) \subset \mathbb{R}^{N-1} \times \mathbb{R}$ for some $a > 0$ such that

$$\Omega_n \cap Z = \{(x', x_N) \in B \times (-a, \infty) \colon x_N < \varphi_n(x')\}$$

and

$$\Omega \cap Z = \{(x', x_N) \in B \times (-a, \infty) \colon x_N < \varphi_0(x')\}$$

with $\varphi_n \colon B \to \mathbb{R}$ Lipschitz continuous for all $n \in \mathbb{N}$.

REMARK 6.4.2. Note that $\varphi_n \to \varphi_0$ in $L_1(B)$. To see this note that because $\Omega \subset \Omega_n$ we have $\varphi_n \geq \varphi_0$ and hence

$$\|\varphi_n - \varphi_0\|_{L_1(B)} = \int_B \varphi_n(x') - \varphi_0(x')\,dx' = |Z \cap (\Omega_n \setminus \Omega)| \leq |\Omega_n \setminus \Omega| \to 0$$

as $n \to \infty$.

The assumption that $\Omega \subset \Omega_n$ is only for simplicity to avoid overly technical proofs. It is sufficient to assume that for every compact set $K \subset \Omega$ there exists $n_0 \in \mathbb{N}$ such that $K \subset \Omega_n$ for all $n > n_0$. The argument is given in [51, Remark 5.10(a)].

We define $R_n(\lambda)$ and $R(\lambda)$ as in Definition 2.5.2, where $R(\lambda)$ is associated with the Dirichlet problem (5.1.2).

THEOREM 6.4.3. *Suppose Ω_n, Ω are bounded Lipschitz domains satisfying Assumption 6.4.1. Moreover, assume that for every $x \in \partial\Omega$ the corresponding functions $\varphi_n : B \to \mathbb{R}$ satisfy*

$$\lim_{n\to\infty} \left|\{y \in B : |\nabla\varphi_n(y)| < t\}\right| = 0$$

for all $t > 0$. Finally suppose that $b_{0n} \geq \beta$ for some constant $\beta > 0$. If $1 < p < \infty$ and $\lambda \in \varrho(-A)$, then for n large enough $\lambda \in \varrho(-A_n)$ and $R_n(\lambda) \to R(\lambda)$ in $\mathcal{L}(L_p(\mathbb{R}^N), L_q(\mathbb{R}^N))$ for all $q \in [1, m(p))$. Here $R(\lambda)$ is the resolvent associated with the Dirichlet problem (5.1.2) and $m(p)$ is given by (6.1.3).

PROOF. As in the proof of Theorem 6.3.3 we only need to consider $p > N$ and $\lambda > \lambda_{\mathcal{A}}$, where $\lambda_{\mathcal{A}}$ is given by (5.1.8). The other cases follow from Theorem 4.3.4 and the uniform a priori estimates from Theorem 2.4.1 in conjunction with Table 2.2. Hence assume that $p > N$, that $\lambda > \lambda_{\mathcal{A}}$ and that $f_n \rightharpoonup f$ weakly in $L_p(\mathbb{R}^N)$. We set $u_n := R_n(\lambda) f_n$. Convergence of a subsequence to a solution u of $\mathcal{A}u + \lambda u = f$ in $L_p(\mathbb{R}^N)$ follows from Corollary 6.2.3. Hence we only need to show that u satisfies Dirichlet boundary conditions, that is, $u \in H_0^1(\Omega)$. Since the Dirichlet problem has a unique solution, u is the only possible accumulation point for the sequence (u_n), and therefore the whole sequence converges.

The boundary conditions are local, so we only need to look at a neighbourhood of every boundary point. Fix a cylinder Z and functions φ_n as in Assumption 6.4.1. Because the domains are Lipschitz domains, it is sufficient to show that u has zero trace on $\partial\Omega \cap Z$ (see [104, Théorème 2.4.2]). We know that $u_n \rightharpoonup u$ weakly in $H^1(\Omega \cap Z)$. By the compactness of the trace operator $\gamma \in \mathcal{L}(H^1(\Omega \cap Z), L_2(\partial\Omega \cap Z))$ (see [104, Théorème 2.6.2]) we know that $\gamma(u_n) \to \gamma(u)$ in $L_2(\partial\Omega \cap Z)$. From now on we simply write u_n, u for the traces. We need to show that

$$\lim_{n\to\infty} \|u_n\|_{L_2(\partial\Omega\cap Z)} = 0. \tag{6.4.1}$$

We do that in two steps. To express the boundary integrals in the coordinates chosen we need the Jacobians

$$g_n := \sqrt{1 + |\nabla\varphi_n|^2}.$$

By Rademacher's theorem (see [68, Section 3.1.2]) the gradient exists almost everywhere. We can write

$$\|u_n\|_{L_2(\partial\Omega\cap Z)} = \left(\int_B \left|u_n(x', \varphi_0(x'))\right|^2 g_0(x')\,dx'\right)^{1/2} \tag{6.4.2}$$

$$\leq \|g_0\|_\infty^{1/2} \left(\int_B \left|u_n(x', \varphi_n(x')) - u_n(x', \varphi_0(x'))\right|^2 dx'\right)^{1/2} \tag{6.4.3}$$

$$+ \|g_0\|_\infty^{1/2} \left(\int_B \left|u_n(x', \varphi_n(x'))\right|^2 dx'\right)^{1/2}. \tag{6.4.4}$$

We show separately that each term on the right-hand side of the above inequality converges to zero as $n \to \infty$. For the first term we use Fubini's theorem and the fundamental theorem of calculus to write

$$\int_B \bigl|u_n(x', \varphi_n(x')) - u_n(x', \varphi_0(x'))\bigr|\, dx' = \int_B \Bigl|\int_{\varphi_0(x')}^{\varphi_n(x')} \frac{\partial}{\partial x_N} u_n(x', x_N)\, dx_N\Bigr|\, dx'$$

$$\leq \int_B \int_{\varphi_0(x')}^{\varphi_n(x')} \bigl|\nabla u_n(x', x_N)\bigr|\, dx_N\, dx' \leq |A_n|^{1/2} \|\nabla u_n\|_{L_2(\Omega\cap Z)}, \tag{6.4.5}$$

where A_n is the region between graph(φ_0) and graph(φ_n) as shown in Figure 6.5. By assumption $|\Omega_n \setminus \Omega| \to 0$ and so $|A_n| \to 0$. Furthermore, by (6.2.1) and the definition of u_n the sequence $\|\nabla u_n\|_{L_2(\partial\Omega\cap Z)}$ is bounded. Since u_n is not necessarily continuously differentiable, (6.4.5) needs to be justified. First note that u_n is continuous on $Z \cap \Omega_n$ (see [76, Theorem 8.24]). Because Ω_n is Lipschitz, u_n can be extended to a function in $H^1(Z)$. Since such functions can be represented by a function such that $u_n(x', \cdot)$ is absolutely continuous in the coordinate directions, we can indeed apply the fundamental theorem of calculus as done above (see [101, Section 1.1.3] or [68, Section 4.9.2]). Because

$$\Bigl(\int_B \bigl|u_n(x', \varphi_n(x')) - u_n(x', \varphi_0(x'))\bigr|^2\, dx'\Bigr)^{1/2}$$

$$\leq 2\|u_n\|_\infty^{1/2} \Bigl(\int_B \bigl|u_n(x', \varphi_n(x')) - u_n(x', \varphi_0(x'))\bigr|\, dx'\Bigr)^{1/2}$$

the bound (6.2.1) implies that (6.4.3) converges to zero.

We next prove that (6.4.4) converges to zero as well. For $t > 0$ we set

$$[g_n \geq t] := \{y \in B \colon g_n(y) \geq t\} \quad \text{and} \quad [g_n < t] := \{y \in B \colon g_n(y) < t\}$$

and write

$$\int_B |u_n(x', \varphi_n(x'))|^2\, dx'$$

$$= \int_{[g_n \leq t]} |u_n(x', \varphi_n(x'))|^2\, dx' + \int_{[g_n > t]} |u_n(x', \varphi_n(x'))|^2\, dx'.$$

By (6.2.1) there exists a constant $M_1 > 0$ such that

$$\int_{[g_n \leq t]} \bigl|u_n(x', \varphi_n(x'))\bigr|^2\, dx' \leq \|u_n\|_\infty^2 |[g_n \leq t]| \leq M_1 |[g_n \leq t]|$$

and that

$$\int_{[g_n > t]} \bigl|u_n(x', \varphi_n(x'))\bigr|^2\, dx'$$

$$\leq \frac{1}{t} \int_B \bigl|u_n(x', \varphi_n(x'))\bigr|^2 g_n(x')\, dx' \leq \frac{1}{t} \|u_n\|_{L_2(\partial\Omega\cap Z)}^2 \leq \frac{M_1}{t}$$

for all $t > 0$. Therefore

$$\int_B \bigl|u_n(x', \varphi_n(x'))\bigr|^2\, dx' \leq M_1 \Bigl(|[g_n \leq t]| + \frac{1}{t}\Bigr).$$

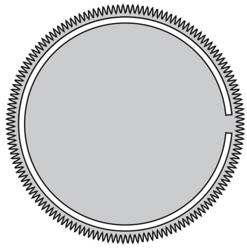

Fig. 6.6. Perturbation of a disc with limit problem a Robin problem.

Fix now $\varepsilon > 0$ and choose $t > 0$ such that $M_1/t < \varepsilon/2$. By assumption there exists $n_0 \in \mathbb{N}$ such that $|[g_n < t]| < \varepsilon/2M_1$ for all $n > n_0$. Hence from the above

$$\int_B \left|u_n(x', \varphi_n(x'))\right|^2 dx' < \varepsilon$$

for all $n > n_0$. Therefore (6.4.4) converges to zero as claimed and (6.4.1) follows. □

REMARK 6.4.4. (a) The assumption that $\partial\Omega_n$ is a graph over $\partial\Omega$ is essential in the above theorem. The domain shown in Figure 6.6 has a very fast oscillating outside boundary. If we shrink the connection from the outside ring with the disc to a point and the ring itself to a circle, then the assumptions of Theorem 6.3.3 are satisfied and the limit problem is the Robin problem (6.3.3).

(b) It is possible to work with diffeomorphisms flattening the boundary locally as shown in Figure 6.7, but the proof is more complicated. We refer to [51] for details.

6.5. *Boundary homogenisation: Limit is a Robin problem*

As in Section 6.4 we look at oscillating boundaries. The oscillations however are slower, and it turns out that the limit problem of (5.1.1) is a Robin problem of the form

$$\begin{aligned} \mathcal{A}u + \lambda u &= f \quad \text{in } \Omega, \\ \frac{\partial}{\partial \nu_{\mathcal{A}}} u + g b_0 u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{6.5.1}$$

with first order coefficient gb_0 rather than just b_0. Now the resolvent $R(\lambda)$ is the pseudo-resolvent associated with (6.5.1). Suppose that Assumption 6.4.1 is satisfied and that φ_n are the functions associated with the parametrisation of $\partial\Omega$ and $\partial\Omega_n$. The function g is associated with the limit of the Jacobians

$$g_n := \sqrt{1 + |\nabla\varphi_n|^2}.$$

The assumption that $\Omega \subset \Omega_n$ is again only for simplicity, see [51, Remark 5.10(a)] on how to overcome it. The assumption that $\partial\Omega_n$ is a graph over $\partial\Omega$ is again essential with a similar example as in Remark 6.4.4(a). This can, in a generalised sense, be as shown in Figure 6.7 (see [51] for details).

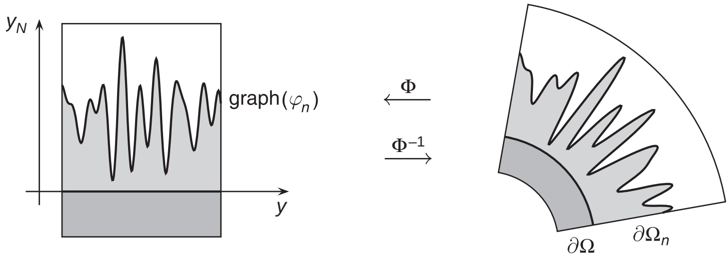

Fig. 6.7. Simultaneously parametrised domains.

THEOREM 6.5.1. *Suppose Ω_n, Ω are bounded Lipschitz domains satisfying Assumption 6.4.1 and that $b_0 \in C(\mathbb{R}^N)$. Set $b_{0n} := b_0|_{\partial\Omega_n}$ with $b_0 \geq \beta_0$ for some constant $\beta > 0$. Moreover, assume that $g \in L_\infty(\partial\Omega)$ such that for every $x \in \partial\Omega$ the corresponding functions $\varphi_n \colon B \to \mathbb{R}$ satisfy*

$$\lim_{n\to\infty} \frac{\sqrt{1+|\nabla\varphi_n|^2}}{\sqrt{1+|\nabla\varphi_0|^2}} = \lim_{n\to\infty} \frac{g_n}{g_0} = g$$

weakly in $L_r(B)$ for some $r \in (1,\infty)$, weakly in $L_\infty(B)$ or strongly in $L_1(B)$. If $1 < p < \infty$ and $\lambda \in \varrho(-A)$, then for n large enough $\lambda \in \varrho(-A_n)$ and $R_n(\lambda) \to R(\lambda)$ in $\mathcal{L}(L_p(\mathbb{R}^N), L_q(\mathbb{R}^N))$ for all $q \in [1, m(p))$. Here $R(\lambda)$ is the resolvent associated with the Robin problem (6.5.1) and $m(p)$ is given by (6.1.3).*

PROOF. As in the previous cases considered we only need to look at $p > N$ and $\lambda > \lambda_{\mathcal{A}}$ with $\lambda_{\mathcal{A}}$ defined by (5.1.8). Given that $f_n \rightharpoonup f$ weakly in $L_p(\mathbb{R}^N)$, we show that $R_n(\lambda) f_n \to R_(\lambda) f$ in $L_p(\mathbb{R}^N)$. Since $R(\lambda)$ is compact, Theorem 4.3.4 and the uniform a priori estimates from Theorem 2.4.1 in conjunction with Table 2.2 complete the proof. Assume that $p > N$, that $\lambda > \lambda_{\mathcal{A}}$ and that $f_n \rightharpoonup f$ weakly in $L_p(\mathbb{R}^N)$. We set $u_n := R_n(\lambda) f_n$. By Corollary 6.2.3 the sequence u_n is bounded in $H^1(\Omega) \cap L_\infty(\mathbb{R}^N)$. It has a subsequence converging to some function $u \in H^1(\Omega)$ in $L_p(\mathbb{R}^N)$. Since (6.5.1) has a unique solution it follows that the whole sequence converges if we can show that u solves (6.5.1). To simplify notation we assume that u_n converges.

We assumed that Ω is a Lipschitz domain and therefore the restrictions of functions in $C_c^\infty(\mathbb{R}^n)$ to Ω are dense in $H^1(\Omega)$. An argument similar to that used in the proof of Proposition 5.1.2 shows that

$$\lim_{n\to\infty} a_{0n}(u_n, \psi) = a_0(u, \psi)$$

where $a_{0n}(\cdot\,,\cdot)$ is the form corresponding to $(\mathcal{A}_n, \mathcal{B}_n)$ excluding the boundary integral as in Definition 2.1.2. We only need to show that

$$\lim_{n\to\infty} \int_{\partial\Omega_n} b_{0n} u_n \psi \, d\sigma = \int_{\partial\Omega} b_0 g u \psi \, d\sigma.$$

By a partition of unity it is sufficient to consider $\psi \in C_c^\infty(\mathbb{R}^N)$ in a cylinder $Z = B \times (-a, \infty)$ as in Assumption 6.4.1. In these coordinates the above becomes

$$\lim_{n\to\infty} \int_B (b_0 u_n \psi)(x', \varphi_n(x')) g_n(x')\, dx' = \int_B (b_0 g u \psi)(x', \varphi_0(x')) g_0(x')\, dx', \tag{6.5.2}$$

where $(b_0 u_n \psi)(x', \varphi_n(x'))$ is the product of the functions $b_0 u_n \psi$ evaluated at the point $(x', \varphi_n(x'))$. First note that

$$u_n(x', \varphi_n(x')) \to u(x', \varphi_0(x'))$$

in $L_1(B)$ by a similar argument as the one used to prove that (6.4.3) converges to zero. The product $b_0\psi$ is continuous and bounded on B and therefore the corresponding superposition operator on $L_1(B)$ is continuous (see [6, Theorem 3.1 and 3.7]). Since $\varphi_n \to \varphi$ in $L_1(B)$ by Remark 6.4.2 we therefore conclude that

$$(b_0\psi)(x', \varphi_n(x')) \to (b_0\psi)(x', \varphi_0(x'))$$

in $L_1(B)$. Since b_0, ψ and u_n are bounded in $L_\infty(B)$ it follows that

$$(b_0 u_n \psi)(x', \varphi_n(x')) \to (b_0 u \psi)(x', \varphi_0(x'))$$

in $L_s(B)$ for all $s \in [1, \infty)$. By assumption

$$g_n(x') = \frac{g_n(x')}{g_0(x')} g_0(x') \rightharpoonup g(x, \varphi_0(x')) g_0(x')$$

weakly in $L_r(B)$ for some $r \in (1, \infty)$, weakly* in $L_\infty(B)$ or strongly in $L_1(B)$. If we combine everything, then (6.5.2) follows. □

REMARK 6.5.2. (a) From the above the function new weight g is larger than or equal to one, so $b_0 g \geq b_0$ always. This reflects the physical description mentioned in Section 6.1, where we argued that a larger surface area of the oscillating boundary will lead to better cooling. We cannot approach a Neumann problem for that reason. The best we can do is to have $g = 1$. If the oscillations are too fast, then heuristically we have "$g = \infty$" and the limit problem is a Dirichlet problem as in Theorem 6.4.3.

(b) Given a Lipschitz domain it is possible to construct a sequence of C^∞ domains satisfying the assumptions of the above theorem in such a way that $g = 1$, that is, the boundary conditions of the limit domain are unchanged. For details see Section 8.3.

7. Neumann problems on varying domains

7.1. *Remarks on Neumann problems*

We saw in Section 2.4.3 that there are no smoothing properties for the Neumann problem uniformly with respect to the domains. This makes dealing with Neumann boundary conditions rather more difficult. In particular, we saw that for Dirichlet and Robin problems the resolvent operators converge in the operator norm. This is not in general the case for Neumann problems. In particular, we cannot expect the spectrum to converge as in the case of the other boundary conditions, which means the resolvent operator only converges strongly (that is, pointwise) in $\mathcal{L}(L_2(\mathbb{R}^N))$. In this exposition we only prove a result similar to those for Dirichlet and Robin problems. It is beyond the scope of these notes to give a comprehensive treatment of the other phenomena. We refer to the literature, in particular to the work of Arrieta [13,17], the group of Bucur, Varchon and Zolésio [29,30] with necessary and sufficient conditions for domains in the plane in [28]. There is other work by Jimbo [87–90] and references therein. Other references include [32,33,47,49,81].

7.2. *Convergence results for Neumann problems*

Given open sets $\Omega_n \subset \mathbb{R}^N$ $(N \geq 2)$ we consider convergence of solutions of the Neumann problems

$$\begin{aligned} \mathcal{A}_n u + \lambda u &= f_n \qquad \text{in } \Omega_n, \\ \frac{\partial}{\partial \nu_{\mathcal{A}}} u &= 0 \qquad \text{on } \partial\Omega_n \end{aligned} \tag{7.2.1}$$

to a solution the Neumann problem

$$\begin{aligned} \mathcal{A} u + \lambda u &= f \qquad \text{in } \Omega, \\ \frac{\partial}{\partial \nu_{\mathcal{A}}} u &= 0 \qquad \text{on } \partial\Omega_n \end{aligned} \tag{7.2.2}$$

on a domain Ω as $n \to \infty$. On the operators $\mathcal{A}_n, \mathcal{A}$ we make the same assumptions as in the case of the Dirichlet and Robin problems, namely those stated in Assumption 5.1.1. To get a result in the spirit of the others proved so far we make the following assumptions on the domains. The conditions are far from optimal, but given the difficulties mentioned in Section 7.1, we refer to the literature cited there for more general conditions.

ASSUMPTION 7.2.1. Suppose that Ω_n, Ω are bounded open sets with the following properties.

(1) There exists a compact set $K \subset \bar{\Omega}$ of capacity zero such that for every neighbourhood U of K there exists $n_0 \in \mathbb{N}$ with $\Omega \subset \Omega \cup U$ for all $n > n_0$.
(2) $|\Omega_n \cap \Omega^c| \to 0$ as $n \to \infty$.
(3) $\{u|_\Omega : u \in C_c^\infty(\mathbb{R}^N)\}$ is dense in $H^1(\Omega)$.
(4) There exists $d > 2$, $c_a > 0$ and $\lambda_0 \geq 0$ such that
$$\|u\|_{2d/(d-2)}^2 \leq c_a(a_n(u,u) + \lambda_0\|u\|_2)$$
for all $u \in H^1(\Omega_n)$ and all $n \in \mathbb{N}$. Here $a_n(\cdot\,,\cdot)$ is given as in Definition 2.1.2 without the boundary integral.

REMARK 7.2.2. (a) Condition (1) allows to cut holes into Ω shrinking to a set K of capacity zero. The holes cannot be arbitrary since otherwise (4) is violated. However

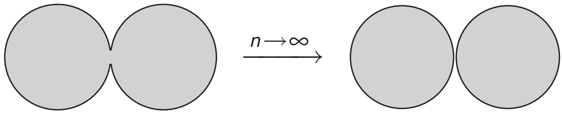

Fig. 7.1. Approximation of touching spheres, preserving a uniform cone condition.

if the holes have a fixed shape and are just contracted by a scalar factor, then (4) is satisfied (see [53, Section 2]).

(b) Condition (4) is satisfied if for instance all domains Ω_n, Ω satisfy a cone condition uniformly with respect to $n \in \mathbb{N}$, that is, the angle and length of the cone defining the cone condition is the same for all $n \in \mathbb{N}$ (see [2, Lemma 5.12]). But as the example with the holes in (a) shows this is only a sufficient condition for (4). Such an approach was used in [37] for instance.

(c) We might think that under the above condition we can have examples like the dumbbell shaped domains in Figure 6.1. As it turns out, condition (4) cannot be satisfied for such a case, because (4) implies convergence in the operator norm as we prove below. Convergence in the operator norm, by Corollary 4.3.2, every finite system of eigenvalues converges, but for dumbbell shaped domains or other exterior perturbations this is not the case (see [13,87]). However, if we replace the dumbbell by two touching balls opened up slightly near the touching balls, then we can ensure that a uniform cone condition is satisfied. Moreover, (3) holds because the union of two balls has a smooth boundary except at a set of capacity zero, where the balls touch (See Figure 7.1). The other conditions are obviously also satisfied.

(d) To get convergence of solutions (but not necessarily of the spectrum) we could work with conditions similar to the Mosco conditions stated in Assumption 5.2.1 and 5.2.2 in the case of the Dirichlet problem. The conditions are explicitly used and stated in [28, Section 2].

We define the resolvent operators $R_n(\lambda)$ and $R(\lambda)$ as in Definition 2.5.2 with $(\mathcal{A}_n, \mathcal{B}_n)$ and $(\mathcal{A}, \mathcal{B})$ being the operators associated with (7.2.1) and (7.2.2), respectively. By Theorem 2.2.2, the Neumann problem (6.1.1) is uniquely solvable for all $\lambda \geq \lambda_0 := \lambda_{\mathcal{A}} + \alpha_0/2$ and all $f_n \in L_2(\Omega_n)$. This means $R_n(\lambda)$ and $R(\lambda)$ is well defined for $n \geq n_0$.

THEOREM 7.2.3. *Suppose that Assumption 7.2.1 holds. If $\lambda \in \varrho(-A)$, then $\lambda \in \varrho(A_n)$ for all n large enough. Moreover, for every $p \in (1, \infty)$ we have $R_n(\lambda) \to R(\lambda)$ in $\mathcal{L}(L_p(\mathbb{R}^N), L_q(\mathbb{R}^N))$ for all $q \in [p, m(p))$, where $m(p)$ is defined by* (2.4.5).

PROOF. Fix $\lambda \geq \lambda_{\mathcal{A}} + \alpha_0/2$ and let $f_n \rightharpoonup f$ weakly in $L_2(\mathbb{R}^N)$. Set $u_n := R_n(\lambda) f_n$ and $u := R(\lambda) f$. Note that by assumption and (2.4.7) the operator $R(\lambda)$ is compact. By Theorem 4.3.4 and (2.4.2) it is therefore sufficient to prove that $u_n \to u$ in $L_2(\mathbb{R}^N)$. To do so first note that by Theorem 2.2.2 we have

$$\|u_n\|_{H^1(\Omega_n)} \leq \frac{2}{\alpha_0} \|f_n\|_{(H^1(\Omega_n))'} \leq \frac{2}{\alpha_0} \|f_n\|_{L_2(\Omega_n)}$$

for all $n \in \mathbb{N}$. Hence $\|u_n\|_{H^1(\Omega)}$ is uniformly bounded. Hence there is a subsequence (u_{n_k}) such that $(\tilde{u}_{n_k})$ and also $(\widetilde{\nabla u_{n_k}})$ converge weakly in $L_2(\mathbb{R}^N)$. Here $\tilde{u}_{n_k}$ and $\widetilde{\nabla u_{n_k}}$ are the extensions of u_{n_k} and ∇u_{n_k} by zero outside Ω_n. If we can show that $u \in H^1(\Omega)$ and that u solves (7.2.2), then the whole sequence converges since the limit problem admits a unique solution. For simplicity we denote the subsequence chosen again by (u_n).

First note that our assumptions make it possible to apply Proposition 6.2.2 and thus $u \in H^1(\Omega)$ and $u_n \to u$ in $L_2(\mathbb{R}^N)$. Note also that this function has support in $\bar{\Omega}$. We show that u solves (7.2.2). By assumption $\operatorname{cap}(\bar{\Omega} \cap \Omega_n^c) \to 0$ and so there exists $\varphi_n \in C_c^\infty(\mathbb{R}^N)$ such that $0 \le \varphi_n \le 1$, $\varphi_n = 1$ in a neighbourhood of $\bar{\Omega} \cap \Omega_n^c$ and $\varphi_n \to 0$ in $H^1(\mathbb{R}^N)$. Let now $\psi \in C_c^\infty(\Omega)$ and set $\psi_n := \psi(1 - \varphi_n)$. Then $\psi_n \to \psi$ in $H^1(\mathbb{R}^N)$. Now

$$a_n(u_n, \psi_n) + \langle u_n, \psi_n \rangle = \langle f_n, \psi_n \rangle$$

for all $n \in \mathbb{N}$. Every term in the above identity involves a pair of a strongly and a weakly convergent sequence and therefore an argument similar to that in the proof of Proposition 5.1.2 shows that we can pass to the limit to get

$$a(u, \psi) + \langle u, \psi \rangle = \langle f, \psi \rangle.$$

Because the restrictions of functions in $C_c^\infty(\Omega)$ to Ω are dense in $H^1(\Omega)$ we conclude that u is the weak solution of (7.2.2) as claimed. □

8. Approximation by smooth data and domains

The above can be used to approximate problems on nonsmooth domains by a sequence of problems on smooth domains. This is a useful tool to get results for nonsmooth domains, using results on smooth domains. Such techniques were for instance central in [12,54,86, 93,98]. The technique can be used to prove isoperimetric inequalities for eigenvalues, given they are known for smooth domains and involve constants independent of the geometry of the domain. A recent collection of such inequality for which the technique could be applied appears in [18]. Such an approach was also used in [60] for Robin boundary conditions.

8.1. *Approximation by operators having smooth coefficients*

Consider an operator $\mathcal{A}$ as in Section 2.1 with diffusion matrix $A_0 = [a_{ij}]$, drift terms $a = (a_1, \dots, a_N)$ and $b = (b_1, \dots, b_N)$, and potential c_0 in $L^\infty(\mathbb{R}^N)$. Also assume that $\mathcal{A}$ satisfies the ellipticity condition (2.1.3). If $\mathcal{A}$ is only given on an open set Ω, then extend it to an operator on $\mathbb{R}^N$ as in Remark 2.1.1.

Define the nonnegative function $\varphi \in C_c^\infty(\mathbb{R}^N)$ by

$$\varphi(x) := \begin{cases} c \exp\left(\dfrac{1}{|x|^2 - 1}\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \ge 1 \end{cases}$$

with $c > 0$ chosen such that

$$\int_{\mathbb{R}^N} \varphi(x)\,dx = 1.$$

For all $n \in \mathbb{N}$ define φ_n by $\varphi_n(x) := n^N \varphi(nx)$. Then $(\varphi_n)_{n\in\mathbb{N}}$ is an approximate identity in $\mathbb{R}^N$. For all $n \in \mathbb{N}$ and $i, j = 1, \dots, N$ we set

$$a_{ij}^{(n)} := \varphi_n * a_{ij}, \qquad a_i^{(n)} := \varphi_n * a_i, \qquad b_i^{(n)} := \varphi_n * b_i \quad \text{and} \quad c_{0n} := \varphi_n * c_0.$$

We then define $\mathcal{A}_n$ by

$$\mathcal{A}_n u := -\operatorname{div}(A_n \nabla u + a_n u) + b_n \cdot \nabla u + c_{0n} u \tag{8.1.1}$$

with $A_n = [a_{ij}^{(n)}]$, $a_n = (a_1^{(n)}, \dots, a_N^{(n)})$, $b_n = (b_1^{(n)}, \dots, b_N^{(n)})$. The following proposition shows that the family of operators $\mathcal{A}_n$ in particular satisfies Assumption 5.1.1.

PROPOSITION 8.1.1. *The family of operators $\mathcal{A}_n$ as defined above has coefficients of class C^∞ and satisfies* Assumption 5.1.1. *Moreover, $\mathcal{A}_n$ has the same ellipticity constant α_0 as $\mathcal{A}$ and*

$$\lambda_{\mathcal{A}_n} := \|c_{0n}^-\|_\infty + \frac{1}{2\alpha_0}\|a_n + b_n\|_\infty^2 \le \lambda_{\mathcal{A}} \tag{8.1.2}$$

for all $n \in \mathbb{N}$, where $\lambda_{\mathcal{A}}$ is defined by (2.1.10).

PROOF. Let $g \in L_\infty(\mathbb{R}^N)$. By the properties of convolution

$$\begin{aligned}
-\|g^-\|_\infty &= -\int_{\mathbb{R}^N} \varphi_n(y)\|g^-\|_\infty\,dy \le \int_{\mathbb{R}^N} \varphi_n(y) g(y-x)\,dy \\
&= \varphi_n * g \le \int_{\mathbb{R}^N} \varphi_n(y)\|g^+\|_\infty\,dy = \|g^+\|_\infty
\end{aligned}$$

for all $n \in \mathbb{N}$ if we use that $\|\varphi_n\|_1 = 1$. Here $g^+ := \max\{u, 0\}$ and $g^- := \max\{-u, 0\}$ are the positive and negative parts of g. In particular, from the above

$$\|\varphi_n * g\|_\infty \le \|g\|_\infty \quad \text{and} \quad \|(\varphi_n * g)^-\|_\infty \le \|g^-\|_\infty.$$

Assuming that the ellipticity condition (2.1.3) holds we get

$$\begin{aligned}
\xi^T A_n(x)\xi &= \sum_{i,j=1}^N a_{ij}^{(n)}(x)\xi_j\xi_i = \sum_{i,j=1}^N \int_{\mathbb{R}^N} \varphi_n(y) a_{ij}(x-y)\,dy\,\xi_j\xi_i \\
&= \int_{\mathbb{R}^N} \varphi_n(y) \sum_{i,j=1}^N a_{ij}(x-y)\xi_j\xi_i\,dy \ge \int_{\mathbb{R}^N} \varphi_n(y)\alpha_0|\xi|^2\,dy = \alpha_0|\xi|^2
\end{aligned}$$

for all $n \in \mathbb{N}$. Hence $\mathcal{A}_n$ is elliptic with the same ellipticity constant as $\mathcal{A}$. Using the smoothing properties of convolution, the coefficients of $\mathcal{A}_n$ are of class C^∞. Moreover, they converge to the corresponding coefficients of $\mathcal{A}$ almost everywhere (see [68, Section 4.2.1, Theorem 1]). □

8.2. *Approximation by smooth domains from the interior*

The purpose of this section is to prove that every open set in $\mathbb{R}^N$ can be exhausted by a sequence of smoothly bounded open sets. This fact is frequently used, but proofs are often not given, only roughly sketched or very technical (see [63, Section II.4.2, Lemma 1] and [67, Theorem V.4.20]). We give a simple proof based on the existence of suitable cutoff functions and Sard's lemma. The idea is similar to the proof of [67, Theorem V.4.20], where the existence of a sequence of approximating domains with analytic boundary is shown. For open sets U, V we write

$$U \subset\subset V$$

if U is bounded and $\bar{U} \subset V$.

PROPOSITION 8.2.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set. Then there exists a sequence of bounded open sets Ω_n with boundary of class C^∞ such that $\Omega_n \subset\subset \Omega_{n+1} \subset\subset \Omega$ for all $n \in \mathbb{N}$ and $\Omega = \bigcup_{n\in\mathbb{N}} \Omega_n$. If Ω is connected we can choose Ω_n to be connected as well.*

PROOF. Given an open set $\Omega \subset \mathbb{R}^N$, define

$$V_n := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > 1/n,\ |x| < n\}$$

for all $n \in \mathbb{N}$. Then $U_n := V_n$ is open and $U_n \subset\subset U_{n+1} \subset\subset \Omega$ for all $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} U_n = \Omega$. Next choose cutoff functions $\psi_n \in C_c^\infty(U_{n+1})$ such that $0 \le \psi \le 1$ with $\psi = 1$ on $\bar{U}_n$. By Sard's lemma (see [85, Theorem 3.1.3]) we can choose regular values $t_n \in (0, 1)$ of ψ_n for every $n \in \mathbb{N}$. Then for every $n \in \mathbb{N}$ we set

$$E_n := \{x \in \Omega : \psi_n(x) > t_n\}.$$

Let Ω_n consist of the connected components of E_n containing U_n. With this choice $U_n \subset\subset \Omega_n \subset\subset U_{n+1}$, and since t_n is a regular value of ψ_n, by the implicit function theorem, $\partial\Omega_n$ is of class C^∞. By the properties of U_n also $\Omega_n \subset\subset \Omega_{n+1} \subset\subset \Omega$ for all $n \in \mathbb{N}$ and $\Omega = \bigcup_{n\in\mathbb{N}} \Omega_n$.

If Ω is connected we can choose a sequence U_n of connected subsets of V_n with $U_n \subset\subset U_{n+1} \subset\subset \Omega$ for all $n \in \mathbb{N}$. Then the Ω_n constructed above are connected as required. We finish this proof by showing that we can indeed choose U_n connected if Ω is connected. There exists $n_0 \in \mathbb{N}$ such that $V_n \neq \emptyset$ for all $n \ge n_0$. Fix $x_0 \in V_{n_0}$ and denote by U_{n_0} the connected component of V_{n_0} containing x_0. For $n > n_0$ we inductively define U_n to be the connected component of V_n containing U_{n-1}. Then $U := \bigcup_{n\in\mathbb{N}} U_n$ is a nonempty open set. If we show that $\Omega \setminus U$ is open, then the connectedness of Ω implies that $U = \Omega$. Let $x \in \Omega \setminus U$. Since Ω is open there exists $m \in \mathbb{N}$ such that $B(x, 2/m) \subset \Omega$. Hence $B(x, 1/m) \subset V_m$ and so $B(x, 1/m) \cap U_n = \emptyset$ for all $n \in \mathbb{N}$, since otherwise $B(x, 1/m) \subset U_n$ for some n. Therefore $B(x, 1/m) \subset \Omega \setminus U$ and thus $\Omega \setminus U$ is open. □

8.3. *Approximation from the exterior for Lipschitz domains*

Approximation from the inside by smooth domains is a useful technique for the Dirichlet problem. The situation is more difficult for Robin problems, where we saw in Section 6 that the limit problem very much depends on the boundary of the domains Ω_n. We want to state an existence theorem on a sequence of smooth domains where the boundary measure converges to the correct measure of the limit domain. The result originally goes back to Nečas [103]. We state it as proved in [66, Theorem 5.1].

THEOREM 8.3.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$. Then there exists a sequence of domains Ω_n of class C^∞ satisfying Assumption 6.4.1. Moreover, the functions $\varphi_n\colon B \to \mathbb{R}$ from Assumption 6.4.1 can be chosen such that*

(1) $\varphi_n \in C^\infty(B)$ *for* $n \geq 1$;

(2) $\varphi_n \to \varphi_0$ *uniformly on* B;

(3) $\|\varphi_n\|_\infty$ *is uniformly bounded with respect to* $n \in \mathbb{N}$;

(4) $\nabla\varphi_n \to \nabla\varphi_0$ *in* $L_p(B)$ *for all* $p \in [1, \infty)$.

The situation is shown in Figure 6.5. Note that Condition (4) is the most difficult to achieve.

REMARK 8.3.2. We can apply the above theorem to get a sequence of smooth domains approaching a given Lipschitz domain by smooth domains in such a way that Theorem 6.5.1 applies with $g = 1$, that is, the boundary conditions on the limit domain are the same as on the domains Ω_n.

Let φ_n be as in the above theorem and set

$$g_n := \sqrt{1 + |\nabla\varphi_n|^2}.$$

Since for $0 \leq b < a$ the function $t \mapsto \sqrt{t + a^2} - \sqrt{t + b^2}$ is decreasing as a function of $t \geq 0$ we get

$$\sqrt{1 + a^2} - \sqrt{1 + b^2} \leq a - b.$$

Therefore

$$\left|\sqrt{1 + |\nabla\varphi_n|^2} - \sqrt{1 + |\nabla\varphi_0|^2}\right| \leq \big||\nabla\varphi_n| - |\nabla\varphi_0|\big| \leq |\nabla\varphi_n - \nabla\varphi_0|,$$

and so from (4) in the above theorem

$$\left\|\frac{g_n}{g_0} - 1\right\|_p \leq \|g_n - g_0\|_p \leq \|\nabla\varphi_n - \nabla\varphi_0\|_p \to 0$$

as $n \to \infty$ for all $p \in [1, \infty)$. Hence

$$\lim_{n\to\infty} \frac{g_n}{g_0} = 1$$

in $L_p(\Omega)$ for all $p \in [1, \infty)$ as required in Theorem 6.5.1.

9. Perturbation of semi-linear problems

9.1. *Basic convergence results for semi-linear problems*

Consider the boundary value problem

$$\begin{aligned} \mathcal{A}u &= f(x, u(x)) && \text{in } \Omega, \\ \mathcal{B}u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{9.1.1}$$

and perturbations

$$\begin{aligned} \mathcal{A}_n u &= f(x, u(x)) && \text{in } \Omega_n, \\ \mathcal{B}_n u &= 0 && \text{on } \partial\Omega_n. \end{aligned} \tag{9.1.2}$$

We assume throughout that $\mathcal{A}_n, \mathcal{A}$ satisfy Assumption 5.1.1. We considered linear problems with various boundary operators $\mathcal{B}_n$. Let $R_n(\lambda)$ and $R(\lambda)$ be the resolvent operators corresponding to the boundary value problems $(\mathcal{A}_n, \mathcal{B}_n)$ and the relevant limit problem $(\mathcal{A}, \mathcal{B})$ as given in Definition 2.5.2. Finally, let A_n, A be the operators induced by $(\mathcal{A}_n, \mathcal{B}_n)$ and $(\mathcal{A}, \mathcal{B})$ as defined in Section 2.2. In all cases we proved that the following assumptions hold.

ASSUMPTION 9.1.1. Let Ω be bounded. For every $\lambda \in \varrho(-A)$ there exists $n_0 \in \mathbb{N}$ such that $\lambda \in \varrho(-A_n)$ for all $n > n_0$. Moreover, for every $p \in (1, \infty)$

$$\lim_{n\to\infty} R_n(\lambda) = R(\lambda)$$

in $\mathcal{L}(L_p(\mathbb{R}^N), L_q(\mathbb{R}^N)) \cap \mathcal{L}(L_p(\mathbb{R}^N))$ for all $q \in [p, m(p))$. Here $m(p)$ is given by (2.4.5) with d depending on to the type of problem under consideration (see Tables 2.1–2.3).

We summarise the various cases below.

1. (1) Dirichlet boundary conditions on Ω_n and the limit domain Ω (see Section 5.2, in particular Theorem 5.2.6).
2. (2) Robin boundary conditions on Ω_n and Ω, perturbing the original boundary $\partial\Omega$ only very slightly (see Section 6.3). The dumbbell problem shown in Figure 6.1 is an example.
3. (3) Robin boundary conditions on Ω_n with very fast oscillating boundary. On the limit domain Ω we have Dirichlet boundary conditions (see Section 6.4).
4. (4) Robin boundary conditions on Ω_n with moderately fast oscillating boundary. On the limit domain Ω we have Robin boundary conditions with a different boundary coefficient (see Section 6.5).
5. (5) Neumann boundary conditions and an additional assumption on the domains Ω_n. In the limit we also have Neumann boundary conditions (see Section 7).

As seen in Section 3.1 we can rewrite (9.1.1) and (9.1.2) as fixed point equations in $L_p(\Omega)$ and $L_p(\Omega_n)$, respectively. We want write the problems as equations on $L_p(\mathbb{R}^N)$.

Suppose that $f\colon \mathbb{R}^N \times \mathbb{R}$ satisfies Assumption 3.1.1 and let F be the corresponding superposition operator. In the spirit of Proposition 3.1.3 we let

$$G(u) := R(\lambda)(F(u) + \lambda u) \quad \text{and} \quad G_n(u) := R_n(\lambda)(F(u) + \lambda u)$$

for $\lambda \geq 0$ big enough so that $R_n(\lambda)$, $R(\lambda)$ are well defined. The only difference to the definition given in Proposition 3.1.3 is, that G acts on $L_p(\mathbb{R}^N)$ rather than $L_p(\Omega)$, and similarly G_n. This allows us to work in a big space independent of Ω. By Proposition 3.1.3, finding a solution of (9.1.1) in $L_p(\Omega)$ is equivalent to finding a solution to the fixed point equation

$$u = G(u) \tag{9.1.3}$$

in $L_p(\mathbb{R}^N)$ for $p \geq 2d/(d-2)$. Similarly, (9.1.2) is equivalent to

$$u = G_n(u) \tag{9.1.4}$$

in $L_p(\mathbb{R}^N)$. We now prove some basic properties of G_n, G.

PROPOSITION 9.1.2. *Suppose that $f\colon \mathbb{R}^N \times \mathbb{R}$ satisfies Assumption 3.1.1, and that γ, p are such that (3.1.5) holds. Let Ω be bounded and suppose that Assumption 9.1.1 is satisfied. Then*

(1) *$G, G_n \in C(L_p(\mathbb{R}^N, L_p(\mathbb{R}^N))$ for all $n \in \mathbb{N}$;*
(2) *If $u_n \to u$ in $L_p(\mathbb{R}^N)$, then $G_n(u_n) \to G(u)$ in $L_p(\mathbb{R}^N)$;*
(3) *For every bounded sequence (u_n) in $L_p(\mathbb{R}^N)$, the sequence $(G_n(u_n))_{n\in\mathbb{N}}$ has a convergent subsequence in $L_p(\mathbb{R}^N)$.*

Moreover, the map G_n is compact if Ω_n is bounded.

PROOF. By the growth condition (3.1.5) and Assumption 9.1.1 it follows that $R_n(\lambda) \to R(\lambda)$ in $\mathcal{L}(L_p(\mathbb{R}^N), L_{p/\gamma}(\mathbb{R}^N)) \cap \mathcal{L}(L_p(\mathbb{R}^N))$. The first assertion (1) then directly follows from Proposition 3.1.3. Also the compactness of G and G_n if Ω and Ω_n are bounded follows from Proposition 3.1.3. Statement (2) follows from the continuity of F proved in Lemma 3.1.2 and the assumptions on $R_n(\lambda)$. To prove (3) note that by Proposition 3.1.3 the sequence $(F(u_n))_{n\in\mathbb{N}}$ is bounded for every bounded sequence (u_n) in $L_p(\Omega)$. Hence there exists a subsequence such that $F(u_{n_k}) + \lambda u_{n_k} \rightharpoonup g$ weakly in $L_{p/\gamma}(\mathbb{R}^N) + L_p(\mathbb{R}^N)$. By Theorem 4.3.4

$$G_{n_k}(u_{n_k}) = R_{n_k}(\lambda)(F(u_{n_k}) + \lambda u_{n_k}) \to R(\lambda)g$$

in $L_p(\mathbb{R}^N)$, completing the proof of the proposition. □

REMARK 9.1.3. Sometimes, when working with positive solutions and comparison principles it is useful to make sure $f(x, \xi) + \lambda\xi$ is increasing as a function of ξ, and that $R(\lambda)$ is a positive operator. Both can be achieved by choosing $\lambda > 0$ large enough if $f \in C^1(\mathbb{R}^N \times \mathbb{R})$.

REMARK 9.1.4. For the assertions of the above propositions to be true Assumption 3.1.1 is not necessary. We discuss some examples. For instance consider a linear f of the form

$$f(x, u) = w_n u$$

with $w_n \rightharpoonup w$ weakly in $L_q(\mathbb{R}^N)$ for some $q > d/2$. Such cases arise in dealing with semi-linear problems (see for instance [50, Lemma 2]). More generally, suppose that γ, p satisfy (3.1.5) and that $q > p/(\gamma - \delta)$. If

$$f(x, u) = w_n(x)|u|^{\delta-1}u$$

with $w_n \rightharpoonup w$ weakly in $L_q(\mathbb{R}^N)$, then the assertions of the above propositions hold in $L_p(\mathbb{R}^N)$.

THEOREM 9.1.5. *Suppose that $f\colon \mathbb{R}^N \times \mathbb{R}$ satisfies Assumption 3.1.1, and that γ, p are such that (3.1.5) holds. Further suppose that u_n are solutions of (9.1.2) such that the sequence (u_n) is bounded in $L_p(\mathbb{R}^N)$. Finally suppose that Assumption 9.1.1 is satisfied. Then there exists a subsequence (u_{n_k}) converging to a solution u of (9.1.1) in $L_q(\mathbb{R}^N)$ for all $q \in [p, \infty)$.*

PROOF. We know from the above discussion that $u_n = G_n(u_n)$ for all $n \in \mathbb{R}^N$. Since (u_n) is bounded, Proposition 9.1.2 property (3) implies that there exists a subsequence (u_{n_k}) such that

$$u_{n_k} = G_{n_k}(u_{n_k}) \to u$$

in $L_p(\mathbb{R}^N)$. Now by Proposition 9.1.2 property (2) we get that $G_{n_k}(u_{n_k}) \to G(u)$ in $L_p(\mathbb{R}^N)$. Theorem 3.2.1 implies that (u_n) is bounded in $L_\infty(\mathbb{R}^N)$. Hence by interpolation, convergence is in $L_q(\mathbb{R}^N)$ for all $q \in [p, \infty)$. □

REMARK 9.1.6. (a) If we assume that $1 \leq \gamma < (d+2)/(d-2)$, then a bound in $L_2(\mathbb{R}^N)$ is sufficient in the above theorem.

(b) Solutions to semi-linear problems are in general not unique, so we do not expect the whole sequence to converge. If we happen to know by some means that (u_n) has at most one limit, then the whole sequence converges.

We next look at problems *without growth conditions* on the nonlinearity, but instead assume that the solutions on Ω_n are bounded uniformly with respect to $n \in \mathbb{N}$. We still assume that $f\colon \mathbb{R}^N \times \mathbb{R}$ is Carathéodory (see Assumption 3.1.1 for what this means).

THEOREM 9.1.7. *Suppose that $f\colon \mathbb{R}^N \times \mathbb{R}$ is a Carathéodory function, and that* $\sup_{n\in\mathbb{N}} |\Omega_n| < \infty$. *Further suppose that u_n are solutions of (9.1.2) such that the sequence (u_n) is bounded in $L_\infty(\mathbb{R}^N)$. Finally suppose that Assumption 9.1.1 is satisfied. Then there exists a subsequence (u_{n_k}) converging to a solution u of (9.1.1) in $L_p(\mathbb{R}^N)$ for all $p \in [1, \infty)$.*

PROOF. Let $M := \sup_{n\in\mathbb{N}} \|u_n\|_\infty$. Let $\psi \in C^\infty(\mathbb{R})$ be a monotone function with $\psi(\xi) = \psi$ if $\xi \leq M$ and $\psi(\xi) = M + 1$ if $|\xi| \geq M + 1$. Then define $\tilde{f}(x, \xi) := f(x, \psi(\xi))$ and by $\tilde{F}$ the corresponding superposition operator. Then $\tilde{f}$ is bounded and therefore satisfies

Assumption 3.1.1 with $\gamma = 1$. Since $\|u_n\|_\infty \leq M$ we clearly have $\tilde{F}(u_n) = F(u_n)$ and so we can replace F by $\tilde{F}$ in the definition of G_n, G without changing the solutions. Hence we can apply Theorem 9.1.5 to conclude that $u_n \to u$ in $L_2(\mathbb{R}^N)$. By the uniform L_∞-bound convergence is in $L_p(\mathbb{R}^N)$ for all $p \in [2, \infty)$, and by the uniform boundedness of $|\Omega_n|$ also for $p \in [1, 2)$. □

REMARK 9.1.8. An L_∞-bound as required above follows from an L_p-bound under suitable growth conditions on the nonlinearity f as shown in Section 3.2.

9.2. *Existence of nearby solutions for semi-linear problems*

Suppose that the limit problem (9.1.1) has a solution. In this section we want to prove that under certain circumstances the perturbed problem (9.1.2) has a solution nearby. In the abstract framework this translates into the question whether the fixed point equation (9.1.4) has, at least for n large enough, a solution near a given solution of (9.1.3). Of course, we do not expect this for arbitrary solutions. Note that the results here do not just apply to domain perturbation problems, but to other types of perturbations having similar properties as well.

One common technique to prove existence of such solutions is by means of the *Leray–Schauder degree* (see [65, Chapter 2.8] or [99, Chapter 4]). We assume that G is a compact operator, that is, if it maps bounded sets onto relatively compact sets. Then, if $U \subset E$ is an open bounded set such that $u \neq G_\Omega(u)$ for all $u \in \partial U$, then the Leray–Schauder degree, $\deg(I - G_\Omega, U, 0) \in \mathbb{Z}$, is well defined. If we deal with positive solutions we can use the degree in cones as in [3,44]. In order to do that we need some more assumptions. These are satisfied for the concrete case of semi-linear boundary value problems as shown in Proposition 9.1.2 with $E = L_p(\mathbb{R}^N)$ for some $p \in (1, \infty)$.

ASSUMPTION 9.2.1. Suppose E is a Banach space and suppose
(1) $G, G_n \in C(E, E)$ are compact for all $n \in \mathbb{N}$;
(2) If $u_n \to u$ in E, then $G_n(u_n) \to G(u)$ in E.
(3) For every bounded sequence (u_n) in E the sequence $(G_n(u_n))_{n\in\mathbb{N}}$ has a convergent subsequence in E.

The following is the main result of this section. The basic idea of the proof for specific domain perturbation problems goes back to [45]. The proof given here is a more abstract version of the ones found in [52,8] for some specific domain perturbation problems.

THEOREM 9.2.2. *Suppose that G_n, G satisfy Assumption 9.2.1. Moreover, let $U \subset E$ be an open bounded set such that $G(u) \neq u$ for all $u \in \partial U$. Then there exists $n_0 \in \mathbb{N}$ such that $G_n(u) \neq u$ for all $u \in \partial U$ and*

$$\deg(I - G, U, 0) = \deg(I - G_n, U, 0) \tag{9.2.1}$$

for all $n \geq n_0$.

PROOF. We use the homotopy invariance of the degree (see [65, Section 2.8.3] or [99, Theorem 4.3.4]) to prove (9.2.1). We define the homotopies $H_n(t, u) := tG_n(u) + (1-t)G(u)$

for $t \in [0, 1]$, $u \in E$ and $n \in \mathbb{N}$. To prove (9.2.1) it is sufficient to show that there exists $n_0 \in \mathbb{N}$ such that

$$u \neq H_n(t, u) \tag{9.2.2}$$

for all $n \geq n_0$, $t \in [0, 1]$ and $u \in \partial U$. Assume to the contrary that there exist $n_k \to \infty$, $t_{n_k} \in [0, 1]$ and $u_{n_k} \in \partial U$ such that $u_{n_k} = H_{n_k}(t_{n_k}, u_{n_k})$ for all $k \in \mathbb{N}$. As U is bounded in E, Assumption 9.2.1 (1) and (3) guarantee that $t_{n_k} \to t_0$ in $[0, 1]$, $G_{n_k}(u_{n_k}) \to v$ and $G(u_{n_k}) \to w$ in E for some $v, w \in E$ if we pass to a further subsequence. Hence

$$\begin{aligned} u_{n_k} &= H_{n_k}(t_{n_k}, u_{n_k}) = t_{n_k} G_{n_k}(u_{n_k}) + (1 - t_{n_k}) G(u_{n_k}) \\ &\xrightarrow{k\to\infty} u := t_0 v + (1 - t_0) w \end{aligned}$$

in E and so $u_{n_k} \to u$ in E and $u \in \partial U$ since ∂U is closed. Now the continuity of G and Assumption 9.2.1 part (2) imply that $v = w = G(u)$, so that $u_{n_k} \to u = t_0 G(u) + (1 - t_0) G(u) = G(u)$. Hence $u = G(u)$ for some $u \in \partial U$, contradicting our assumptions. Thus there exists $n_0 \in \mathbb{N}$ such that (9.2.2) is true for all $n \geq n_0$, completing the proof of the theorem. □

Of course, we are most interested in the case $\deg(I - G_\Omega, U, 0) \neq 0$. Then, by the solution property of the degree (see [99, Theorem 4.3.2]), (9.1.3) has a solution in U. As a corollary to Theorem 9.2.2 we get the existence of a solution of (9.1.4) in U.

COROLLARY 9.2.3. *Suppose that G_n, G satisfy Assumption 9.2.1 and that $U \subset E$ is open and bounded with $u \neq G(u)$ for all $u \in \partial U$. If $\deg(I - G, U, 0) \neq 0$, then there exists $n_0 \in \mathbb{N}$ such that (9.1.4) has a solution in U for all $n \geq n_0$.*

Now we consider an isolated solution u_0 of (9.1.3) and recall the definition of its *index*. Denote by $B_\varepsilon(u_0)$ the open ball of radius $\varepsilon > 0$ and centre u_0 in E. Then $\deg(I - G, B_\varepsilon(u_0), 0)$ is defined for small enough $\varepsilon > 0$. Moreover, by the excision property of the degree $\deg(I - G, B_\varepsilon(u_0), 0)$ stays constant for small enough $\varepsilon > 0$. Hence the fixed point index of u_0,

$$\mathrm{i}(G, u_0) := \lim_{\varepsilon\to 0} \deg(I - G, B_\varepsilon(u_0), 0)$$

is well defined.

THEOREM 9.2.4. *Suppose that G_n, G satisfy Assumption 9.2.1. If u_0 is an isolated solution of (9.1.3) with $\mathrm{i}(G, u_0) \neq 0$, then for n large enough there exist solutions u_n of (9.1.4) such that $u_n \to u_0$ in E as $n \to \infty$.*

PROOF. By assumption there exists $\varepsilon_0 > 0$ such that

$$\mathrm{i}(G, u_0) = \deg(I - G, B_\varepsilon(u_0), 0) \neq 0$$

for all $\varepsilon \in (0, \varepsilon_0)$. Hence by Corollary 9.2.3 problem (9.1.4) has a solution in $B_\varepsilon(u_0)$ for all $\varepsilon \in (0, \varepsilon_0)$ if only n large enough. Hence a sequence as required exists. □

Without additional assumptions it is possible that there are several different sequences of solutions of (9.1.4) converging to u_0. However, if $G \in C^1(E, E)$ and u_0 is *nondegenerate*, that is, the linearised problem

$$v = DG(u_0)v \tag{9.2.3}$$

has only the trivial solution, then u_n is unique for large $n \in \mathbb{N}$.

THEOREM 9.2.5. *Suppose that $G_n, G \in C^1(E, E)$ satisfy Assumption 9.2.1. Further assume that $DG_n(u_n) \to DG(u)$ in $\mathcal{L}(E)$. If u_0 is a nondegenerate solution of (9.1.3), then there exists $\varepsilon > 0$ such that (9.1.4) has a unique solution in $B_\varepsilon(u_0)$ for all n large enough and this solution is nondegenerate.*

PROOF. As u_0 is nondegenerate $I - DG(u_0)$ is invertible with bounded inverse. Moreover, since G is compact [65, Proposition 8.2] implies that $DG(u_0)$ is compact as well. By [99, Theorem 5.2.3 and Theorem 4.3.14] $\mathrm{i}(G, u_0) = \pm 1$, so by Theorem 9.2.4 there exists a sequence of solutions u_n of (9.1.3) with $u_n \to u$ as $n \to \infty$. As the set of invertible linear operators is open in $\mathcal{L}(E)$ we conclude that $I - DG_n(u_n)$ is invertible for n sufficiently large. Hence u_n is nondegenerate for n sufficiently large.

We now show uniqueness. Suppose to the contrary that there exist solutions u_n and v_n of (9.1.4) converging to u_0 with $u_n \neq v_n$ for all $n \in \mathbb{N}$ large enough. As $G_n \in C^1(E, E)$ we get

$$G(u_n) - G(v_n) = \int_0^1 DG_n(tu_n + (1-t)v_n)\,dt\,(u_n - v_n).$$

Hence if we set $w_n := \frac{u_n - v_n}{\|u_n - v_n\|}$, then $\|w_n\| = 1$ and

$$w_n = \frac{G_n(u_n) - G_n(v_n)}{\|u_n - v_n\|} = \int_0^1 DG_n(tu_n + (1-t)v_n)\,dt\,w_n \tag{9.2.4}$$

for all $n \in \mathbb{N}$. As $u_n, v_n \to u_0$ we get $tu_n + (1-t)v_n \to u_0$ in E for all $t \in [0, 1]$ and hence by assumption $DG_n(tu_n + (1-t)v_n) \to DG(u_0)$ in $\mathcal{L}(E)$ for all $t \in [0, 1]$. By the continuity of DG_n and the compactness of $[0, 1]$ there exists $t_n \in [0, 1]$ such that

$$\sup_{t \in [0,t]} \left\| DG_n(tu_n + (1-t)v_n) \right\| = \left\| DG_n(t_n u_n + (1-t_n)v_n) \right\|.$$

Since $t_n u_n + (1-t_n)v_n \to u_0$ by assumption $DG_n(tu_n + (1-t)v_n) \to DG(u_0)$ in $\mathcal{L}(E)$ and hence

$$\sup_{\substack{t \in [0,t] \\ n \in \mathbb{N}}} \left\| DG_n(tu_n + (1-t)v_n) \right\| < \infty.$$

By the dominated convergence theorem

$$\int_0^1 DG_n(tu_n + (1-t)v_n)\,dt \to DG(u_0)$$

in $\mathcal{L}(E)$. As the set of invertible operators is open we see that

$$I - \int_0^1 DG_n(tu_n + (1-t)v_n)\,dt$$

has a bounded inverse for n sufficiently large. But this contradicts (9.2.4) and therefore uniqueness follows. □

9.3. *Applications to boundary value problems*

We now discuss how to apply the abstract results in the previous section to boundary value problems. In addition to the assumptions made on f in Section 9.1 we assume that the corresponding superposition operator is differentiable. Conditions for that can be found in [6, Theorem 3.12]. We also assume that there exists a bounded set B such that

$$\Omega_n, \Omega \subset B$$

for all $n \in \mathbb{N}$.

THEOREM 9.3.1. *Suppose that $f \in C(\mathbb{R}^N \times \mathbb{R})$ satisfies Assumption 3.1.1, and that γ, p are such that (3.1.5) holds. Further suppose that Assumption 9.1.1 is satisfied, and that $u \in L_p(\mathbb{R}^N)$ is a nondegenerate isolated solution of (9.1.1). Then, there exists $\varepsilon > 0$ such that, for n large enough, equation (9.1.2) has a unique solution $u_n \in L_p(\Omega)$ with $\|u_n - u\|_p < \varepsilon$. Moreover, $u_n \to u \in L_q(\mathbb{R}^N)$ for all $q \in (1, \infty)$.*

PROOF. By Proposition 9.1.2 all assumptions of Theorem 9.2.5 are satisfied if we choose $E = L_p(\mathbb{R}^N)$. Hence there exist $\varepsilon > 0$ and $u_n \in L_p(\mathbb{R}^N)$ as claimed. By Theorem 3.2.1 the sequence (u_n) is bounded in $L_\infty(\mathbb{R}^N)$ and therefore the convergence is in $L_q(\mathbb{R}^N)$ for all $q \in [p, \infty)$. Since the measure of Ω_n is uniformly bounded, convergence takes place in $L_q(\mathbb{R}^N)$ for $q \in [1, p)$ as well. □

REMARK 9.3.2. (a) We could for instance choose $f(u) := |u|^{\gamma-1}u$, or a nonlinearity with that growth behaviour. Then we can choose p big enough such that (3.1.5) is satisfied. If $\gamma < (d+2)/(d-2)$, then the theorem applies to all nondegenerate weak solutions.

(b) The above theorem does not necessarily imply that (9.1.2) has only one solution near the solution u of (9.1.1) in $L_2(\mathbb{R}^N)$, because a solution in $L_2(\mathbb{R}^N)$ does not need to be close to u in $L_p(\mathbb{R}^N)$. However, if $\gamma < (d+2)/(d-2)$, then u_n is the unique weak solution of (9.1.2) in the ε-neighbourhood of u in $L_2(\mathbb{R}^N)$.

We next want to look at a problem without any growth conditions on f. In such a case we have to deal with solutions in $L_\infty(\mathbb{R}^N)$ only. The idea is to truncate the nonlinearity and apply Theorem 9.3.1.

THEOREM 9.3.3. *Suppose that $f \in C(\mathbb{R}^N \times \mathbb{R})$ satisfies Assumption 3.1.1, and that γ, p are such that (3.1.5) holds. Further suppose that Assumption 9.1.1 is satisfied, and that $u \in L_\infty(\mathbb{R}^N)$ is a nondegenerate isolated solution of (9.1.1). Then, there exists $\varepsilon > 0$ such that, for n large enough, equation (9.1.2) has a unique solution $u_n \in L_\infty(\Omega)$ with $\sup_{n\in\mathbb{N}} \|u_n\|_\infty < \infty$ and $u_n \to u \in L_q(\mathbb{R}^N)$ for all $q \in [1, \infty)$.*

PROOF. Fix $p > d/2$ with d from Assumption 9.1.1. Because $u \in L_\infty(\mathbb{R}^N)$ and $f \in C(\bar{B} \times \mathbb{R})$ we have $F(u) \in L_p(\mathbb{R}^N)$, where F is the superposition operator induced by f. By Theorem 2.4.1 and the assumptions on $(\mathcal{A}_n, \mathcal{B}_n)$ there exists a constant C independent of $n \in \mathbb{N}$ such that

$$\|u\|_\infty \le M := C(\|F(u)\|_p + \lambda_0\|u\|_p) + \|u\|_p.$$

Let $\psi \in C^\infty(\mathbb{R})$ be a monotone function with

$$\psi(\xi) = \begin{cases} \xi & \text{if } \xi \le M+1, \\ \operatorname{sgn}\xi(M+2) & \text{if } |\xi| \ge M+2. \end{cases}$$

Then define $\tilde{f}(x,\xi) := f(x,\psi(\xi))$. Then $\tilde{f}$ is a bounded function on $B \times \mathbb{R}$ and $\tilde{f}(x,\xi) = f(x,\xi)$ whenever $|\xi| \le M+1$. As $\tilde{f}$ is bounded we can apply Theorem 9.3.1. Hence there exists $\varepsilon > 0$ such that, for n large enough, equation (9.1.2) has a unique solution $u_n \in L_p(\Omega)$ with $\|u_n - u\|_p < \varepsilon$. Moreover, $u_n \to u \in L_p(\mathbb{R}^N)$. By what we have seen above

$$\|u_n\|_\infty \le C(\|\tilde{F}(u_n)\|_p + \lambda_0\|u_n\|_p) + \|u_n\|_p$$

for all $n \in \mathbb{N}$. Because $\tilde{f}$ is bounded, Lemma 3.1.2 shows that $\tilde{F}(u_n) \to \tilde{F}(u)$ in $L_p(\mathbb{R}^N)$. Hence, for n large enough

$$\|u_n\|_\infty < M+1.$$

By definition of $\tilde{F}$ it follows that $\tilde{F}(u_n) = F(u_n)$ for n large enough, so u_n is a solution of (9.1.2). Convergence in $L_q(\mathbb{R}^N)$ for $q \in [p,\infty)$ follows by interpolation and for $q \in [1,p)$ since B has bounded measure. □

The main difficulty in applying the above results is to show that the solution of (9.1.1) is nondegenerate. A typical example is to look at a problem on two or more disjoint balls and get multiple solutions by taking different combinations of solutions. For instance if we have a trivial and a nontrivial solution, then on two disjoint balls we get four solutions: the trivial solution on both, the trivial on one and the nontrivial on the other and vice versa, and the nontrivial solution on both. We can then connect the domains either as a dumbbell domain with a small strip (see Figure 6.1) or touching balls (see Figure 7.1). The dumbbell example works for Dirichlet and Robin boundary conditions, but not for Neumann boundary conditions. For Neumann boundary conditions, the example of the touching spheres applies.

To illustrate the above we give an example to the Gelfand equation from combustion theory (see [72, §15]) due to [45]. The example shows that a simple equation can have multiple solutions on simply connected domains.

EXAMPLE 9.3.4. Consider the Gelfand equation

$$\begin{aligned} -\Delta u &= \mu e^u && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega \end{aligned} \tag{9.3.1}$$

on a bounded domain of class C^2. If $\mu > 0$, then $\mu e^u > 0$ for all u and thus by the maximum principle every solution of (9.3.1) is positive. By [73, Theorem 1] positivity implies that all solutions are radially symmetric if Ω is a ball.

It is well known that there exists $\mu_0 > 0$ such that (9.3.1) has a minimal positive solution for $\mu \in [0, \mu_0]$ and no solution for $\mu > \mu_0$ (see [4,41]). Moreover, for $\mu \in (0, \mu_0)$ this minimal solution is nondegenerate (see [41, Lemma 3]). Let now $\Omega = B_0 \cup B_1$ be the union of two balls B_0 and B_1 of the same radius and Ω_n the dumbbell-like domains as shown in Figure 6.1. If $N = 2$ and $\mu \in (0, \mu_0)$, then there exists a second solution for the problems on B_0 and B_1. In fact, the two solutions are the only solutions on a ball if $N = 2$ (see [91, p. 242] or [72, §15, p. 359]). Note that this is not true for $N \geq 3$ as shown in [91]. Equation (8) on page 415 together with the results in Section 2 in [43] imply that there is bifurcation from every degenerate solution. Since we know that there is no bifurcation in the interval $(0, \mu_0)$, it follows that the second solution is also nondegenerate.

We now show that there are possibly more than two solutions on (simply connected) domains other than balls. Looking at $\Omega = B_0 \cup B_1$ we have four nontrivial nondegenerate solutions of (9.3.1). Hence by Theorem 9.3.3 there exist at least four nondegenerate solutions of (9.3.1) on Ω_n for n large. Note that similar arguments apply to the nonlinearity $|u|^{p-1}u$ for p subcritical as discussed in [45].

9.4. *Remarks on large solutions*

If all solutions of (9.1.1) are nondegenerate it is tempting to believe that the number of solutions of (9.1.2) is the same for n sufficiently large. However, this is not always true, and Theorem 9.3.1 only gives a lower bound for the number of solutions of the perturbed problem. If there are more solutions on Ω_n, then Theorem 9.1.5 implies that their L_∞-norm goes to infinity as otherwise they converge to one of the solutions on Ω and hence are unique. To prove precise multiplicity of solutions of the perturbed problem, the task is to find a universal bound on the L_∞ norm valid for all solutions to the nonlinear problem. This is quite different from the result in Theorem 3.2.1 which just shows that under suitable growth conditions on the nonlinearity, a bound in $L_p(\mathbb{R}^N)$ implies a bound in $L_\infty(\mathbb{R}^N)$. However, it is still unclear whether in general there is a uniform bound on the L_p-norm for some p. Such uniform bounds are very difficult to get in general. We prove their existence for solutions to

$$\begin{aligned} -\Delta u &= f(u(x)) && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{9.4.1}$$

provided f is sublinear in the sense that

$$\lim_{|\xi|\to\infty} \frac{|f(\xi)|}{|\xi|} = 0. \tag{9.4.2}$$

The above class clearly includes all bounded nonlinearities.

PROPOSITION 9.4.1. *Suppose that $f \in C(\mathbb{R})$ satisfies (9.4.2). Then there exists a constant M depending only on the function f and the diameter of Ω such that $\|u\|_\infty < M$ for every weak solution of (9.4.1).*

PROOF. By (9.4.2), for every $\varepsilon > 0$ there exists $\alpha > 0$ such that

$$\frac{|f(\xi)|}{|\xi|} < \varepsilon$$

whenever $|\xi| > \alpha$. Setting $m_\varepsilon := \sup_{|\xi| \le \alpha} |f(\xi)|$ we get

$$|f(\xi)| \le m_\varepsilon + \varepsilon|\xi|$$

for all $\xi \in \mathbb{R}$. In particular f satisfies (3.1.4) with $\gamma = 1$, and so by Theorem 3.2.1 it is sufficient to prove a uniform L_2-bound for all weak solutions of (9.4.1). If u is a weak solution of (9.4.1), then from the above

$$\begin{aligned}\|\nabla u\|_2^2 &= \int_\Omega f(u)u\,dx \le \int_\Omega |f(u)||u|\,dx \le \int_\Omega (m_\varepsilon + \varepsilon|u|)|u|\,dx \\ &\le \int_\Omega m_\varepsilon|u|\,dx + \varepsilon\|u\|_2^2 \le (m_\varepsilon|\Omega|)^{1/2}\|u\|_2 + \varepsilon\|u\|_2^2.\end{aligned}$$

Using an elementary inequality we conclude that

$$\|\nabla u\|_2^2 \le \varepsilon^{-1} m_\varepsilon|\Omega| + 2\varepsilon\|u\|_2^2.$$

If D denotes the diameter of Ω, then from Friedrich's inequality (2.1.7)

$$\|u\|_2^2 \le D^2\|\nabla u\|_2^2 \le \varepsilon^{-1}D^2 m_\varepsilon|\Omega| + 2D^2\varepsilon\|u\|_2^2.$$

We next choose $\varepsilon := 1/4D^2$ and so we get

$$\|u\|_2^2 \le 8D^4 m_\varepsilon|\Omega|$$

for every weak solution of (9.4.1). The right-hand side of the above inequality only depends on the quantities listed in the proposition, and hence the proof is complete. □

A priori estimates similar to the above can be obtained also for superlinear problems, but they involve the shape of the domain. An extensive discussion of the phenomena can be found in [45, Section 5] as well as [46,48,35]. The techniques to prove uniform a priori bounds are derived from Gidas and Spruck [75,74].

We complete this section by showing a simple example where large solutions occur. More sophisticated examples are in [46,48,35], in particular the example of the dumbbell as in Figure 6.1, where large solutions occur for the equation

$$\begin{aligned} -\Delta u &= |u|^{p-1}u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{9.4.3}$$

for $1 < p < (N+2)/(N-2)$. We give a simple example for the above equation, where large solutions occur, similar to examples given in [45, Section 5].

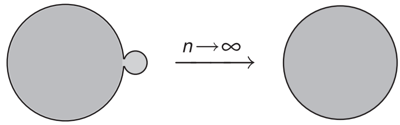

Fig. 9.1. Attaching a small shrinking ball.

EXAMPLE 9.4.2. By [114, Section I.2] the above equation has a positive solution u_r of $\Omega = B_r$ which is a ball of radius r. By a simple rescaling, it turns out that $\|u_r\|_\infty \to \infty$ as $r \to 0$. That solution is unique and nondegenerate by [45, Theorem 5], if $N = 2$. Then consider a domain Ω_n constructed from two touching balls $B_1 \cup B_{1/n}$ with a small connection as shown in Figure 9.1. When we let $n \to \infty$, then from Theorem 5.4.5 we get that $\Omega_n \to \Omega := B_1$ in the sense of Mosco. Since on every ball there are two solutions, the trivial solution and a nontrivial solution, there are four solutions on the union $B_1 \cup B_{1/n}$. If we make a small enough connection between the balls, then by Theorem 9.3.1 there are still at least four solutions. Hence we can construct a sequence of domains Ω_n with the required property. However, we know that there are precisely two solutions on Ω, the trivial and the nontrivial solution. The solutions on Ω_n not converging to one of the two solutions on Ω are such that $\|u_n\|_\infty \to \infty$.

9.5. *Solutions by domain approximation*

Consider now the nonlinear Dirichlet problem

$$\begin{aligned} \mathcal{A}u &= f(u) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{9.5.1}$$

on some domain Ω, bounded or unbounded. We assume that $f \in C^1(\mathbb{R})$. We then let Ω_n be open sets such that $\Omega_n \to \Omega$ and consider the sequence of problems

$$\begin{aligned} \mathcal{A}_n u &= f(u) \quad \text{in } \Omega_n, \\ u &= 0 \quad \text{on } \partial\Omega_n. \end{aligned} \tag{9.5.2}$$

The difference to the results in Section 9.3 is that we allow unbounded domains Ω. One possibility is to use the technique to prove results on nonsmooth or unbounded domains by approximation by smooth bounded domains from inside applying the results from Section 8.

THEOREM 9.5.1. *Suppose that $f \in C^1(\mathbb{R})$ and that $\Omega_n \to \Omega$ in the sense of Mosco. Suppose that u_n are weak solutions of (9.5.2) and that the sequence (u_n) is bounded in $H^1(\mathbb{R}^N) \cap L_\infty(\mathbb{R}^N)$. Then there exists a subsequence (u_{n_k}) converging to a solution u of (9.5.1) weakly in $H^1(\mathbb{R}^N)$ and strongly in $L_{p,\mathrm{loc}}(\mathbb{R}^N)$ for all $p \in [2, \infty)$.*

PROOF. By the boundedness of (u_n) in $H^1(\mathbb{R}^N)$ there exists a subsequence (u_{n_k}) converging to $u \in H^1(\mathbb{R}^N)$ weakly. Since $H^1(\mathbb{R}^N)$ is compactly embedded into

$L_{2,\mathrm{loc}}(\mathbb{R}^N)$ we have $u_{n_k} \to u$ in $L_{2,\mathrm{loc}}(\mathbb{R}^N)$ as $k \to \infty$. Because (u_n) is bounded in $L_\infty(\mathbb{R}^N)$, convergence takes place in $L_{p,\mathrm{loc}}(\mathbb{R}^N)$ for all $p \in [2, \infty)$. Since we have a uniform bound on (u_n) which is bounded in $L_\infty(\mathbb{R}^N)$ we can also truncate the nonlinearity f as in the proof of Theorem 9.3.3 and assume it is bounded. Then from Lemma 3.1.2 we get that $f(u_{n_k}) \to f(u)$ in $L_{2,\mathrm{loc}}(\mathbb{R}^N)$.

Now fix $\varphi \in C_c^\infty(\Omega)$. Then there exists a ball B with $\operatorname{supp}\varphi \subset B$. As $\Omega_n \to \Omega$, using Proposition 5.3.3, there exists $\varphi_n \in C_c^\infty(\Omega_n \cap B)$ such that $\varphi_n \to \varphi$ in $H^1(\mathbb{R}^N)$. Hence by using Proposition 5.1.2 and the fact that the support of φ_n is in B for all $n \in \mathbb{N}$ we get

$$\lim_{k\to\infty} \langle f(u_{n_k}), \varphi_n\rangle = \langle f(u), \varphi\rangle.$$

Since $u_{n_k} \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$ and $\varphi_n \to \varphi$ in $H^1(B)$ we have

$$\lim_{k\to\infty} a_{n_k}(u_{n_k}, \varphi_n) = a(u, \varphi).$$

Finally, by assumption $a(u_{n_k}, \varphi_{n_k}) = \langle f(u_{n_k}), \varphi_{n_k}\rangle$ for all $k \in \mathbb{N}$ we get $a(u, \varphi) = \langle f(u), \varphi\rangle$ for all $\varphi \in C_c^\infty(\Omega)$, that is, u is a weak solution of (9.5.1). □

Note that in general we cannot expect the solution u whose existence the above theorem proves to be nonzero if Ω_n or Ω are unbounded, or more precisely if the resolvents of the linear problems do not converge in the operator norm.

REMARK 9.5.2. Note that under suitable growth conditions, a uniform L_p-bound or even an L_2-bound on the solutions of (9.5.2) implies a uniform L_∞-bound by Theorem 3.2.1.

9.6. *Problems on unbounded domains*

Suppose that $\Omega \subset \mathbb{R}^N$ is an unbounded domain. By approximation by a sequence of bounded domains Ω_n we want to construct a nonnegative weak solution of

$$\begin{aligned} -\Delta u &= |u|^{p-2}u && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned} \tag{9.6.1}$$

with subcritical growth $p \in (2, 2N/(N-2))$. We also assume that the spectral bound of the Dirichlet Laplacian is positive. It is given as the infimum of the Rayleigh coefficient

$$s = \inf_{u \in H_0^1(\Omega)} \frac{\|\nabla u\|_2^2}{\|u\|_2^2} > 0. \tag{9.6.2}$$

Examples of such a domain are infinite strips of the form $\mathbb{R} \times U$ with U a domain in $\mathbb{R}^{N-1}$.

To construct a solution we let Ω_n be bounded open sets such that $\Omega_n \subset \Omega_{n+1} \subset \Omega$ for all $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} \Omega_n = \Omega$. We then show that a subsequence of the positive solutions of

$$\begin{aligned} -\Delta u &= |u|^{p-2}u \quad \text{in } \Omega_n \\ u &= 0 \quad \text{on } \partial\Omega_n \end{aligned} \tag{9.6.3}$$

converges to a solution of (9.6.1) as $n \to \infty$. The solution of (9.6.3) can be obtained by means of a constrained variational problem. We let

$$M_n := \{u \in H_0^1(\Omega_n) : u > 0 \text{ and } \|u\|_p^p = p\}$$

and the functional

$$J(u) := \frac{1}{2}\int_{\Omega_n} |\nabla u|^2 \, dx.$$

Then J has a minimiser $v_n \in M_n$ for every $n \in \mathbb{N}$, and that minimiser is a positive weak solution of

$$\begin{aligned} -\Delta v &= \mu_n |v|^{p-2}v \quad \text{in } \Omega_n \\ v &= 0 \quad \text{on } \partial\Omega_n \end{aligned}$$

with

$$\mu_n = 2J(v_n) = \|\nabla v_n\|_2^2.$$

The function

$$u_n := \mu_n^{1/(p-1)} v_n$$

then solves (9.6.3). For a proof of these facts we refer to [114, Section I.2]. We also get a bound on the solutions u_n, namely

$$\begin{aligned} \|\nabla u_n\|_2^2 = \mu_n^{2/(p-1)} \|\nabla v_n\|_2^2 &= \|\nabla v_n\|_2^{4/(p-1)} \|\nabla v_n\|_2^2 \\ &= \|\nabla v_n\|^{2(p+1)/(p-1)} = (2J(v_n))^{(p+1)/(p-1)} \end{aligned} \tag{9.6.4}$$

since $M_n \subset M_{n+1}$ for all $n \in \mathbb{N}$ we also have $J(v_{n+1}) \leq J(v_n)$. The sequences (∇v_n) and therefore (∇u_n) are bounded in $L_2(\Omega, \mathbb{R}^N)$. By (9.6.2) it follows that (u_n) and (v_n) are bounded sequences in $H_0^1(\Omega)$.

Note that the nonlinearity $|u|^{p-2}u$ satisfies Assumption 3.1.1 with $g = 0$ and $\gamma = p - 1 < (N+2)/(N-2)$. Since we can choose λ_0 (see Table 2.1), Proposition 3.2.1 shows that (u_n) is a bounded sequence in $L_\infty(\Omega)$. Hence there exists $S > 0$ such that $\|u\|_\infty^{p-2} \leq S$ for all $n \in \mathbb{N}$. By (9.6.2) and since u_n is a weak solution of (9.6.3)

$$s\|\nabla u_n\|_2^2 = s\|u_n\|_p^p \leq s\|u_n\|_\infty^{p-2}\|u_n\|_2^2 \leq \|u_n\|_\infty^{p-2}\|\nabla u_n\|_2^2.$$

Hence $0 < s \leq \|u_n\|_\infty^{p-2}$ for all $n \in \mathbb{N}$. We have thus proved the following proposition.

PROPOSITION 9.6.1. *Let s, S be as above. Then for every $n \in \mathbb{N}$ the problem* (9.6.3) *has a positive solution u_n such that (u_n) is bounded in $H_0^1(\Omega) \cap L_\infty(\Omega)$. Moreover, $\|\nabla u_n\|_2$ is decreasing and $0 < s \leq \|u_n\|_\infty^{p-2} \leq S$ for all $n \in \mathbb{N}$.*

The next task is to show that u_n converges to a solution of (9.6.1). We have seen that (u_n) is bounded in $H_0^1(\Omega) \cap L_\infty(\Omega)$. It therefore follows from Theorem 9.5.1 that a subsequence of (u_n) converges to a solution of (9.6.1).

PROPOSITION 9.6.2. *Let u_n be the solution of* (9.6.3) *as constructed above. Then there exists a subsequence of (u_{n_k}) converging weakly in $H^1(\Omega)$ to a solution u of* (9.6.1).

The main question is whether the solution u is nonzero. If the embedding $H_0^1(\Omega) \hookrightarrow L_2(\Omega)$ is compact this is easy to see. If the embedding $H_0^1(\Omega) \hookrightarrow L_2(\Omega)$ is not compact, then we cannot expect u to be nonzero in general. However, if we assume that Ω is symmetric like for instance an infinite strip with cross-section of finite measure, we will prove the existence of a nonzero solution of (9.6.1) by exploiting the symmetry to generate compactness of a minimising sequence. We look at domains of the form

$$\Omega = \mathbb{R} \times U,$$

where $U \subset \mathbb{R}^{N-1}$ is a bounded open set. If we let $\Omega_n = (-n, n) \times U$, then by a result of Gidas–Ni–Nirenberg (see [70, Theorem 3.3]), the solutions u_n of (9.6.3) is symmetric with respect to the plane $\{0\} \times \mathbb{R}^N$ and decreasing as $|x_1|$ increases. By [97, Théorème III.2] the sequence (u_n) is compact in $L_q(\Omega)$ for $2 < q < 2N/(N-2)$. In the case of $N = 2$ a similar result appears in [115, Section 4]. It therefore follows that there exists a subsequence (u_{n_k}) converging strongly in $L_p(\Omega)$. Using u_{n_k} as a test function we get

$$\lim_{k\to\infty} \|\nabla u_{n_k}\|_2^2 = \lim_{k\to\infty} \|u_{n_k}\|_p^p = \|u\|_p^p = \|\nabla u\|_2^2.$$

As $u_{n_k} \rightharpoonup u$ weakly in $H_0^1(\Omega)$ it follows that $u_{n_k} \to u$ strongly in $H_0^1(\Omega)$. By Proposition 9.6.1 the sequence (u_n) is bounded in $L_\infty(\Omega)$ and so (u_n) is bounded in $C^\mu(\Omega)$ (see [76, Theorem 8.22]) as well. Hence, by Arzela–Ascoli's theorem the subsequence also converges locally uniformly on Ω. The symmetry guarantees that the maximum of (u_n) is in $U \times \{0\}$ and therefore the lower bound on $\|u_n\|_\infty^{p-2}$ from Proposition 9.6.1 implies that $u \neq 0$.

However, without the symmetry, the solution may converge to zero. As an example look at the semi-strip $\Omega = (0, \infty) \times U$ which we exhaust by the domains $\Omega_n = (0, 2n) \times U$. Then the above proposition applies. The solutions on Ω_n are just translated solutions on $(-n, n) \times U$. However, the maximum of the function u_n is in $\{n\} \times U$, and moves to infinity as $n \to \infty$. Because the solutions decrease away from the maximum, they converge to zero in $L_{2,\mathrm{loc}}((0, \infty) \times U)$. This shows that the symmetry was essential for concluding that the limit solution is nonzero. The lower bound on $\|u_n\|_\infty^{p-2}$ from Proposition 9.6.1 and local uniform convergence do not help to get a nonzero solution.

Similarly we could look at (9.6.1) on the whole space $\Omega = \mathbb{R}^N$. We can then write Ω as a union of concentric balls Ω_n, and try to use the spherical symmetry to get some compactness from [97] similarly as above. However, since by [75] the equation (9.6.1) has no positive solution on $\mathbb{R}^N$, such an attempt must fail.

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CHAPTER 2

Singular Solutions of Semi-Linear Elliptic Problems

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Abstract

We are concerned in this survey with singular solutions to semi-linear elliptic problems. An example of the type of equations we are interested in is the Gelfand–Liouville problem $-\Delta u = \lambda e^u$ on a smooth bounded domain Ω of $\mathbb{R}^N$ with zero Dirichlet boundary condition. We explore up to what degree known results for this problem are valid in other situations with a similar structure, with emphasis on the extremal solution and its properties. Of interest is the question of identifying conditions such that the extremal solution is singular. We find that, in the problems studied, there is a strong link between these conditions and Hardy-type inequalities.

Keywords: Singular solution, Blow-up solution, Stability, Perturbation of singular solutions

Contents

HANDBOOK OF DIFFERENTIAL EQUATIONS
Stationary Partial Differential Equations, volume 6
Edited by M. Chipot

1. Introduction

In this survey we are interested in singular solutions to semi-linear partial differential equations of the form

$$\begin{cases} -\Delta u = \lambda g(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $\lambda > 0$ and $g : [0, \infty) \to \mathbb{R}$ satisfies

$$g \text{ is smooth increasing, convex, } g(0) > 0 \tag{1.2}$$

and superlinear at $+\infty$ in the sense

$$\lim_{u \to +\infty} \frac{g(u)}{u} = +\infty. \tag{1.3}$$

Some typical examples are $g(u) = e^u$ and $g(u) = (1 + u)^p$ with $p > 1$.

We are also interested in some variants of (1.1) such as

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = \lambda g(u) & \text{on } \Gamma_1 \\ u = 0 & \text{on } \Gamma_2, \end{cases} \tag{1.4}$$

where $\lambda > 0$ and $\Omega \subset \mathbb{R}^N$ is a smooth, bounded domain and Γ_1, Γ_2 is a partition of $\partial\Omega$ into surfaces separated by a smooth interface, and ν is the exterior unit normal vector.

We shall consider as well the fourth-order equation

$$\begin{cases} \Delta^2 u = \lambda g(u) & \text{in } B \\ u = 0 & \text{on } \partial B \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial B, \end{cases} \tag{1.5}$$

where B is the unit ball in $\mathbb{R}^N$.

Equations of the form (1.1) have been studied in various contexts and applications. Liouville [85] considered this equation with $g(u) = e^u$ in connection to surfaces with constant Gauss curvature. The exponential nonlinearity in dimension 3 appears in connection with the equilibrium of gas spheres and the structure of stars, see Emden [53], Fowler [60] and Chandrasekhar [29]. Later Frank-Kamenetskii [61] obtained a model like (1.1) with $g(u) = (1 - \varepsilon u)^m e^{u/(1+\varepsilon u)}$ in combustion theory. Also in connection with combustion theory, Barenblatt, in a volume edited by Gelfand [69], studied the case $g(u) = e^u$ in a ball in dimensions 2 and 3. Since then, this problem has attracted the attention of many researchers [10,19,20,24,34,35,62–64,76,79,83,93,94].

Boundary value problems of the form (1.4) with exponential nonlinearity arise in conformal geometry when prescribing Gaussian curvature of a 2-dimensional domain and curvature of the boundary, see for instance Li, Zhu [84] and the references therein. The study of conformal transformations in manifolds with boundary in higher dimensions also gives rise to nonlinear boundary conditions, see Cherrier [31] and Escobar [54–56]. A related motivation is the study of Sobolev spaces and inequalities, specially the Sobolev trace theorem, see Aubin [6] and the surveys of Rossi [105] and Druet, Hebey [52]. In connection with physical models (1.4), exponential nonlinearity appears in corrosion modelling where there is an exponential relationship between boundary voltages and boundary normal currents. See [21,78,92,107] and [46] for the derivation of this and related corrosion models and references to the applied literature. Nonlinear boundary conditions appear also in some models of heat propagation, where u is the temperature and the normal derivative $\frac{\partial u}{\partial \nu}$ in (1.4) is the heat flux. In [86] the authors derive a similar model in a combustion problem where the reaction happens only at the boundary of the container.

Higher-order equations have attracted the attention of many researchers in the last few years. In particular fourth-order equations with an exponential nonlinearity have been studied in 4 dimensions, in a setting analogous to Liouville's equation by Wei [108], Djadli and Malchiodi [48] and Baraket *et al.* [7]. In higher dimensions Arioli *et al.* [4] considered the bilaplacian together with the exponential nonlinearity in the whole of $\mathbb{R}^N$ and Arioli *et al.* [5] studied (1.5) for $g(u) = e^u$ in ball, which is the natural fourth-order analogue of the classical Gelfand problem (1.1) with $g(u) = e^u$.

A general objective concerning equations (1.1), (1.4) and (1.5) is to study the structure of all solutions (λ, u) and the existence and qualitative properties of singular solutions. These problems share the same basic result:

THEOREM 1.1. *For problems* (1.1), (1.4) *and* (1.5) *there exists a finite parameter* $\lambda^* > 0$ *such that:*

(1) *if* $0 < \lambda < \lambda^*$ *then there exists a minimal bounded solution* u_λ,
(2) *if* $\lambda > \lambda^*$ *then there is no bounded solution.*

We call λ^* the extremal parameter. The branch u_λ with $0 < \lambda < \lambda^*$ is increasing in λ and the linearization of the nonlinear equation around the minimal solution is stable. As $\lambda \to \lambda^*$ the increasing limit $u^* = \lim_{\lambda \uparrow \lambda^*} u_\lambda$ exists pointwise and is a solution with parameter λ^* in a weak sense to be given later on (the exact definition depends on the problem). Depending on the situation, u^* maybe bounded or singular.

Some questions that we are interested in are:

– Can one determine in each situation whether u^* is singular or not?
– Are there singular solutions for $\lambda > \lambda^*$?
– What are the singular solutions for $\lambda < \lambda^*$?
– What happens to the singular solutions under perturbations of the equation?

In what follows we shall review in more detail some of the literature related with the previous questions. Then we shall consider in more detail recent works of the author and some collaborators: Dupaigne, Montenegro and Guerra, [43–45].

1.1. *Basic properties*

Theorem 1.1 and the properties mentioned after its statement can be obtained by the method of sub and supersolutions. Indeed, the three problems (1.1), (1.4) and (1.5) have a maximum principle. Partly due to this reason we restrict the analysis of (1.5) to the ball, since the maximum principle for Δ^2 in this domain with Dirichlet boundary conditions $u = \frac{\partial u}{\partial \nu} = 0$ holds [15].

To be more concrete we sketch the argument for equation (1.1). We remark that for λ positive, 0 is a subsolution which is not a solution and for small positive λ one can take as a supersolution the solution to

$$\begin{cases} -\Delta\zeta = 1 & \text{in } \Omega \\ \zeta = 0 & \text{on } \partial\Omega. \end{cases}$$

Defining λ^* as the supremum of the values such that a classical solution exists, we see that $\lambda^* > 0$ and for any $0 < \lambda < \lambda^*$ there is a bounded solution u_λ, which is minimal among all classical solutions.

To show that λ^* is finite let φ_1 be a positive eigenfunction of $-\Delta$ with Dirichlet boundary condition and eigenvalue $\lambda_1 > 0$. Suppose that u is a classical solution to (1.1) and multiply this equation by φ_1. Integrating and using (1.2), (1.3), which implies $g(u) \geq cu$ for some $c > 0$, we find

$$\lambda_1 \int_\Omega u\varphi_1 = \lambda \int_\Omega g(u)\varphi_1 \geq \lambda c \int_\Omega u\varphi_1 \tag{1.6}$$

which shows that $\lambda \leq \lambda_1/c$. Since there is a constant C such that $g(u) \geq 4\lambda_1 u/\lambda^* - C$ for all $u > 0$, if $\lambda^*/2 < \lambda < \lambda^*$ we have

$$\lambda_1 \int_\Omega u_\lambda\varphi_1 = \lambda \int_\Omega g(u_\lambda)\varphi_1 \geq 2\lambda_1 \int_\Omega u_\lambda\varphi_1 - C' \tag{1.7}$$

for some constant C'. This shows that $\int_\Omega u_\lambda\varphi_1 \leq C$ and implies that $u^* = \lim_{\lambda\to\lambda^*} u_\lambda$ exists a.e.

An important property of the minimal branch of solutions is its stability, that is,

$$\mu_1(-\Delta - \lambda g'(u_\lambda)) > 0, \quad \forall\, 0 \leq \lambda < \lambda^*, \tag{1.8}$$

where $\mu_1(-\Delta - \lambda g'(u))$ denotes the first eigenvalue of the operator $-\Delta - \lambda g'(u_\lambda)$ with Dirichlet boundary conditions. We recall that

$$\mu_1 = \inf_{\varphi \in C_0^\infty(\Omega)} \frac{\int_\Omega |\nabla\varphi|^2 - \lambda g'(u_\lambda)\varphi^2}{\int_\Omega \varphi^2} \tag{1.9}$$

and that there exists a first positive eigenfunction of $-\Delta - \lambda g'(u)$, that is,

$$\begin{cases} -\Delta\psi_1 - \lambda g'(u_\lambda)\psi_1 = \mu_1\psi_1 & \text{in } \Omega \\ \psi_1 > 0 & \text{in } \Omega \\ \psi_1 = 0 & \text{on } \partial\Omega, \end{cases}$$

where we may normalize $\|\psi_1\|_{L^2(\Omega)} = 1$ (see [70]).

Fix $0 \le \lambda < \lambda^*$ and let us show that $\mu_1 > 0$. Since $\lambda < \lambda^*$ we may fix $\lambda < \bar\lambda < \lambda^*$ and write $\bar u = u_{\bar\lambda}$, that is, the minimal solution with parameter $\bar\lambda$. Then by the positivity and convexity of g we have

$$-\Delta(\bar u - u_\lambda) = \bar\lambda g(\bar u) - \lambda g(u_\lambda) > \lambda(g(\bar u) - g(u_\lambda)) \ge \lambda g'(u_\lambda)(\bar u - u_\lambda).$$

Multiplying this inequality by ψ_1 and integrating by parts we find

$$\mu_1 \int_\Omega (\bar u - u_\lambda)\psi_1 > 0.$$

But the integral above is positive because $\psi_1 > 0$ and $\bar u > u_\lambda$ by the strong maximum principle, and we conclude that $\mu_1 > 0$.

Actually the stability characterizes the minimal solution, that is, if (λ, u) is a classical solution to (1.1) such that $\mu_1(-\Delta - \lambda g'(u)) > 0$ then necessarily $u = u_\lambda$. Indeed, since u_λ is the minimal solution we have immediately $u_\lambda \le u$. Now, by convexity of g

$$-\Delta(u_\lambda - u) = \lambda(g(u_\lambda) - g(u)) \ge \lambda g'(u)(u_\lambda - u).$$

Since $\mu_1(-\Delta - \lambda g'(u)) > 0$ the operator $-\Delta - \lambda g'(u)$ satisfies the maximum principle and we deduce that $u_\lambda \ge u$.

The implicit function theorem can also be applied to problems (1.1), (1.4) and (1.5). It implies that starting from the trivial solution $(0, 0)$ there exists a maximal interval $[0, \lambda_0)$ and a C^1 curve of solutions $u(\lambda)$ defined in this interval. Then it is possible to prove that this curve is exactly the branch of minimal solutions u_λ as constructed above and that $\lambda^* = \lambda_0$. For the results here we refer to [69,34,79,35].

1.2. *A second-order semi-linear equation*

In this section we recall some facts related to (1.1), in particular reviewing a few cases where the solution structure is completely known, sufficient conditions for $u^* \in L^\infty$ in general domains, examples where $u^* \notin L^\infty$, and then some properties of the extremal solution such as its stability and uniqueness.

Let us start recalling some of the results for the case $g(u) = e^u$ in the unit ball. In dimension 1 this problem was first studied by Liouville [85]. Bratu [17] found an explicit solution when $N = 2$. Later Chandrasekhar [29] and Frank-Kamenetskii [61] considered $N = 3$ and Barenblatt [69] proved that in dimension 3 for $\lambda = 2$ there are infinitely many solutions. Joseph and Lundgren [76], using phase-plane analysis, gave a complete description of the classical solutions to (1.1) when Ω is the unit ball and $g(u) = e^u$ or $g(u) = (1 + u)^p$, $p > 1$.

THEOREM 1.2 (Joseph and Lundgren [76]). *Let Ω be the unit ball in $\mathbb{R}^N$, $N \ge 1$ and $g(u) = e^u$. Then*

- *If $N = 1, 2$ for any $0 < \lambda < \lambda^*$ there are exactly 2 solutions, while for $\lambda = \lambda^*$ there is a unique solution, which is classical.*

– *If* $3 \le N \le 9$ *we have that* u^* *is bounded and* $\lambda^* > \lambda_0$, *where* $\lambda_0 = 2(N-2)$. *For* $\lambda = \lambda_0$ *there are infinitely many solutions that converge to* $U(x) = -2\log|x|$, *which is a singular solution with parameter* λ_0. *For* $|\lambda - \lambda_0| \neq 0$ *but small there are a large number of solutions.*
– *If* $N \ge 10$ *then* $\lambda^* = 2(N-2)$ *and* $u^* = -2\log|x|$. *Moreover for any* $0 < \lambda < \lambda^*$ *there is only one solution.*

When Ω *is the unit ball in* $\mathbb{R}^N$, $N \ge 3$ *and* $g(u) = (1+u)^p$, $p > 1$ *then:*

– *When* $1 < p \le \frac{N+2}{N-2}$ *there are exactly two solutions for any* $0 < \lambda < \lambda^*$, *while for* $\lambda = \lambda^*$ *there is a unique solution, which is classical.*
– *When* $p > \frac{N+2}{N-2}$ *and* $N < 2 + \frac{4p}{p-1} + 4\sqrt{\frac{p}{p-1}}$ *we have that* u^* *is bounded and* $\lambda^* > \lambda_p$, *where* $\lambda_p = \frac{2}{p-1}(N - \frac{2}{p-1})$. *For* $\lambda = \lambda_p$ *there are infinitely many solutions that converge to* $U_p = |x|^{-\frac{2}{p-1}} - 1$, *which is a singular solution with parameter* λ_p. *For* $|\lambda - \lambda_p| \neq 0$ *but small there are a large number of solutions.*
– *If* $p > \frac{N+2}{N-2}$ *and* $N \ge 2 + \frac{4p}{p-1} + 4\sqrt{\frac{p}{p-1}}$ *then* $\lambda^* = \lambda_p$ *and* $u^* = U_p$. *Moreover for any* $0 < \lambda < \lambda^*$ *there is only one solution.*

For general domains Crandall and Rabinowitz [35] showed that if u^* is a classical solution then the branch of minimal solutions (λ, u_λ) can be continued as curve $s \in (-\delta, \delta) \to (\lambda(s), u_s)$ that "bends back", that is, u_s coincides with the minimal branch for $-\delta < s \le 0$, $\lambda(0) = \lambda^*$, $u_0 = u^*$ and for $0 < s < \delta$ we have $\lambda(s) < \lambda^*$ while u_s is a second solution associated to $\lambda(s)$. These authors and also Mignot and Puel [93, 94] gave sufficient conditions for u^* to be a classical solution in general domains for some nonlinearities.

THEOREM 1.3 (Crandall–Rabinowitz [35], Mignot–Puel [93]). *Let* $\Omega \subseteq \mathbb{R}^N$ *be a bounded smooth domain.*

(1) *If* $g(u) = e^u$ *then* u^* *is classical provided* $N \le 9$.
(2) *When* $g(u) = (1+u)^p$ *with* $p > 1$, u^* *is classical when*

$$N < 2 + \frac{4p}{p-1} + 4\sqrt{\frac{p}{p-1}}.$$

The conditions on p and N in Theorem 1.3 are optimal if Ω is the unit ball by the results of Joseph and Lundgren. A basic fact about the branch of minimal solutions that is important in the proof of this result is that u_λ is stable, in the sense that the first Dirichlet eigenvalue of the linearized operator $-\Delta - \lambda g'(u_\lambda)$ is positive, that is, $\mu_1 > 0$, where μ_1 is given by (1.9). In particular

$$\lambda \int_\Omega g'(u_\lambda)\varphi^2 \le \int_\Omega |\nabla\varphi|^2 \quad \forall \varphi \in C_0^\infty(\Omega). \tag{1.10}$$

Let us sketch briefly the proof of Theorem 1.3 in the case of the exponential nonlinearity $g(u) = e^u$. The aim is to obtain estimates for the minimal solution u_λ for $0 < \lambda < \lambda^*$ that are independent of λ. Let $j > 0$ and take $\varphi = e^{ju_\lambda} - 1$. Then from (1.10) we have

$$j^2 \int_\Omega e^{2ju_\lambda}|\nabla u_\lambda|^2 \ge \lambda \int_\Omega e^{u_\lambda}(e^{ju_\lambda} - 1)^2. \tag{1.11}$$

Multiplying equation (1.1) by $e^{2ju_\lambda} - 1$ and integrating yields

$$2j \int_\Omega e^{2ju_\lambda} |\nabla u_\lambda|^2 = \lambda \int_\Omega e^{u_\lambda}(e^{2ju_\lambda} - 1). \tag{1.12}$$

Combining (1.11) and (1.12) we see that if $j < 2$ then there is some C independent of λ such that

$$\int_\Omega e^{(2j+1)u_\lambda} \le C.$$

Thus $\|u_\lambda\|_{L^q} \le C$ with C_q independent of λ for any $q < 5$. Hence, if $N \le 9$ by the Sobolev and Morrey embedding theorems we have that $\|u_\lambda\|_{L^\infty} \le C$, and this shows that u^* is bounded, and consequently smooth.

Brezis and Vázquez [20] posed the question of finding whether u^* is bounded for general $g(u)$. The result in this direction that holds for the most general nonlinearity and domain is:

THEOREM 1.4 (Cabré [22]). *Let Ω be a smooth, bounded, strictly convex domain in $\mathbb{R}^N$ with $N \le 4$. If g satisfies* (1.2), (1.3) *then the extremal solution u^* to* (1.1) *is bounded.*

Before this result, Nedev [96] had proved that u^* is bounded if $N \le 3$, without any restriction on the domain. It is not known if the extremal solution u^* is singular for some domains and nonlinearities in dimension $5 \le N \le 9$. Cabré and Capella [24] settled this question in the radial case (see [23] for a related result in the entire space):

THEOREM 1.5 (Cabré–Capella [24]). *Suppose g satisfies* (1.2), (1.3) *and let $\Omega = B_1$ be the unit ball in $\mathbb{R}^N$, $N \le 9$. Then u^* is bounded.*

The proof of [24] is based on a rewriting of the stability inequality (1.10) in a form that makes it independent of g. Indeed, let u_λ denote the minimal solution in $\Omega = B_1$, which is radial, and let us write u'_λ for the radial derivative $\frac{du_\lambda}{dr}$. Let $\eta \in C_0^\infty(B_1)$ and consider $\varphi = \eta u'_\lambda$ in (1.10). Then

$$\int_{B_1} \nabla u'_\lambda \nabla(u'_\lambda \eta^2) + (u'_\lambda)^2 |\nabla \eta|^2 \ge \lambda \int_{B_1} g'(u_\lambda)(u'_\lambda)^2 \eta^2. \tag{1.13}$$

But u'_λ satisfies

$$-\Delta u'_\lambda + \frac{N-1}{r^2} u'_\lambda = \lambda g'(u_\lambda) u'_\lambda.$$

Multiplying this equation by $u'_\lambda \eta^2$ and integrating by parts we find

$$\int_{B_1} \nabla u'_\lambda \nabla(u'_\lambda \eta^2) + \int_{B_1} \frac{N-1}{r^2}(u'_\lambda)^2 \eta^2 = \lambda \int_{B_1} g'(u_\lambda)(u'_\lambda)^2 \eta^2. \tag{1.14}$$

Combining (1.13) and (1.14) we obtain

$$\int_{B_1} (u'_\lambda)^2 \left(|\nabla \eta|^2 - \frac{N-1}{r^2} \eta^2 \right) \ge 0 \quad \forall \eta \in C_0^\infty(B_1). \tag{1.15}$$

This form of the stability can be used to deduce from it weighted integrability for u'_λ. Indeed, by density we can argue that it holds for $\eta = r^{-a}$ for $a < \frac{N-2}{2}$, but it is only useful to choose a such that $|\nabla\eta|^2 - \frac{N-1}{r^2}\eta^2 \geq 0$. Now, if $\eta = r^{-a}$ then

$$|\nabla\eta|^2 - \frac{N-1}{r^2}\eta^2 = \left(a^2 - N - 1\right) r^{-2a-2}.$$

Then for any $0 < a < \sqrt{N-1}$, from (1.15) we deduce

$$\int_0^1 (u'_\lambda)^2 r^{N-2a-3}\, dr \leq C. \tag{1.16}$$

We note that C depends on a but not on λ. From (1.16) we can deduce now that if $N < 10$ then $\|u_\lambda\|_{L^\infty} \leq C$ with a constant independent of λ. Indeed, let $\beta > 0$ to be fixed later on and $0 < r < 1$. Since $u_\lambda(1) = 0$

$$u_\lambda(r) = -\int_r^1 u'_\lambda(s)\, ds \leq \left(\int_r^1 u'_\lambda(s)^2 s^\beta\right)^{1/2} \left(\int_r^1 s^{-\beta}\, ds\right)^{1/2}.$$

Observe that $N - 2\sqrt{N-1} - 3 < 1$ whenever $N < 10$. Thus for $N < 10$, we may choose $N - 2\sqrt{N-1} - 3 < \beta < 1$ and it follows that

$$u_\lambda(r) \leq \left(\int_0^1 u'_\lambda(s)^2 s^\beta\, ds\right)^{1/2} \left(\int_0^1 s^{-\beta}\, ds\right)^{1/2} \leq C$$

with C independent of r and λ. This shows that u^* is bounded and hence a classical solution.

The argument of [22] for a general strictly convex domain in $\mathbb{R}^N$, $N \leq 4$ follows the same idea as for the radial case, but this time the role u'_λ is taken by $|\nabla u_\lambda|$. The proof is more involved because the equation satisfied by $|\nabla u_\lambda|$ is more complicated.

To continue the discussion of the properties of u^* we shall define precisely the notion of weak solution we will use when dealing with (1.1), and we adopt the one introduced by Brezis et al. [19]:

DEFINITION 1.6. A function $u \in L^1(\Omega)$ is a weak solution to (1.1) if $g(u)\delta(x) \in L^1(\Omega)$ and

$$-\int_\Omega u\Delta\zeta = \lambda\int_\Omega g(u)\zeta \quad \text{for all } \zeta \in C^2(\overline{\Omega}),\ \zeta = 0 \text{ on } \partial\Omega,$$

where

$$\delta(x) = dist(x, \partial\Omega).$$

It is not difficult to show that $u^* = \lim_{\lambda\to\lambda^*} u_\lambda$ is a weak solution in the above sense. Moreover Nedev [96] proved that in any dimension $u^* \in L^p(\Omega)$ for any $p < \frac{N}{N-4}$ if $N > 4$, for any $p < +\infty$ if $N = 4$ and $u^* \in L^\infty$ for $N \leq 3$.

A question of interest is whether weak solutions may exist for $\lambda > \lambda^*$. Brezis *et al.* [19] showed that this is not the case for (1.1):

THEOREM 1.7 (Brezis–Cazenave–Martel–Ramiandrisoa [19]). *If* $\lambda > \lambda^*$ *then* (1.1) *has no weak solution.*

This result can be restated as follows: if (1.1) has a weak solution for some $\lambda > 0$ then for any $0 < \lambda' < \lambda$, equation (1.1) has a classical solution. The proof of this assertion in [19] is based on a truncation method specially adapted to the nonlinearity. Suppose u is a weak supersolution of (1.1) with parameter λ. In [19] they consider a C^2 concave function $\phi : [0, \infty) \to [0, \infty)$ and set

$$v = \phi(u).$$

Assuming for a moment that u is smooth we can compute

$$\Delta v = \Delta\phi(u) = \phi'(u)\Delta u + \phi''(u)|\nabla u|^2 \le \phi'(u)\Delta u.$$

If ϕ' is bounded, the inequality

$$\Delta v \le \phi'(u)\Delta u$$

can be proved in the sense of distributions when $u, \Delta u \in L^1(\Omega)$. Then, given $0 < \lambda' < \lambda$ we seek a concave, bounded ϕ such that v becomes a supersolution to (1.1) with parameter λ'. If u is a weak solution, then

$$-\Delta v \ge -\phi'(u)\Delta u = \lambda\phi'(u)g(u)$$

and we would like to have

$$\lambda\phi'(u)g(u) \ge \lambda' g(\phi(u)).$$

In particular it is sufficient to achieve equality and directly integrating the ODE yields

$$\phi(u) = H^{-1}\left(\frac{\lambda}{\lambda'}H(u)\right), \tag{1.17}$$

where

$$H(t) = \int_0^t \frac{ds}{g(s)}.$$

It can be checked that ϕ defined by (1.17) is concave, increasing with a bounded derivative. Moreover it is bounded if $\int_0^\infty \frac{ds}{g(s)} < +\infty$ and this leads to a proof of the statement in this case. If on the contrary, $\int_0^\infty \frac{ds}{g(s)} = +\infty$, then still v has better regularity that u, and repeating this construction a finite number of times shows that for $\lambda'' < \lambda'$ a bounded supersolution exists, see the details in [19].

Using the same truncation method and a delicate argument Martel [87] was able to prove the uniqueness of u^*.

THEOREM 1.8 (Martel [87]). *If* $\lambda = \lambda^*$ *then* (1.1) *has a unique weak solution.*

Going back to the discussion of whether u^* is bounded or not, we have seen some ideas to prove that under certain conditions u^* is bounded. But there are few situations where it is known that u^* is singular. One of these examples is the case when Ω is the unit ball in $\mathbb{R}^N$, $N \geq 10$ and $g(u) = e^u$. In [76] it is shown through phase-plane analysis that $u^* = -2\log|x|$. Brezis–Vázquez [20] found a new proof of this fact, showing a connection with Hardy's inequality which we recall:

$$\frac{(N-2)^2}{4}\int_{\mathbb{R}^N}\frac{\varphi^2}{|x|^2} \leq \int_{\mathbb{R}^N}|\nabla\varphi|^2 \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^N), \quad (N \geq 3). \tag{1.18}$$

This connection is a characterization of singular energy solutions.

THEOREM 1.9 (Brezis–Vázquez [20]). *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded smooth domain. Suppose $u \in H_0^1(\Omega)$ is a singular weak solution to (1.1) for some $\lambda > 0$ such that*

$$\lambda\int_\Omega g'(u)\varphi^2 \leq \int_\Omega |\nabla\varphi|^2 \quad \textit{for all } \varphi \in C_0^\infty(\Omega). \tag{1.19}$$

Then $u = u^*$ and $\lambda = \lambda^*$.

When $\Omega = B_1(0)$ in $\mathbb{R}^N$ with $N \geq 10$ and $g(u) = e^u$ the explicit solution $U = -2\log|x|$ with parameter $\lambda_0 = 2(N-2)$ satisfies condition (1.19) thanks to Hardy's inequality (1.18). Thus the previous result immediately yields $u^* = U$ and $\lambda^* = \lambda_0$. The same idea applies when $g(u) = (1+u)^p$, $p > 1$ in the unit ball: the solution $u = |x|^{-\frac{2}{p-1}} - 1$ satisfies (1.19) when $N \geq 2 + \frac{4p}{p-1} + 4\sqrt{\frac{p}{p-1}}$.

The idea of the proof of Theorem 1.9 is as follows. First we remark that $\lambda \leq \lambda^*$ by Theorem 1.7. If $\lambda = \lambda^*$ then the uniqueness result Theorem 1.8 implies that $u = u^*$. So we have to rule out the case $\lambda < \lambda^*$, which we do by contradiction. By density we see that (1.19) holds for $\varphi \in H_0^1(\Omega)$. Since by hypothesis $u \in H_0^1(\Omega)$ we are allowed to take $\varphi = u - u_\lambda$, where u_λ denotes the minimal solution. We obtain, after integration by parts and using the equations for u and u_λ,

$$\int_\Omega (g(u_\lambda) - (g(u) + g'(u)(u_\lambda - u)))(u - u_\lambda) \leq 0.$$

But the integrand is nonnegative since $u > u_\lambda$ a.e. and g is convex. This implies

$$g(u_\lambda) = g(u) + g'(u)(u_\lambda - u) \quad \text{a.e. in } \Omega.$$

It follows that g is linear in intervals of the form $[u_\lambda(x), u(x)]$ for a.e. $x \in \Omega$. The union of such intervals is an interval and coincides with $[0, \infty)$ because $u_\lambda = 0$ on $\partial\Omega$ and u is unbounded, contradicting (1.3).

1.3. *Perturbation of singular solutions*

In the search for nonradial examples where the extremal solution is singular, a natural approach is to consider perturbations of the radial case. Let us consider the Gelfand problem in dimension $N \geq 3$, that is

$$\begin{cases} -\Delta u = \lambda e^u & \text{in } \Omega \subset \mathbb{R}^N \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.20}$$

In dimension $N = 3$ and when $\Omega = B$ is the unit ball, there are infinitely many singular solutions, with a unique singular point which can be prescribed near the origin. This result was announced by H. Matano and proved by Rébaï [101]. Similar results hold when the nonlinearity is $g(u) = (1+u)^p$.

THEOREM 1.10 (Rébaï [101]). *Let B be the unit ball in $\mathbb{R}^3$. Then there exists $\varepsilon > 0$ such that for any $\xi \in B_\varepsilon$ there is a solution (λ, u) of*

$$\begin{cases} \Delta u = \lambda e^u & \text{in } B \setminus \{\xi\} \\ u = 0 & \text{on } \partial B \end{cases} \tag{1.21}$$

which has a nonremovable singularity at ξ.

The solution in the above result has the behavior $u(x) \sim -2\log|x-\xi|$ and it can be seen that (1.21) holds in the sense of distributions.

Pacard [98] proved that for $N > 10$, there exist a dumbbell shaped domain Ω and a positive solution u of $-\Delta u = e^u$ in Ω having prescribed singularities at finitely many points, but $u = 0$ may not hold on $\partial\Omega$. Rébaï [102] extended this result to the case $N = 3$. When the exponential nonlinearity is replaced by $g(u) = u^\alpha$, Mazzeo and Pacard [90] proved that for any exponent α lying in a certain range and for any bounded domain Ω, there exist solutions of $-\Delta u = u^\alpha$ in Ω with $u = 0$ on $\partial\Omega$, with a nonremovable singularity on a finite union of smooth manifolds without boundary. Further results in this direction can be found in [103,99] and their references.

We are interested in the existence of singular solutions to (1.21) in domains in $\mathbb{R}^N$, $N \geq 4$ which are perturbations of the unit ball. Given a C^2 map $\psi : \overline{B}_1 \to \mathbb{R}^N$ and $t \in \mathbb{R}$ define

$$\Omega_t = \{x + t\psi(x) : x \in B_1\}.$$

We work with $|t|$ sufficiently small in order that Ω_t is a smooth bounded domain diffeomorphic to B_1 and we consider the Gelfand problem in Ω_t:

$$\begin{cases} -\Delta u = \lambda e^u & \text{in } \Omega_t \\ u = 0 & \text{on } \partial\Omega_t. \end{cases} \tag{1.22}$$

Our main result is:

THEOREM 1.11. *Let $N \geq 4$. Then there exists $\delta > 0$ (depending on N and ψ) and a curve $t \in (-\delta, \delta) \mapsto (\lambda(t), u(t))$ such that $(\lambda(t), u(t))$ is a solution to (1.22) and $\lambda(0) = 2(N-2)$, $u(0) = \log \frac{1}{|x|^2}$. Moreover there exists $\xi(t) \in B_1$ such that*

$$\left\| u(x,t) - \log \frac{1}{|x-\xi(t)|^2} \right\|_{L^\infty(\Omega_t)} + |\lambda(t) - 2(N-2)| \to 0 \quad \text{as } t \to 0. \tag{1.23}$$

The behavior of the singular solution at the origin is characterized as follows:

$$u(x,t) = \ln \frac{1}{|x-\xi(t)|^2} + \log\left(\frac{\lambda(0)}{\lambda(t)}\right) + \varepsilon(|x-\xi(t)|),$$

where $\lim_{s\to 0} \varepsilon(s) = 0$ (see [43, Corollary 1.4]).

Once Theorem 1.11 is established it implies that for small t the extremal solution is singular in dimension $N \geq 11$.

COROLLARY 1.12. *Let $N \geq 11$ and $(\lambda(t), u(t))$ be the singular solution of Theorem 1.11. Then $u(t)$ is the extremal solution in Ω_t and $\lambda(t)$ the extremal parameter.*

Indeed, let $u = u(t)$ denote the solution of (1.22) obtained in Theorem 1.11. Since $N \geq 11$ we have $2(N-2) < (N-2)^2/4$ and it follows from (1.23) that if $|t|$ is chosen small enough,

$$\lambda(t) e^{\left\| u - \log \frac{1}{|x-\xi(t)|^2} \right\|_{L^\infty(\Omega_t)}} < \frac{(N-2)^2}{4}.$$

Hence for $\varphi \in C_0^\infty(\Omega_t)$,

$$\lambda(t) \int_{\Omega_t} e^u \varphi^2 \leq \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{\varphi^2}{|x-\xi(t)|^2} \leq \int_{\mathbb{R}^N} |\nabla\varphi|^2,$$

by Hardy's inequality (1.18) and thanks to Theorem 1.9, $u(t)$ is the extremal solution of (1.22).

The proof of Theorem 1.11 is by linearization around the singular solution $-2\log|x|$. First we change variables to replace (1.22) with a problem in the unit ball. The map $id + t\psi$ is invertible for t small and we write the inverse of $y = x + t\psi(x)$ as $x = y + t\tilde{\psi}(t, y)$. Define v by

$$u(y) = v(y + t\tilde{\psi}(t, y)).$$

Then

$$\Delta_y u = \Delta_x v + L_t v,$$

where L_t is a second-order operator given by

$$L_t v = 2t \sum_{i,k} v_{x_i x_k} \frac{\partial \tilde{\psi}_k}{\partial y_i} + t \sum_{i,k} v_{x_k} \frac{\partial^2 \tilde{\psi}_k}{\partial y_i^2} + t^2 \sum_{i,j,k} v_{x_j x_k} \frac{\partial \tilde{\psi}_j}{\partial y_i} \frac{\partial \tilde{\psi}_k}{\partial y_i}.$$

We look for a solution of the form

$$v(x) = \log \frac{1}{|x-\xi|^2} + \phi, \quad \lambda = c^* + \mu, \tag{1.24}$$

where $c^* = 2(N-2)$. Then (1.22) is equivalent to

$$\begin{cases} -\Delta\phi - L_t\phi - \dfrac{c^*}{|x-\xi|^2}\phi = \dfrac{c^*}{|x-\xi|^2}(e^\phi - 1 - \phi) + \dfrac{\mu}{|x-\xi|^2}e^\phi \\ \qquad\qquad + L_t\Big(\log \dfrac{1}{|x-\xi|^2}\Big) \quad \text{in } B \\ \phi = -\log \dfrac{1}{|x-\xi|^2} \quad \text{on } \partial B. \end{cases} \tag{1.25}$$

Here the unknowns are ϕ, ξ and μ. From Hardy's inequality (1.18) we see that whenever $c^* < \frac{(N-2)^2}{4}$, which holds if $N \geq 11$, if the right-hand side of (1.25) belongs to $L^2(B)$ then there is a unique solution in $H_0^1(B)$. But typically solutions are singular at the origin, with a behavior $|x-\xi|^{-\alpha}$ for some $\alpha > 0$ (see Baras and Goldstein [9], Dupaigne [50]). Thus, although the linear operator $-\Delta - \frac{c^*}{|x-\xi|^2}$ may be coercive in $H_0^1(B)$, this functional setting is not useful since the nonlinear term that appears on the right-hand side of (1.25), namely $\frac{c^*}{|x-\xi|^2}(e^\phi - 1 - \phi)$, is too strong. Our approach is to consider other functional spaces, more precisely, weighted Hölder spaces specially adapted to the singularity. It turns out that the singular linear operator has a right inverse in these spaces if the data satisfies some orthogonality conditions. More precisely, if one wants solutions such that $|\phi(x)| \leq C|x-\xi|^\nu$, the number and type of orthogonality conditions that appear depend on ν and the value c^*. In our case we would like $\nu = 0$ and c^* is given, and as we will see, this requires $N+1$ orthogonality conditions (if $N \geq 4$). Fortunately we have $N+1$ free parameters: μ and ξ in (1.24), and this is the reason not to force the position of the singularity of v. If $N = 3$ then only one orthogonality condition is required. This explains that in Theorem 1.10 the position of the singularity can be prescribed arbitrarily near the origin, while μ or equivalently λ has to be adjusted.

The proof of Theorem 1.11, which is presented in Section 2 is divided into the following steps. First, in Section 2.1 we study the Laplacian with a potential which is the inverse square to a point ξ. The main result is the solvability of the associated linear equation in weighted Hölder spaces. The analysis in this section is related to the work of Mazzeo and Pacard [90], see also [28,89]. We also study the differentiability properties of the solution with respect to ξ and we show that the previous results hold for perturbations of the Laplacian with the same singular potential. Then the proof itself of Theorem 1.11 is in Section 2.2.

A similar result can be obtained for power-type nonlinearities: given $p > 1$, consider the problem

$$\begin{cases} -\Delta u = \lambda(1+u)^p & \text{in } \Omega_t \\ u = 0 & \text{on } \partial\Omega_t. \end{cases} \tag{1.26}$$

When $t = 0$, i.e. when the domain is the unit ball, it is known (see Theorem 1.2 or [76,20]) that the extremal solution is unbounded and given by $u^* = |x|^{-2/(p-1)} - 1$ if and only if $N \geq 11$ and

$$N \geq 6 + \frac{4}{p-1} + 4\sqrt{\frac{p}{p-1}}.$$

THEOREM 1.13. *Let $N \geq 11$ and $p > 1$ such that $N > 6 + \frac{4}{p-1} + 4\sqrt{\frac{p}{p-1}}$. Given t small, let $u^*(t)$ denote the extremal solution to* (1.26)*. Then there exists $t_0 = t_0(N, \psi, p) > 0$ such that if $|t| < t_0$, $u^*(t)$ is singular.*

Going back to (1.20) naturally the question arises whether if $N \geq 10$ for any convex smooth, bounded domain $\Omega \subseteq \mathbb{R}^N$ the extremal solution u^* is singular. The restriction of convexity is reasonable since if Ω is an annulus it is easily seen that with no restriction on N the extremal solution u^* is smooth. This question, which appears in [20], was considered by Dancer [36, p. 54–56] who showed that in any dimension there are thin convex domains such that the extremal solution is bounded. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with smooth boundary. We assume furthermore that Ω is convex and $\partial\Omega$ is *uniformly convex*, i.e. its principal curvatures are bounded away from zero. Write $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ and $x = (x_1, x_2) \in \mathbb{R}^N$ with $x_1 \in \mathbb{R}^{N_1}$, $x_2 \in \mathbb{R}^{N_2}$. For $\varepsilon > 0$ set

$$\Omega_\varepsilon = \{x = (y_1, \varepsilon y_2) : (y_1, y_2) \in \Omega\} \tag{1.27}$$

and consider the Gelfand problem in Ω_ε:

$$\begin{cases} -\Delta u = \lambda e^u & \text{in } \Omega_\varepsilon \\ \quad u = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{1.28}$$

THEOREM 1.14. *Given $\varepsilon > 0$, let u_ε^* be the extremal solution to* (1.28)*. If $N_2 \leq 9$ then there exists $\varepsilon_0 = \varepsilon_0(N, \Omega) > 0$ such that if $\varepsilon < \varepsilon_0$, u_ε^* is smooth.*

The idea of the proof is to fix the domain by setting

$$v_\varepsilon(y_1, y_2) = u(y_1, \varepsilon y_2).$$

Then v_ε is defined in $\overline{\Omega}$ and satisfies

$$\begin{cases} -(\varepsilon^2 \Delta_{y_1} + \Delta_{y_2}) v_\varepsilon = \varepsilon_j^2 \lambda e^{v_\varepsilon} & \text{in } \Omega \\ \qquad\qquad v_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.29}$$

where Δ_{y_i} denotes the Laplacian with respect to the variables y_i, $i = 1, 2$. After taking $\varepsilon \to 0$ one obtains an equation in each "slice" $\Omega_\alpha = \{y_2 : (\alpha, y_2) \in \Omega\}$ which lives in $\mathbb{R}^{N_2}$ with $N_2 \leq 9$. For all these equations there is an a priori bound for stable solutions as seen, for instance, from the proof of Theorem 1.3. We get a contradiction with this a priori bound, and at the same time manage to prove the convergence as $\varepsilon \to 0$ by selecting for each $\varepsilon > 0$ small a value λ_ε such that the minimal solution u_ε of (1.28) with parameter λ_ε satisfies

$$\max_{\overline{\Omega}_\varepsilon} u_\varepsilon = M, \tag{1.30}$$

where M is a suitably large fixed number. This is possible, if we argue by contradiction, that is, assuming there is a sequence of $\varepsilon \to 0$ such that $u_\varepsilon^* \notin L^\infty(\Omega_\varepsilon)$. For the purpose of proving convergence of v_ε it is important to establish: for some constant C_0 we have

$$\lambda_\varepsilon^* \le \frac{C_0}{\varepsilon^2} \tag{1.31}$$

and for some constant C independent of ε

$$\|\nabla v_\varepsilon\|_{L^\infty(\Omega)} \le C. \tag{1.32}$$

For the last property we use the uniform convexity of Ω, which allows us to find $R > 0$ large enough so that for any $y_0 \in \partial\Omega$ there exists $z_0 \in \mathbb{R}^N$ such that the ball $B_R(z_0)$ satisfies $\Omega \subset B_R(z_0)$ and $y_0 \in \partial B_R(z_0)$. For convenience write for $\varepsilon > 0$

$$L_\varepsilon = \varepsilon^2 \Delta_{y_1} + \Delta_{y_2}.$$

Define $\zeta(y) = R^2 - |y - z_0|^2$ so that $\zeta \ge 0$ in Ω and $-L_\varepsilon \zeta = 2\varepsilon N_1 + 2N_2$. From (1.31) we have the uniform bound $\varepsilon^2 \lambda_\varepsilon \le C$. It follows from (1.29) and the maximum principle that $v_\varepsilon \le C\zeta$ with C independent of ε and y_0. Since $v_\varepsilon(y_0) = \zeta(y_0) = 0$, this in turn implies that

$$|\nabla v_\varepsilon(y_0)| \le C \quad \forall y_0 \in \partial\Omega. \tag{1.33}$$

Then, since the linearization of (1.29) around v_ε has a positive first eigenvalue, we deduce (1.32). A complete proof can be found in [43], see also [36].

1.4. *Reaction on the boundary*

We consider the problem (1.4), that is,

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = \lambda g(u) & \text{on } \Gamma_1 \\ u = 0 & \text{on } \Gamma_2, \end{cases} \tag{1.34}$$

where $\lambda > 0$ is a parameter, $\Omega \subset \mathbb{R}^N$ is a smooth, bounded domain and Γ_1, Γ_2 is a partition of $\partial\Omega$ into surfaces separated by a smooth interface. We will assume that

$$g \text{ is smooth, nondecreasing, convex, } g(0) > 0, \tag{1.35}$$

$$\liminf_{t \to +\infty} \frac{g'(t)t}{g(t)} > 1. \tag{1.36}$$

We recall that the branch of minimal solutions is stable in the sense that for $0 \le \lambda < \lambda^*$:

$$\inf_{\varphi \in C^1(\overline{\Omega}), \varphi = 0 \text{ on } \Gamma_2} \frac{\int_\Omega |\nabla \varphi|^2 \, dx - \lambda \int_{\Gamma_1} f'(u_\lambda) \varphi^2 \, ds}{\int_{\Gamma_1} \varphi^2 \, ds} > 0. \tag{1.37}$$

Assumption (1.36) is not essential, but it simplifies some of the arguments and holds for the examples $g(u) = e^u$, $g(u) = (1+u)^p$, $p > 1$. It allows us to say immediately that u^* is an energy solution in the following sense.

DEFINITION 1.15. We say that u is an energy solution to (1.34) if $u \in H^1(\Omega)$, $g(u) \in L^1(\Gamma_1)$ and

$$\int_\Omega \nabla u \nabla \varphi = \lambda \int_{\Gamma_1} g(u) \varphi \quad \forall \varphi \in C^1(\overline{\Omega}).$$

Indeed, from the stability of the minimal solutions u_λ

$$\lambda \int_{\Gamma_1} g'(u_\lambda) u_\lambda^2 \le \int_\Omega |\nabla u_\lambda|^2 = \lambda \int_{\Gamma_1} g(u_\lambda) u_\lambda.$$

By the hypothesis (1.36) for some $\sigma > 0$ and $C > 0$

$$(1+\sigma) g(u) u \le g'(u) u^2 + C \quad \forall u \ge 0.$$

It follows that there exists C independent of λ such that

$$\lambda \int_{\Gamma_1} g(u_\lambda) u_\lambda \le C$$

and hence

$$\int_\Omega |\nabla u_\lambda|^2 \le C. \tag{1.38}$$

This shows that $u^* \in H^1(\Omega)$. Moreover $g(u^*) \in L^1(\Gamma_1)$. Indeed, let φ be the solution to

$$\begin{cases} \Delta \varphi = 0 & \text{in } \Omega \\ \dfrac{\partial \varphi}{\partial \nu} = 1 & \text{on } \Gamma_1 \\ \varphi = 0 & \text{on } \Gamma_2. \end{cases}$$

Then

$$\int_\Omega \nabla u_\lambda \nabla \varphi = \lambda \int_{\Gamma_1} g(u_\lambda).$$

From (1.38) we deduce $\|g(u_\lambda)\|_{L^1(\Gamma_1)} \le C$ with C independent of λ and the assertion follows.

We are interested in determining whether the extremal solution u^* is bounded or singular in the cases $g(u) = e^u$ and $g(u) = (1+u)^p$, $p > 1$. For this purpose we remark that, as for (1.1) (cf. Theorem 1.9), the stability of a singular energy solution implies that it is the extremal one.

LEMMA 1.16. *Suppose g satisfies* (1.35), (1.36). *Assume that $v \in H^1(\Omega)$ is an unbounded solution of* (1.34) *for some $\lambda > 0$ such that*

$$\lambda \int_{\Gamma_1} g'(v)\varphi^2 \le \int_\Omega |\nabla\varphi|^2 \quad \forall \varphi \in C^1(\overline{\Omega}), \varphi = 0 \text{ on } \Gamma_2. \tag{1.39}$$

Then $\lambda = \lambda^*$ and $v = u^*$.

We shall give in Section 3.1 a proof of this fact under hypothesis (1.36). We note here, though, that the argument is simpler than for Theorem 1.9 because we know immediately that $u^* \in H^1(\Omega)$ and we do not need to rely on a uniqueness result for u^* similar to Theorem 1.8. The advantage of this approach is that Lemma 1.16 holds also under more general conditions, which include the case that Ω has a corner at the interface $\Gamma_1 \cap \Gamma_2$.

For smooth domains the uniqueness of u^* holds only assuming that g satisfies (1.2) and (1.3) and in a more general class of weak solutions. We will discuss this in Section 3.2. In fact, in that section we will develop some tools and results in the context of problem (1.34), that are now classical for (1.1). These are basically the notion of weak solution and the nonexistence of weak solutions for $\lambda > \lambda^*$ as in Brezis *et al.* [19], the regularity results for u^* in low dimensions of Nedev [96] and the uniqueness of u^* in the class of weak solutions, see Martel [87]. Throughout that section we will assume that g satisfies only (1.2) and (1.3).

We would like to construct singular solutions for some nonlinearities, and as a model case we consider first $g(u) = e^u$. Probably the simplest singular solution one may construct is

$$u_0(x) = \int_{\partial\mathbb{R}^N_+} K(x, y) \log \frac{1}{|y|}\, dy \quad \text{for } x \in \mathbb{R}^N_+, \tag{1.40}$$

where

$$K(x, y) = \frac{2x_N}{N\omega_N}|x - y|^{-N} \tag{1.41}$$

is the Green's function for the Dirichlet problem in $\mathbb{R}^N_+$ on the half space $\mathbb{R}^N_+ = \{(x', x_N) \,/\, x_N > 0\}$. Then u_0 is harmonic in $\mathbb{R}^N_+$ and

$$u_0(x) = \log \frac{1}{|x|} \quad \text{for } x \in \partial\mathbb{R}^N_+, x \ne 0.$$

A calculation, see [45], shows the following:

LEMMA 1.17.

$$\frac{\partial u_0}{\partial \nu} = \lambda_{0,N} e^{u_0} \quad \text{on } \partial\mathbb{R}^N_+,$$

where

$$\lambda_{0,N} = \begin{cases} (N-3)\dfrac{\sqrt{\pi}\,\Gamma(\frac{N}{2}-\frac{3}{2})}{2\Gamma(\frac{N}{2}-1)} & \text{if } N \ge 4, \\ 1 & \text{if } N = 3. \end{cases} \tag{1.42}$$

Let

$$\Omega_0 = \{x \in \mathbb{R}^N_+ : u_0(x) > 0\} \quad \Gamma_1 = \partial\Omega \cap \partial\mathbb{R}^N_+ \quad \Gamma_2 = \partial\Omega \setminus \partial\mathbb{R}^N_+.$$

The boundary $\partial\Omega_0$ is not smooth itself but Γ_1, Γ_2 are, and it can be checked that Theorem 1.1 still holds in this case.

Since the singular solution has the form $u_0(x) = -\log|x|$ for $x \in \partial\mathbb{R}^N_+$ its linearized stability is equivalent, by scaling, to

$$\int_{\mathbb{R}^N_+} |\nabla\varphi|^2 \geq \lambda_{0,N} \int_{\partial\mathbb{R}^N_+} \frac{\varphi^2}{|x|}, \quad \forall\varphi \in C_0^\infty(\overline{\mathbb{R}^N_+}).$$

Let us recall here Kato's inequality: for $N \geq 3$

$$\int_{\mathbb{R}^N_+} |\nabla\varphi|^2 \geq H_N \int_{\partial\mathbb{R}^N_+} \frac{\varphi^2}{|x|}, \quad \forall\varphi \in C_0^\infty(\overline{\mathbb{R}^N_+}), \tag{1.43}$$

where the best constant

$$H_N := \inf\left\{ \frac{\int_{\mathbb{R}^N_+} |\nabla\varphi|^2}{\int_{\partial\mathbb{R}^N_+} \frac{\varphi^2}{|x|}} : \varphi \in H^1(\mathbb{R}^N_+), \varphi\Big|_{\partial\mathbb{R}^N_+} \not\equiv 0 \right\} \tag{1.44}$$

is given by

$$H_N = 2\frac{\Gamma(\frac{N}{4})^2}{\Gamma(\frac{N-2}{4})^2} \qquad \forall N \geq 3, \tag{1.45}$$

and Γ is the Gamma function. A proof of it was given by Herbst [73] and we will give later on in Section 3.3 a self-contained proof of (1.43). Actually we are able to improve this inequality in a similar fashion as was done by Brezis and Vázquez [20] or Vázquez and Zuazua [106] for (1.18) (see also [11,20,42,68,106] for other improved versions of Hardy's inequality).

It is not difficult to verify that $\lambda_{0,N} \leq H_N$ if and only if $N \geq 10$ (a proof can be found in [45]). Thus we have:

THEOREM 1.18. *Let $f(u) = e^u$. In any dimension $N \geq 10$ there exists a domain $\Omega \subset \mathbb{R}^N$ and a partition in smooth sets Γ_1, Γ_2 of $\partial\Omega$ such that $u^* \notin L^\infty(\Omega)$.*

Naturally the question becomes whether for all $N \leq 9$ and all domains $\Omega \subseteq \mathbb{R}^N$ one has $u^* \in L^\infty(\Omega)$. A first attempt using the ideas of Crandall–Rabinowitz [35] does not yield the optimal condition on the dimension. For convenience, let $u = u_\lambda$ be the minimal solution of (1.34). Working as in [35] we take $\varphi = e^{ju} - 1$, $j > 0$ in (1.37) and multiply (1.34) by $\psi = e^{2ju} - 1$. We obtain

$$\frac{\lambda}{j^2} \int_{\Gamma_1} e^u \left(e^{ju} - 1\right)^2 ds \le \frac{\lambda}{2j} \int_{\Gamma_1} e^u \left(e^{2ju} - 1\right) ds.$$

It follows that

$$\begin{aligned} \left(\frac{1}{j} - \frac{1}{2}\right) \int_{\Gamma_1} e^{(2j+1)u} \, ds &\le \frac{2}{j} \int_{\Gamma_1} e^{(j+1)u} \, ds \\ &\le \frac{2}{j} \int_{\Gamma_1 \cap A} e^{(j+1)u} \, ds + \frac{2}{j} \int_{\Gamma_1 \cap B} e^{(j+1)u} \, ds, \end{aligned}$$

where $A = [(1/j - 1/2) e^{(2j+1)u} < \frac{4}{j} e^{(j+1)u}]$ and $B = [(1/j - 1/2) e^{(2j+1)u} \ge \frac{4}{j} e^{(j+1)u}]$. Given $j \in (0, 2)$, we see that u remains uniformly bounded on A, while

$$\frac{2}{j} \int_{\Gamma_1 \cap B} e^{(j+1)u} \, ds \le \frac{1}{2} \left(\frac{1}{j} - \frac{1}{2}\right) \int_{\Gamma_1} e^{(j+1)u} \, ds.$$

We conclude that e^u is bounded in $L^{2j+1}(\partial\Omega)$ independently of λ. If $2j + 1 > N - 1$ we obtain by elliptic estimates a bound for u in $C^\alpha(\overline{\Omega})$, for some $\alpha \in (0, 1)$. Thus if $N < 6$ we can choose $j \in (0, 2)$ such that $N - 1 < 2j + 1 < 5$ and obtain a bound for u in $C^\alpha(\overline{\Omega})$ independent of λ.

The above argument proves

PROPOSITION 1.19. *Let $g(u) = e^u$ and assume $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain such that $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 \subset \partial\mathbb{R}^N_+$ and $\Gamma_2 \subset \mathbb{R}^N_+$. Assume further that $N < 6$. Then the extremal solution u^* of* (1.34) *belongs to $L^\infty(\Omega)$.*

We are able to overcome this difficulty under some assumptions on the domain, showing that the method used to prove Proposition 1.19 is not suitable for problem (1.34). In Section 3.4 we will give a proof of:

THEOREM 1.20. *Let $g(u) = e^u$, $N \le 9$ and suppose $\Omega \subset \mathbb{R}^N_+$ is an open, bounded set such that $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 \subset \partial\mathbb{R}^N_+$ and $\Gamma_2 \subset \mathbb{R}^N_+$, Ω is symmetric with respect to the hyperplanes $x_1 = 0, \ldots, x_{N-1} = 0$, and Ω is convex with respect to all directions $x_1, \ldots, x_{N-1}$. Then the extremal solution u^* of* (1.34) *belongs to $L^\infty(\Omega)$.*

Our proof is based on a lower bound of the form:

$$\liminf_{x \to 0, x \in \Gamma_1} \frac{u^*(x)}{\log(1/|x|)} \ge 1. \tag{1.46}$$

Then we show that this behavior is too singular in low dimensions $N \le 9$ for the extremal solution to be weakly stable. Our proof of (1.46) is a simple blow-up argument, but is limited to the exponential nonlinearity.

Next we look at (1.34) in the case $g(u) = (1+u)^p$, $p > 1$. Given $0 < \alpha < N-1$ define

$$w_\alpha(x) = \int_{\partial\mathbb{R}^N_+} K(x,y)|y|^{-\alpha} dy \quad \text{for } x \in \mathbb{R}^N_+, \tag{1.47}$$

where K is defined by (1.41). Clearly, $w_\alpha > 0$ in $\mathbb{R}^N_+$. Moreover w_α is harmonic in $\mathbb{R}^N_+$ and w_α extends to a function belonging to $C^\infty(\overline{\mathbb{R}^N_+} \setminus \{0\})$ with

$$w_\alpha(x) = |x|^{-\alpha} \quad \text{for all } x \in \partial\mathbb{R}^N_+ \setminus \{0\}. \tag{1.48}$$

It is not difficult to verify that for some constant $C(N,\alpha)$ we have

$$\frac{\partial w_\alpha}{\partial \nu}(x) = C(N,\alpha)|x|^{-\alpha-1} \qquad \forall x \in \partial\mathbb{R}^N_+ \setminus \{0\}.$$

In Section 3.5 we shall prove

Lemma 1.21. *For* $0 < \alpha < N-1$ *we have:*

$$C(N,\alpha) = 2\frac{\Gamma\left(\frac{\alpha}{2}+\frac{1}{2}\right)\Gamma\left(\frac{N-1}{2}-\frac{\alpha}{2}\right)}{\Gamma(\frac{\alpha}{2})\Gamma\left(\frac{N-2}{2}-\frac{\alpha}{2}\right)}. \tag{1.49}$$

An heuristic calculation shows that for (1.34) with nonlinearity $g(u) = (1+u)^p$, the expected behavior of a solution u which is singular at $0 \in \partial\Omega$ should be $u(x) \sim |x|^{\frac{1}{p-1}}$. The boundedness of u^* is then related to the value of $C(N, \frac{1}{p-1})$. Observe that $C(N, \frac{1}{p-1})$ is defined for $p > \frac{N}{N-1}$. In the sequel, when writing $C(N, \frac{1}{p-1})$ we will implicitly assume that this condition holds.

Let us write $x = (x', x_N)$ with $x' \in \mathbb{R}^{N-1}$. For the next result we will assume that Ω is convex with respect to x', that is, $(tx', x_N) + ((1-t)y', x_N) \in \Omega$ whenever $t \in [0,1]$, $x = (x', x_N) \in \Omega$ and $y = (y', x_N) \in \Omega$. We shall also denote by Π_N the projection on $\partial\mathbb{R}^N_+$, namely $\Pi_N(x', x_N) = x'$ for all $x = (x', x_N) \in \mathbb{R}^N_+$.

Theorem 1.22. *Consider* (1.34) *with* $g(u) = (1+u)^p$. *Assume* $\Omega \subset \mathbb{R}^N_+$ *is a bounded domain such that* $\partial\Omega = \Gamma_1 \cup \Gamma_2$, *where* $\Gamma_1 \subset \partial\mathbb{R}^N_+$ *and* $\Gamma_2 \subset \mathbb{R}^N_+$, Ω *is convex with respect to* x' *and* $\Pi_N(\Omega) = \Gamma_1$. *If* $p\,C(N, \frac{1}{p-1}) > H_N$ *or* $1 < p < \frac{N}{N-2}$ *then* u^* *is bounded.*

The same result holds if Ω is convex with respect to all directions $x_1, \ldots, x_{N-1}$ and Ω is symmetric with respect to the hyperplanes $x_1 = 0, \ldots, x_{N-1} = 0$. The proof (see [45]) of this result is also through a blow-up argument, but this time we do not prove a lower bound such as (1.46).

As a converse to the previous result we have:

Theorem 1.23. *Consider* (1.34) *with* $g(u) = (1+u)^p$. *If* $p\,C(N, \frac{1}{p-1}) \leq H_N$ *and* $p \geq \frac{N}{N-2}$ *there exists a domain* Ω *such that* u^* *is singular.*

We shall not give the details here but just mention that $u = w_{\frac{1}{p-1}} - 1$ considered in $\Omega = \{x \in \mathbb{R}^N_+ \mid u(x) > 0\}$, with $\Gamma_1 = \partial\Omega \cap \partial\mathbb{R}^N_+$, $\Gamma_2 = \partial\Omega \setminus \partial\mathbb{R}^N_+$ is a singular solution to (1.34). It satisfies the stability condition (1.39) by Kato's inequality (1.43).

The condition $p\,C(N, \frac{1}{p-1}) \le H_N$ is not enough to guarantee that the extremal solution is singular for some domain. Actually this condition can hold for some values of p in the range $\frac{N}{N-1} < p < \frac{N}{N-2}$. In this case a singular solution exists in some domains, but it does not correspond to the extremal one. This is similar to what happens to (1.1) with $g(u) = (1+u)^p$ and p in the range $\frac{N}{N-2} < p < \frac{N+2}{N-2}$. For that problem in the unit ball B_1 there exists a weak solution $u = |x|^{-\frac{2}{p-1}} - 1$ which is not the extremal solution (since it is not in H^1), but for p in the smaller range $\frac{N}{N-2} < p \le \frac{N+2\sqrt{N-1}}{N-4+2\sqrt{N-1}}$ it satisfies condition (1.19), see Theorem 6.2 in [20].

1.5. *A fourth-order variant of the Gelfand problem*

In this section we turn our attention to (1.5) with exponential nonlinearity, that is,

$$\begin{cases} \Delta^2 u = \lambda e^u & \text{in } B \\ u = a & \text{on } \partial B \\ \dfrac{\partial u}{\partial \nu} = b & \text{on } \partial B, \end{cases} \tag{1.50}$$

where $a, b \in \mathbb{R}$. One of the reasons to consider this equation in the unit ball $B = B_1(0)$ is that the maximum principle for Δ^2 with Dirichlet boundary condition ($u = \frac{\partial u}{\partial \nu} = 0$) holds in this domain, see [15], a situation that is not true for general domains [5]. But also most our arguments require the radial symmetry of the solutions. As a consequence u_λ, $0 \le \lambda < \lambda^*$ and u^* are radially symmetric.

Equation (1.50) with $a = b = 0$ was considered recently by Arioli *et al.* [5]. They give a proof of Theorem 1.1 for this problem and show that the minimal solutions of (1.50) are stable in the sense that

$$\int_B (\Delta\varphi)^2 \ge \lambda \int_B e^{u_\lambda} \varphi^2, \quad \forall \varphi \in C_0^\infty(B), \tag{1.51}$$

see [5, Proposition 37]. These authors work with the following class of weak solutions, which we will adopt here: $u \in H^2(B)$ is a weak solution to (1.50) if $e^u \in L^1(B)$, $u = a$ on ∂B, $\frac{\partial u}{\partial \nu} = b$ on ∂B and

$$\int_B \Delta u \Delta\varphi = \lambda \int_B e^u \varphi, \quad \text{for all } \varphi \in C_0^\infty(B).$$

They also show that if $\lambda > \lambda^*$ then (1.50) has no weak solution, but it does not seem to be possible to adapt their proof for problems like (1.5) with a general nonlinearity. The problem stems from the fact that the truncation method, as described after Theorem 1.7 seems not well suited for the fourth-order equation.

Regarding the regularity of u^*, the authors in [5] find a radial singular solution U_σ to (1.50) with $a = b = 0$ associated to a parameter $\lambda_\sigma > 8(N-2)(N-4)$ for dimensions $N = 5, \dots, 16$. Their construction is computer assisted. They show that $\lambda_\sigma < \lambda^*$ if $N \le 10$ and claim to have numerical evidence that this holds for $N \le 12$.

We start here by establishing the fact that the extremal solution u^* is the unique solution to (1.50) in the class of weak solutions. Actually the statement is stronger:

THEOREM 1.24. *If*

$$v \in H^2(B),\ e^v \in L^1(B),\ v|_{\partial B} = a,\ \tfrac{\partial v}{\partial n}|_{\partial B} \le b \tag{1.52}$$

and

$$\int_B \Delta v \Delta \varphi \ge \lambda^* \int_B e^v \varphi \quad \forall\, \varphi \in C_0^\infty(B), \varphi \ge 0, \tag{1.53}$$

then $v = u^*$. *In particular for* $\lambda = \lambda^*$ *problem* (1.50) *has a unique weak solution.*

The proof of this result can be found in Section 4.2, while in Section 4.1 we describe the comparison principles that are useful for the arguments. It is analogous to Theorem 1.8 of Martel [87] for (1.1) but our proof does not seem useful for the general version of this problem (1.5). Again, the reason for this limitation is that truncation method developed in [19] is not well adapted to this fourth-order equation.

The results of [5] are an indication that u^* maybe bounded up to dimension $N \le 12$. We have

THEOREM 1.25. *For any a and b, if $N \le 12$ then the extremal solution u^* of* (1.50) *is smooth.*

Our method of proof is different to the one leading to Theorem 1.3 and is similar to the scheme we used for the problem with reaction on the boundary. Indeed, using the same blow-up argument as for the proof of Theorem 1.20 in Section 3.4 it is possible to show that if u^* is singular then

$$\liminf_{r\to 0} \frac{u^*(r)}{\log(1/r^4)} \ge 1 \tag{1.54}$$

(a complete proof can be found in [44]). Now, if $N \le 4$ the problem is subcritical, and the boundedness of u^* can be proved by other means: no singular solutions exist for positive λ (see [5]) but in dimension $N = 4$ they can blow up as $\lambda \to 0$, see [108].

So assume $5 \le N \le 12$ and that u^* is unbounded. Fix $\sigma > 0$. By (1.54), multiplication of (1.50) by $\varphi = |x|^{4-N+2\varepsilon}$ and integration by parts gives

$$\lambda \int_B e^u |x|^{4-N+2\varepsilon} \ge 4(N-2)(N-4)\omega_N(1-\sigma)\frac{1}{\varepsilon} + O(1), \tag{1.55}$$

where ω_N is the surface area of the unit $N-1$-dimensional sphere S^{N-1} and $O(1)$ represents boundary terms, which are bounded as $\varepsilon \to 0$. Using the weak stability of

u^* (1.60) with $\psi = |x|^{\frac{4-N}{2}+\varepsilon}$ multiplied by an appropriate cut-off function yields

$$\lambda \int_B e^u |x|^{4-N+2\varepsilon} \le \left(\frac{N^2(N-4)^2}{16} + O(\varepsilon) \right) \int_B |x|^{-N+2\varepsilon} = \omega_N \frac{\frac{N^2(N-4)^2}{16}}{2\varepsilon} + O(1), \tag{1.56}$$

since $(\Delta\psi)^2 = (N^2(N-4)^2/16 + O(\varepsilon))|x|^{-N+2\varepsilon}$. From (1.55) and (1.56), and letting $\varepsilon \to 0$ and then $\sigma \to 0$, we find

$$8(N-2)(N-4) \le \frac{N^2(N-4)^2}{16}.$$

This is valid only if $N \ge 13$, a contradiction.

The constant $N^2(N-4)^2/16$ appears in Rellich's inequality [104], which states that if $N \ge 5$ then

$$\int_{\mathbb{R}^N} (\Delta\varphi)^2 \ge \frac{N^2(N-4)^2}{16} \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^4} \qquad \forall \varphi \in C_0^\infty(\mathbb{R}^N). \tag{1.57}$$

The constant $N^2(N-4)^2/16$ is known to be optimal as seen from functions such that $\psi = |x|^{\frac{4-N}{2}+\varepsilon}$. This inequality will play an important role in proving that u^* is singular if $N \ge 13$ and $b = 0$.

Going back to Theorem 1.24 we mention that it can be used to deduce properties of the extremal solution in case it is singular. In [5] the authors say that a radial weak solution u to (1.50) is weakly singular if $\lim_{r\to 0} ru'(r)$ exists. For example, the singular solutions U_σ of [5] verify this condition. As a corollary of Theorem 1.24 we show

COROLLARY 1.26. *The extremal solution u^* to* (1.50) *with $b \ge -4$ is always weakly singular.*

We prove this corollary in Section 4.2. A weakly singular solution either is smooth or exhibits a log-type singularity at the origin. More precisely, if u is a non-smooth weakly singular solution of (1.50) with parameter λ then (see [5]) the following refinement of (1.54) holds:

$$\lim_{r\to 0} u(r) + 4\log r = \log \frac{8(N-2)(N-4)}{\lambda},$$
$$\lim_{r\to 0} ru'(r) = -4.$$

In view of Theorem 1.25, it is natural to ask whether u^* is singular in dimension $N \ge 13$. We show that this is true in the case $a = b = 0$.

THEOREM 1.27. *Let $N \ge 13$ and $a = b = 0$. Then the extremal solution u^* to* (1.50) *is unbounded.*

The proof of Theorem 1.27 is related to Theorem 1.9 and a similar result holds for (1.50):

PROPOSITION 1.28. *Assume that* $u \in H^2(B)$ *is an unbounded weak solution of* (1.50) *satisfying the stability condition*

$$\lambda \int_B e^u \varphi^2 \le \int_B (\Delta\varphi)^2, \quad \forall \varphi \in C_0^\infty(B). \tag{1.58}$$

Then $\lambda = \lambda^*$ *and* $u = u^*$.

See the proof in Section 4.2. When $a = 0$ and $b = -4$ we have an explicit solution

$$\bar{u}(x) = -4 \log |x|$$

associated to $\bar{\lambda} = 8(N-2)(N-4)$. Thanks to Rellich's inequality (1.57) the solution $\bar{u}$ satisfies condition (1.58) when $N \ge 13$. Therefore, by Theorem 1.25 and a direct application of Proposition 1.28 we obtain Theorem 1.27 in the case $b = -4$.

For general values of b we do not know any explicit singular solution to the equation (1.50) and Proposition 1.28 is not useful. We instead find a suitable variant of it (see a proof in Section 4.1):

LEMMA 1.29. (a) *Let* $u_1, u_2 \in H^2(B_R)$ *with* $e^{u_1}, e^{u_2} \in L^1(B_R)$. *Assume that*

$$\Delta^2 u_1 \le \lambda e^{u_1} \quad \text{in } B_R$$

in the sense

$$\int_{B_R} \Delta u_1 \Delta\varphi \le \lambda \int_{B_R} e^{u_1}\varphi \quad \forall \varphi \in C_0^\infty(B_R),\ \varphi \ge 0, \tag{1.59}$$

and $\Delta^2 u_2 \ge \lambda e^{u_2}$ *in* B_R *in the similar weak sense. Suppose also*

$$u_1|_{\partial B_R} = u_2|_{\partial B_R} \quad \text{and} \quad \frac{\partial u_1}{\partial n}|_{\partial B_R} = \frac{\partial u_2}{\partial n}|_{\partial B_R}.$$

Assume furthermore that u_1 *is stable in the sense that*

$$\lambda \int_{B_R} e^{u_1} \varphi^2 \le \int_{B_R} (\Delta\varphi)^2, \quad \forall \varphi \in C_0^\infty(B_R). \tag{1.60}$$

Then

$$u_1 \le u_2 \quad \text{in } B_R.$$

(b) *Let* $u_1, u_2 \in H^2(B_R)$ *be* **radial** *with* $e^{u_1}, e^{u_2} \in L^1(B_R)$. *Assume* $\Delta^2 u_1 \le \lambda e^{u_1}$ *in* B_R *in the sense of* (1.59) *and* $\Delta^2 u_2 \ge \lambda e^{u_2}$ *in* B_R. *Suppose* $u_1|_{\partial B_R} \le u_2|_{\partial B_R}$ *and* $\frac{\partial u_1}{\partial n}|_{\partial B_R} \ge \frac{\partial u_2}{\partial n}|_{\partial B_R}$ *and that the stability condition* (1.60) *holds. Then* $u_1 \le u_2$ *in* B_R.

The idea of the proof of Theorem 1.27 consists in estimating accurately from above the function $\lambda^* e^{u^*}$, and to deduce that the operator $\Delta^2 - \lambda^* e^{u^*}$ has a strictly positive first eigenvalue (in the $H_0^2(B)$ sense). Then, necessarily, u^* is singular. Upper bounds for both λ^* and u^* are obtained by finding suitable sub and supersolutions. For example, if for some λ_1 there exists a supersolution then $\lambda^* \ge \lambda_1$. If for some λ_2 one can exhibit a stable singular subsolution u, then $\lambda^* \le \lambda_2$. Otherwise $\lambda_2 < \lambda^*$ and one can then prove that the minimal solution u_{λ_2} is above u, which is impossible. The bound for u^* also requires a stable singular subsolution.

It turns out that in dimension $N \ge 32$ we can construct the necessary subsolutions and verify their stability by hand. Indeed, assume $a = b = 0$, $N \ge 13$ and let us show

$$u^* \le \bar{u} = -4 \log |x| \quad \text{in } B_1. \tag{1.61}$$

For this define $\bar{u}(x) = -4 \log |x|$. Then $\bar{u}$ satisfies

$$\begin{cases} \Delta^2 \bar{u} = 8(N-2)(N-4)e^{\bar{u}} & \text{in } \mathbb{R}^N \\ \bar{u} = 0 & \text{on } \partial B_1 \\ \dfrac{\partial \bar{u}}{\partial n} = -4 & \text{on } \partial B_1. \end{cases}$$

Observe that since $\bar{u}$ is a supersolution to (1.50) with $a = b = 0$ we deduce immediately that $\lambda^* \ge 8(N-2)(N-4)$.

In the case $\lambda^* = 8(N-2)(N-4)$ we have $u_\lambda \le \bar{u}$ for all $0 \le \lambda < \lambda^*$ because $\bar{u}$ is a supersolution, and therefore $u^* \le \bar{u}$ holds.

Suppose now that $\lambda^* > 8(N-2)(N-4)$. We prove that $u_\lambda \le \bar{u}$ for all $8(N-2)(N-4) < \lambda < \lambda^*$. Fix such λ and assume by contradiction that $u_\lambda \le \bar{u}$ is not true. Note that for $r < 1$ and sufficiently close to 1 we have $u_\lambda(r) < \bar{u}(r)$ because $u_\lambda'(1) = 0$ while $\bar{u}'(1) = -4$. Let

$$R_1 = \inf\{0 \le R \le 1 \mid u_\lambda < \bar{u} \text{ in } (R, 1)\}.$$

Then $0 < R_1 < 1$, $u_\lambda(R_1) = \bar{u}(R_1)$ and $u_\lambda'(R_1) \le \bar{u}'(R_1)$. So u_λ is a solution to the problem

$$\begin{cases} \Delta^2 u = \lambda e^u & \text{in } B_{R_1} \\ u = u_\lambda(R_1) & \text{on } \partial B_{R_1} \\ \dfrac{\partial u}{\partial n} = u_\lambda'(R_1) & \text{on } \partial B_{R_1} \end{cases}$$

while $\bar{u}$ is a stable subsolution to the same problem, because of (1.57) and $8(N-2)(N-4) \le N^2(N-4)^2/16$ for $N \ge 13$. By Lemma 1.29 part (b) we deduce $\bar{u} \le u_\lambda$ in B_{R_1} which is impossible.

An upper bound for λ^* is obtained by considering again a stable, singular subsolution to the problem but with another parameter:

Lemma 1.30. *For $N \ge 32$ we have*

$$\lambda^* \le 8(N-2)(N-4)e^2. \tag{1.62}$$

PROOF. Consider $w = 2(1 - r^2)$ and define

$$u = \bar{u} - w,$$

where $\bar{u}(x) = -4 \log |x|$. Then

$$\Delta^2 u = 8(N-2)(N-4)\frac{1}{r^4} = 8(N-2)(N-4)e^{\bar{u}} = 8(N-2)(N-4)e^{u+w} \\ \le 8(N-2)(N-4)e^2 e^u.$$

Also $u(1) = u'(1) = 0$, so u is a subsolution to (1.50) with parameter $\lambda_0 = 8(N-2)(N-4)e^2$.

For $N \ge 32$ we have $\lambda_0 \le N^2(N-4)^2/16$. Then by (1.57) u is a stable subsolution of (1.50) with $\lambda = \lambda_0$. If $\lambda^* > \lambda_0 = 8(N-2)(N-4)e^2$ the minimal solution u_{λ_0} to (1.50) with parameter λ_0 exists and is smooth. From Lemma 1.29 part (a) we find $u \le u_{\lambda_0}$ which is impossible because u is singular and u_{λ_0} is bounded. Thus we have proved (1.62) for $N \ge 32$. □

With the above remarks we can now prove Theorem 1.27 in the case $N \ge 32$. Combining (1.61) and (1.62) we have that if $N \ge 32$ then $\lambda^* e^{u^*} \le r^{-4}\, 8(N-2)(N-4)e^2 \le r^{-4} N^2(N-4)^2/16$. This and (1.57) show that

$$\inf_{\varphi \in C_0^\infty(B)} \frac{\int_B (\Delta\varphi)^2 - \lambda^* \int_B e^{u^*}\varphi^2}{\int_B \varphi^2} > 0$$

which is not possible if u^* is bounded.

For dimensions $13 \le N \le 31$ it seems difficult to find subsolutions as before explicitly. We adopt then an approach that involves a computer-assisted construction and verification of the desired inequalities. More precisely, first we solve numerically (1.50) by following a branch of singular solutions to

$$\begin{cases} \Delta^2 u = \lambda e^u & \text{in } B \\ u = 0 & \text{on } \partial B \\ \dfrac{\partial u}{\partial \nu} = t & \text{on } \partial B. \end{cases} \tag{1.63}$$

We start with $t = -4$, where an explicit solution is known, and follow this branch to $t = 0$, transforming first (1.63) with an Emden–Fowler-type change of variables, which allows us to work with smooth solutions. This numerical solution, which is represented as a piecewise polynomial function with coefficients in $\mathbb{Q}$ that are kept explicitly, serves as the desired subsolution. The verification of the conditions mentioned before is done with a program in Maple, and in such a way that it guarantees a rigorous proof of the inequalities. This and the proof of Theorem 1.27 for $13 \le N \le 31$ is described in Section 4.3.

For general constant boundary values, it seems more difficult to determine the dimensions for which the extremal solution is singular. Observe that u^* is the extremal solution of (1.50) if and only if $u^* - a$ is the extremal solution of the same equation with boundary condition $u = 0$ on ∂B and so we may assume $a = 0$. But one may ask if Theorem 1.27 still holds for any $N \geq 13$ and any b. Here the situation becomes interesting, because the critical dimension for the boundedness of u^* depends on b and is not always equal to 13.

THEOREM 1.31. (a) *Let $N \geq 13$ and $b \geq -4$. There exists a critical parameter $b^{\max} > 0$ such that the extremal solution u^* is singular if and only if $b \leq b^{\max}$.*

(b) *Let $b \geq -4$. There exists a critical dimension $N^{\min} \geq 13$ such that the extremal solution u^* to* (1.50) *is singular if $N \geq N^{\min}$.*

The proof of this result can be found in [44]. Let us remark that it follows from Theorem 1.31, part (a), that for $b \in [-4, 0]$, the extremal solution is singular if and only if $N \geq 13$. We also deduce from this result that there exist values of b for which $N^{\min} > 13$. We do not know whether u^* remains bounded for $13 \leq N < N^{\min}$.

Finally let us mention that it remains open to describe fully the bifurcation diagram of (1.50), in the spirit if the work of Joseph and Lundgren (Theorem 1.2) for the second-order problem with exponential nonlinearity.

1.6. *Other directions*

The literature on the kind of problems we have mentioned is extensive. Nevertheless we would like mention other related directions which have been the matter of recent studies.

In general domains there are few results on the structure of solutions to (1.1). Let us mention here the results of Dancer [37–39]. For analytic nonlinearities g such that $g(u) \sim u^q e^u$ as $u \to +\infty$ in a bounded smooth domain Ω in $\mathbb{R}^3$ he shows that there is an unbounded connected curve of solutions $\widehat{T} = \{(\lambda(s), u(s)) : s \geq 0\}$ starting from $(0, 0)$ such that $\|u(s)\| + |\lambda(s)| \to +\infty$ as $s \to +\infty$ and $-\Delta - \lambda(s)g'(u(s))$ is invertible except at isolated singularities. This curve has infinitely many bifurcation points outside any compact subset, which include the possibility that the curve "bends back" at some of these points. In [37] Dancer also shows that a sequence of solutions to (1.1) with $g(u) = e^u$ in a bounded smooth domain in three dimensions, remains bounded if and only if their Morse indices are uniformly bounded. This is a consequence of a related result that asserts that any solution to

$$-\Delta u = e^u, \quad u < 0 \quad \text{in } \mathbb{R}^3$$

has infinite Morse index. The proof of [37] uses a result of Bidaut-Verón and Verón [14], that characterizes solutions to

$$-\Delta u = \lambda e^u \quad \text{in } \mathbb{R}^3 \setminus B_1 \tag{1.64}$$

such that

$$e^u \leq \frac{C}{|x|^2} \quad \text{in } \mathbb{R}^3 \setminus B_1. \tag{1.65}$$

In [14] it is proved that any solution to (1.64), (1.65) satisfies

$$\lim_{r\to+\infty}\left(u(r,\theta)-\log\frac{1}{r^2}\right)=2\omega(\theta)+\log\frac{2}{\lambda}\quad\text{in } C^k \text{ of } S^2$$

for any $k \geq 1$, where r, θ are spherical coordinates and ω is a smooth solution to

$$\Delta_{S^2}\omega+e^{2\omega}-1=0\quad\text{on } S^2. \tag{1.66}$$

Here Δ_{S^2} is the Laplace–Beltrami operator on S^2 with the standard metric. It is known that all continuous solutions to (1.66) arise from a single solution and the conformal transformations of S^2, see Chang and Yang [30].

We would like to mention some results for problems similar to (1.1) but where the Laplacian is replaced by a nonlinear operator. For example Clément *et al.* [33] considered the p-Laplacian and k-Hessian operators $S_k(D^2u)$ defined as the sum of all principal $k \times k$ minors of D^2u. Their results were extended by Jacobsen and Schmitt [74,75] and we shall describe them next. Consider

$$\begin{cases} r^{-\gamma}(r^{\alpha}|u'|^{\beta}u')'+\lambda e^u=0 & 0<r<1\\ u>0 & 0<r<1\\ u'(0)=u(1)=0, \end{cases} \tag{1.67}$$

where α, β, γ satisfy

$$\begin{cases} \alpha\geq 0\\ \gamma+1>\alpha\\ \beta+1>0. \end{cases} \tag{1.68}$$

This includes the case of the Laplacian ($\alpha = N-1$, $\beta = 0$, $\gamma = N-1$), the p-Laplacian with $p > 1$ ($\alpha = N-1$, $\beta = p-2$, $\gamma = N-1$) and the k-Hessian operator ($\alpha = N-k$, $\beta = k-1$, $\gamma = N-1$). The main result in [74] characterizes in terms of α, β and γ the multiplicity of solutions as a function of λ.

THEOREM 1.32. *Suppose α, β, γ satisfy* (1.68) *and define*

$$\xi=\gamma+1-\alpha$$
$$\delta=\frac{\gamma+\beta-\alpha+2}{\xi}.$$

Case 1. If $\alpha-\beta-1\leq 0$ there exists a unique $\lambda^*>0$ such that (1.67) *has a unique solution for $\lambda=\lambda^*$, and exactly 2 solutions for $0<\lambda<\lambda^*$.*

Case 2. If $0<\alpha-\beta-1<\frac{4\delta\xi}{\beta+1}$ then (1.67) *has continuum of solutions (λ,u) with $u(0)\to+\infty$ and λ oscillating around $(\alpha-\beta-1)(\delta\xi)^{\beta+1}$.*

Case 3. If $\frac{4\delta\xi}{\beta+1}\leq\alpha-\beta-1$ then the equation has a unique solution for $0<\lambda<(\alpha-\beta-1)(\delta\xi)^{\beta+1}$ and no solution for $\lambda\geq(\alpha-\beta-1)(\delta\xi)^{\beta+1}$. Moreover $u(0)\to+\infty$ as $\lambda\to(\alpha-\beta-1)(\delta\xi)^{\beta+1}$.

The problem (1.1) for the p-Laplacian operator in general smooth, bounded domains, that is,

$$\begin{cases} -\Delta_p u = \lambda g(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has also been the subject of study. We mention the case $g(u) = e^u$ considered by García-Azorero and Peral [65] and García-Azorero *et al.* [66] who showed that the extremal solution is bounded if $N < p + 4p/(p-1)$ and that this condition is optimal. Recently Cabré and Sanchón [27] (see also [25]) also considered this problem for general g, extending the ideas of [19,20] to this setting.

Another direction of interest is the parabolic counterpart of (1.1). Consider

$$\begin{cases} u_t - \Delta u = \lambda g(u) & \text{in } (0,T) \times \Omega \\ u = 0 & \text{on } \partial\Omega \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.69}$$

where g is a nonlinear function, $\lambda > 0$ and $u_0 \geq 0$, $u_0 \in L^\infty(\Omega)$.

It is well known that if $u_0 \in L^\infty(\Omega)$ and g is Lipschitz, then (1.69) has a classical solution defined on a maximal time interval.

Problem (1.69) with exponential nonlinearity was considered by Fujita [62,63]. Lacey [80] and also Bellout [12] proved, under certain extra conditions, that the solution of (1.69) blows up in finite time for $\lambda > \lambda^*$, see also [81]. In this direction we would like to mention the following results due to Brezis *et al.* [19]. Roughly speaking they imply that with initial condition $u_0 = 0$, the solution to the parabolic problem (1.69) is global if and only $\lambda \leq \lambda^*$, that is, if and only if the stationary problem has a weak solution.

THEOREM 1.33 (Brezis *et al.* [19]). *Assume $g : [0,\infty) \to \infty$ is a C^1 convex nondecreasing function such that there exists $x_0 \geq 0$ with $g(x_0) > 0$ and*

$$\int_{x_0}^{\infty} \frac{du}{g(u)} < +\infty. \tag{1.70}$$

Then if (1.69) has a global solution for some $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$ then there is a weak solution to the elliptic problem (1.1).

This result has also a converse.

THEOREM 1.34 (Brezis *et al.* [19]). *Assume $g : [0,\infty) \to \infty$ is a C^1 convex nondecreasing function. If (1.1) has a weak solution w then for any initial condition $u_0 \in L^\infty(\Omega)$, $0 \leq u_0 \leq w$ the solution to (1.69) is global in time.*

Peral and Vázquez [100] considered also the parabolic problem (1.69) with the exponential nonlinearity in $\Omega = B_1$ and with $\lambda = 2(N-2)$, since for this parameter $U(x) = -2\log|x|$ is a weak solution of the stationary problem. They are interested in singular initial conditions and hence they work with the following notion of weak solution: $u \in C((0,\infty); W_0^{1,2}(B_1))$ such that u_t, Δu, $e^u \in L^1([\tau, T] \times B_1)$ for all $0 < \tau < T < +\infty$, equation (1.69) holds a.e. and $u(t,\cdot) \to u_0$ in $L^2(B_1)$ as $t \to 0$. First they take an initial condition u_0 satisfying $0 \le u_0(x) \le U(x)$. They show that (1.69) possesses a minimal and a maximal solution u satisfying $0 \le u(t,x) \le U(x)$. Moreover it becomes classical for $t > 0$. They show that if $3 \le N \le 9$ then any solution satisfying the previous conditions converges to the minimal solution u_λ as $t \to +\infty$. If $N \ge 10$ then $u(t,\cdot) \to U$ as $t \to +\infty$. These authors also study the possibility of having solutions of the parabolic problem above the singular solution U and establish the following

THEOREM 1.35. *Consider* (1.69) *with* $g(u) = e^u$, $\lambda = 2(N-2)$ *and* $\Omega = B_1$. *Then there is no weak solution defined on* $(0,T) \times B_1$ *such that* $u(t,x) \ge U(x)$, *and* $u_0 \not\equiv U$.

The solutions in the above result are shown to blow up completely (such as in Brezis and Cabré [18]) and instantaneously. Dold *et al.* [49] studied the blow-up rate of (1.69) with $\Omega = B_1$ and $g(u) = e^p$ or $g(u) = u^p$, $p > \frac{N+2}{N-2}$. Martel [88] showed that if the initial condition u_0 satisfies $u_0 \in L^\infty(\Omega) \cap W_0^{1,1}(\Omega)$, $u_0 \ge 0$ and $\Delta u_0 + \lambda g(u_0) \ge 0$, then the solution u to (1.69), which is defined on a maximal time interval $[0, T_m)$, blows up completely after T_m if $T_m < +\infty$. This means that for any sequence g_n of bounded approximations of g such that

$$g_n \in C([0,\infty),[0,n)) \quad \text{for all } x \ge 0,\ g_n(x) \uparrow g(x), \text{ as } n \to +\infty$$

the sequence of solutions u_n of (1.69) with g replaced by g_n satisfies

$$\frac{u_n(x,t)}{dist(x,\partial\Omega)} \to +\infty \quad \text{as } n \to +\infty \text{ uniformly for } t \in [T_m + \varepsilon, \infty)$$

for any $\varepsilon > 0$. The hypothesis on the initial condition says, roughly speaking, that $u_t(0) \ge 0$ and hence u is monotone nondecreasing in time, which is seen to be necessary (see below and [59]).

An interesting result of Fila and Poláčik [59] is the following. Consider (1.69) with $g(u) = e^u$ in the unit ball $\Omega = B_1$ and with a radial initial condition $u_0 \in C(\overline{B}_1)$. If $N \le 9$ and the solution u to (1.69) is global, i.e. is a classical solution defined for all times, then u is uniformly bounded, that is,

$$\sup_{t>0, r\in[0,1]} |u(r,t)| < +\infty.$$

In dimensions $N = 1, 2$ this holds for general domains and initial conditions, see [57].

In [59] the authors also show that for $g(u) = e^u$ and also in the radial setting in dimension $3 \le N \le 9$, certain stationary solutions can be connected by solutions that blow up in finite time but can be continued in an L^1 sense. An L^1 solution of the parabolic equation (1.69) is a function $u \in C([0,T]; L^1(\Omega))$ such that $g(u) \in L^1((0,T) \times \Omega)$ and

$$\int_\Omega u\varphi\Big|_\tau^t dx - \int_\tau^t \int_\Omega u\varphi_t\, dx\, ds = \int_\tau^t \int_\Omega (u\Delta\varphi + \lambda g(u)\varphi)\, dx\, ds$$

for all $0 \le \tau < t < T$ and $\varphi \in C^2([0,T] \times \overline{\Omega})$ with $\varphi = 0$ on $[0,T] \times \partial\Omega$. To describe the result [59] we use the notation, following [58]. The solutions to (1.1) with $g(u) = e^u$

in the unit ball B_1 with $3 \le N \le 9$ can be written as a smooth curve

$$(\lambda(s), u(s)), \quad s > 0$$

such that

$$\max_{\overline{B}_1} u(s) = u(s)(0) = s.$$

This curve satisfies

(a) $\lim_{s\to 0} \lambda(s) = 0$, $\lim_{s\to+\infty} \lambda(s) = 2(N-2)$

(b) the critical points of $\lambda(s)$ form a sequence $0 < s_1 < s_2 < \dots$ and the critical values $\lambda(s_j) = \lambda_j$ satisfy

$$\lambda_1 > \lambda_3 \dots > \lambda_{2j+1} \downarrow 2(N-2),$$
$$\lambda_2 < \lambda_4 < \dots \uparrow 2(N-2).$$

For $0 < \lambda < \lambda^*$ let $\tilde{s}_0(\lambda) < \tilde{s}_1(\lambda) < \dots$ denote the sequence of points s such that $\lambda(s) = \lambda$. This sequence is finite if $\lambda \ne 2(N-2)$ and infinite if $\lambda = 2(N-2)$. Write $u_\lambda^j = u(\tilde{s}_j)$. The minimal solution corresponds to $u_\lambda = u_\lambda^0$.

Fila and Poáčik [59] showed that if $\lambda \in (\lambda_2, \lambda_3)$ there exists a smooth initial condition u_0 such that the solution u to (1.69) satisfies:

(1) $u(\cdot, t)$ blows up in finite time T_m,

(2) $u(\cdot, t)$ can be extended to an L^1 global solution (i.e. define on $(0, T)$ for all $T > 0$),

(3) $u(\cdot, t) \to u_\lambda$ as $t \to +\infty$, where $u_\lambda = u_\lambda^0$ is the minimal solution (the convergence is $C^1_{loc}((0, 1]))$

(4) $u(\cdot, t)$ is defined and smooth for all $t \in (-\infty, T_m)$ and $u(\cdot, t) \to u_\lambda^2$.

This solution is called an L^1 connection between the equilibria u_λ^2 and u_λ^0.

Later Fila and Matano [58] extended the results of [59] showing that for any $k \ge 2$ there is an L^1 connection from u_k^λ to u_0^λ. They also show that if an L^1 connection from u_k^λ to u_j^λ exists then $k \ge j + 2$. See also previous work by Ni *et al.* [97], Lacey and Tzanetis [82].

Nonlinear elliptic and parabolic equations such as (1.1) and (1.69) but with explicit singular terms in them have also been a matter of recent studies. Let us mention Brezis and Cabré [18], who showed that if $u \ge 0$ and

$$-\Delta u \ge \frac{u^2}{|x|^2} \quad \text{in } \Omega$$

in the sense of distributions (assuming $u, u^2/|x|^2 \in L^1_{loc}(\Omega)$), in a domain Ω containing the origin, then $u \equiv 0$. Dupaigne [50], Dupaigne and Nedev [51] have studied elliptic equations with a singular potential of the form:

$$\begin{cases} -\Delta u - a(x)u = f(u) + \lambda b(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $a, b, f, \lambda \ge 0$. They characterize, under some assumptions, in terms of the linear operator $-\Delta - a(x)$ and the nonlinearity $f(u)$ the cases where there are solutions for some $\lambda > 0$ or not. For instance if $a(x) = c/|x|^2$, $f(u) = u^p$, $p > 1$, then there is a solution for some $\lambda > 0$ if and only if $c \le (N-2)^2/4$ and $p < p_0 = 1 + 2/a$, where $a = \frac{N-2-\sqrt{(N-2)^2-4c}}{2}$. See also Kalton and Verbitsky [77].

We have mentioned already that the analysis of singular operators such as the Laplacian with a potential given by the inverse square distance to a point has been used to construct singular solutions to a variety of nonlinear problems, [28,89–91,98,99,101–103]. But in fact the same techniques can be applied to construct solutions in exterior domains which in some sense are singular at infinity, or in other words, that decay slowly at infinity. A model equation is

$$\Delta u + u^p = 0, \quad u > 0 \quad \text{in } \mathbb{R}^N \setminus \bar{\mathcal{D}}, \tag{1.71}$$

$$u = 0 \quad \text{on } \partial\mathcal{D}, \qquad \lim_{|x|\to+\infty} u(x) = 0 \tag{1.72}$$

with supercritical p, namely $p > \frac{N+2}{N-2}$.

THEOREM 1.36 ([40,41]). *Let $\mathcal{D}$ be a bounded domain with smooth boundary such that $\mathbb{R}^N \setminus \overline{\mathcal{D}}$ is connected. For any $p > \frac{N+2}{N-2}$ there is a continuum of solutions u_λ, $\lambda > 0$, to* (1.71), (1.72) *such that*

$$u_\lambda(x) = \beta^{\frac{1}{p-1}} |x|^{-\frac{2}{p-1}} (1 + o(1)) \quad \text{as } |x| \to \infty \tag{1.73}$$

and $u_\lambda(x) \to 0$ as $\lambda \to 0$, uniformly in $\mathbb{R}^N \setminus \mathcal{D}$, where

$$\beta = \frac{2}{p-1}\left(N - 2 - \frac{2}{p-1}\right).$$

The idea of the proof is by linearization around $w(x)$, the unique positive radial solution

$$\Delta w + w^p = 0 \quad \text{in } \mathbb{R}^N, \qquad w(0) = 1. \tag{1.74}$$

Note that all radial solutions of $\Delta u + u^p = 0$ defined in all $\mathbb{R}^N$ have the form

$$w_\lambda(x) = \lambda^{\frac{2}{p-1}} w(\lambda|x|), \quad \lambda > 0. \tag{1.75}$$

We look for a solution u_λ in the form of a small perturbation of w_λ. This naturally leads us to study the linearized operator $\Delta + p w_\lambda^{p-1}$ in $\mathbb{R}^N \setminus \mathcal{D}$ under Dirichlet boundary conditions. Since w_λ is small on bounded sets for small λ, an inverse can be found as a small perturbation of an inverse of this operator in the whole $\mathbb{R}^N$ and then, by scaling, it suffices to analyze the case $\lambda = 1$. Thus we need to study $\Delta + p w^{p-1}$ in $\mathbb{R}^N$. Note that at main order one has $w(r) = \beta^{\frac{1}{p-1}} r^{-\frac{2}{p-1}} (1 + o(1))$ as $r \to +\infty$ [72], and hence the singular potential has the form $p\beta/r^2(1 + o(1))$. We construct an inverse in weighted L^∞ norms for $p \ge \frac{N+1}{N-3}$, however if $\frac{N+2}{N-2} < p < \frac{N+1}{N-3}$ the linearized operator is not surjective, having a range orthogonal to the generators of translations. We overcome this difficulty by adjusting the location of the origin. The invertibility analysis for $p \geq \frac{N+1}{N-3}$ is strongly related to one of Mazzeo and Pacard [90] in the construction of singular solutions with prescribed singularities for $\frac{N}{N-2} < p < \frac{N+2}{N-2}$ in bounded domains. At the radial level, supercritical and subcritical in this range are completely dual.

Problems (1.71)–(1.72) has also a fast decay solution, that is a solution u such that $\limsup_{|x|\to+\infty} |x|^{2-N} u(x) < +\infty$, provided $\frac{N+2}{N-2} < p$ and $p - \frac{N+2}{N-2}$ is small, see [41].

A related result for supercritical problems in bounded domains is the following. Consider

$$\Delta u + u^p = 0, \quad u > 0 \quad \text{in } \mathcal{D} \setminus B_\delta(Q), \tag{1.76}$$

$$u = 0 \quad \text{on } \partial\mathcal{D} \cup \partial B_\delta(Q), \tag{1.77}$$

where $\mathcal{D}$ is a bounded domain with smooth boundary, $B_\delta(Q) \subset \mathcal{D}$ and $\delta > 0$ is to be taken small.

THEOREM 1.37 (del Pino and Wei [47]). *There exists a sequence*

$$\frac{N+2}{N-2} < p_1 < p_2 < p_3 < \ldots, \quad \text{with} \lim_{k\to+\infty} p_k = +\infty \tag{1.78}$$

such that if $p > \frac{N+2}{N-2}$ and $p \neq p_j$ for all j, then there is a $\delta_0 > 0$ such that for any $\delta < \delta_0$, Problems (1.76), (1.77) possess at least one solution.

2. Perturbation of singular solutions

2.1. *The Laplacian with the inverse square potential*

We consider the linear problem

$$\begin{cases} -\Delta\phi - \dfrac{c}{|x-\xi|^2}\phi = g & \text{in } B \\ \phi = h & \text{on } \partial B, \end{cases} \tag{2.1}$$

where $B = B_1(0)$, $\xi \in B$ and c is any real number. The main results are Propositions 2.1 and 2.3 below, which assert the solvability of (2.1) in weighted Hölder spaces assuming that the right-hand side verifies certain orthogonality conditions, provided ξ is close to the origin. We use the weighted Hölder spaces that appear in [101,8,28], which are defined as follows. Given Ω a smooth domain, $\xi \in \Omega$, $k \geq 0$, $0 < \alpha < 1$, $0 < r \leq dist(x, \partial\Omega)/2$ and $u \in C^{k,\alpha}_{\text{loc}}(\overline{B} \setminus \{\xi\})$ we define :

$$|u|_{k,\alpha,r,\xi} = \sup_{r\leq|x-\xi|\leq 2r} \sum_{j=0}^{k} r^j |\nabla^j u(x)| + r^{k+\alpha}\left[\sup_{r\leq|x-\xi|,|y-\xi|\leq 2r} \frac{|\nabla^k u(x) - \nabla^k u(y)|}{|x-y|^\alpha}\right].$$

Let $d = dist(\xi, \partial\Omega)$ and for any $\nu \in \mathbb{R}$ let

$$\|u\|_{k,\alpha,\nu,\xi;\Omega} = \|u\|_{C^{k,\alpha}(\overline{\Omega}\setminus B_{d/2}(\xi))} + \sup_{0<r\leq\frac{d}{2}} r^{-\nu}|u|_{k,\alpha,r,\xi}.$$

Define the Banach space

$$C^{k,\alpha}_{\nu,\xi}(\Omega) = \{u \in C^{k,\alpha}_{\mathrm{loc}}(\overline{\Omega}\setminus\{\xi\}) \ : \ \|u\|_{k,\alpha,\nu,\xi;\Omega} < \infty\}.$$

It embeds continuously in the space of bounded functions if $\nu \geq 0$.

For the analysis of (2.1) when $\xi = 0$ it is convenient to decompose all functions in Fourier series. So we recall that the eigenvalues of the Laplace–Beltrami operator $-\Delta$ on S^{N-1} are given by (see [13])

$$\lambda_k = k(N + k - 2), \quad k \geq 0.$$

Let m_k denote the multiplicity of λ_k and $\varphi_{k,l}$, $l = 1, \dots, m_k$ the eigenfunctions associated to λ_k. We normalize these eigenfunctions so that $\{\varphi_{k,l} : k \geq 0,\ l = 1, \dots, m_k\}$ is an orthonormal system in $L^2(S^{N-1})$. We choose the first functions to be

$$\varphi_{0,1} = \frac{1}{|S^{N-1}|^{1/2}}, \quad \varphi_{1,l} = \frac{x_l}{(\int_{S^{N-1}} x_l^2)^{1/2}} = \left(\frac{N}{|S^{N-1}|}\right)^{1/2} x_l, \quad l = 1, \dots, N.$$

Let $r = |x|$ and $\theta = x/|x|$ denote polar coordinates in $\mathbb{R}^N$.

First we study the kernel of the operator $\Delta + c/|x|^2$. Thus we look for solutions to

$$-\Delta w - \frac{c}{|x|^2} w = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\} \tag{2.2}$$

of the form $w(x) = f(r)\varphi_{k,l}(\theta)$ which yields the ODE:

$$f'' + \frac{N-1}{r} f' + \frac{c - \lambda_k}{r^2} f = 0, \quad \text{for } r > 0. \tag{2.3}$$

Equation (2.3) is of Euler-type and it admits a basis of solutions of the form $f(r) = r^{-\alpha_k^\pm}$, where $\alpha_k^\pm$ are the roots of the associated characteristic equation, i.e.

$$\alpha_k^\pm = \frac{N-2}{2} \pm \sqrt{\left(\frac{N-2}{2}\right)^2 - c + \lambda_k}. \tag{2.4}$$

Note that $\alpha_k^\pm$ may have a nonzero imaginary part only for finitely many k's. If k_0 is the first integer k such that $\alpha_k^\pm \in \mathbb{R}$ then

$$\dots < \alpha_{k_0+1}^- < \alpha_{k_0}^- \leq \frac{N-2}{2} \leq \alpha_{k_0}^+ < \alpha_{k_0+1}^+ < \dots,$$

whereas, if $k < k_0$, we denote the imaginary part of α_k^+ by

$$b_k = \sqrt{c - \left(\frac{N-2}{2}\right)^2 - \lambda_k}.$$

For $k \geq 0$, $l = 1, \dots, m_k$, we have a family of real-valued solutions of (2.2), denoted by $w^1 = w^1_{k,l}$, $w_2 = w^2_{k,l}$ and defined on $\mathbb{R}^N \setminus \{0\}$ by:

if $(\frac{N-2}{2})^2 - c + \lambda_k > 0$

$$w^1 = r^{-\alpha_k^+}\varphi_{k,l}(\theta), \quad w^2 = r^{-\alpha_k^-}\varphi_{k,l}(\theta), \tag{2.5}$$

if $(\frac{N-2}{2})^2 - c + \lambda_k = 0$

$$w^1 = r^{-\frac{N-2}{2}}\log r\varphi_{k,l}(\theta), \quad w^2 = r^{-\frac{N-2}{2}}\varphi_{k,l}(\theta), \tag{2.6}$$

if $\frac{N-2}{2})^2 - c + \lambda_k < 0$

$$w^1 = r^{-\frac{N-2}{2}}\sin(b_k\log r)\varphi_{k,l}(\theta), \quad w^2(x) = r^{-\frac{N-2}{2}}\cos(b_k\log r)\varphi_{k,l}(\theta). \tag{2.7}$$

Then the functions $W_{k,l}$ defined by

$$\begin{cases} \text{if } (\frac{N-2}{2})^2 - c + \lambda_k > 0: & W_{k,l}(x) = w^1(x) - w^2(x), \\ \text{if } (\frac{N-2}{2})^2 - c + \lambda_k \le 0: & W_{k,l}(x) = w^1(x), \end{cases} \tag{2.8}$$

solve (2.2) and satisfy

$$W_{k,l}|_{\partial B} = 0.$$

The main result in this section for the case $\xi = 0$ is

PROPOSITION 2.1. *Let $c, \nu \in \mathbb{R}$ and assume*

$$\exists k_1 \text{such that} \quad \alpha_{k_1}^- \in \mathbb{R} \quad \text{and} \quad -\alpha_{k_1}^- < \nu < -\alpha_{k_1+1}^-. \tag{2.9}$$

Let $g \in C^{0,\alpha}_{\nu-2,0}(B)$ and $h \in C^{2,\alpha}(\partial B)$ and consider

$$\begin{cases} -\Delta\phi - \dfrac{c}{|x|^2}\phi = g & \text{in } B \\ \phi = h & \text{on } \partial B. \end{cases} \tag{2.10}$$

Then (2.10) has a solution in $C^{2,\alpha}_{\nu,0}(B)$ if and only if

$$\int_B gW_{k,l} = \int_{\partial B} h\frac{\partial W_{k,l}}{\partial n}, \quad \forall k = 0, \dots, k_1, \forall l = 1, \dots, m_k. \tag{2.11}$$

Under this condition the solution $\phi \in C^{2,\alpha}_{\nu,0}(B)$ to (2.10) is unique and it satisfies

$$\|\phi\|_{2,\alpha,\nu,0;B} \le C(\|g\|_{0,\alpha,\nu-2,0;B} + \|h\|_{C^{2,\alpha}(\partial B)}), \tag{2.12}$$

where C is independent of g and h.

Note that with the hypotheses of Lemma 2.1 we have

$$\nu > -\alpha_{k_1}^- \ge -\frac{N-2}{2}. \tag{2.13}$$

This implies that the integrals on the left-hand side of (2.11) exist.

PROOF OF PROPOSITION 2.1. Write ϕ as

$$\phi(x) = \sum_{k=0}^{\infty} \sum_{l=1}^{m_k} \phi_{k,l}(r)\varphi_{k,l}(\theta), \quad x = r\theta, 0 < r < 1, \theta \in S^{N-1}.$$

Then ϕ solves $-\Delta\phi - \frac{c}{|x|^2}\phi = g$ in $B \setminus \{0\}$ if and only if $\phi_{k,l}$ satisfies the ODE

$$\phi''_{k,l} + \frac{N-1}{r}\phi'_{k,l} + \frac{c-\lambda_k}{r^2}\phi_{k,l} = -g_{k,l} \quad 0 < r < 1, \tag{2.14}$$

for all $k \geq 0$ and $l = 1, \ldots, m_k$, where

$$g_{k,l}(r) = \int_{S^{N-1}} g(r\theta)\varphi_{k,l}(\theta)\, d\theta, \quad 0 < r < 1, \theta \in S^{N-1}.$$

Note that if $\phi \in L^\infty_\nu(B)$ then there exists a constant $C > 0$ independent of r such that

$$|\phi_{k,l}(r)| \leq Cr^\nu. \tag{2.15}$$

Furthermore, $\phi = h$ on ∂B if and only if $\phi_{k,l}(1) = h_{k,l}$ for all k, l, where

$$h_{k,l} = \int_{S^{N-1}} h(\theta)\varphi_{k,l}(\theta)\, d\theta.$$

Step 1. Clearly, $\sup_{0\leq t\leq 1} t^{2-\nu}|g_{k,l}(t)| < \infty$ and observe that (2.11) still holds when g is replaced by $g_{k,l}\varphi_{k,l}$ and h by $h_{k,l}\varphi_{k,l}$. We claim that there is a unique $\phi_{k,l}$ that satisfies (2.14), (2.15) and

$$\phi_{k,l}(1) = h_{k,l}. \tag{2.16}$$

We also have

$$|\phi_{k,l}(r)| \leq C_k r^\nu \left(\sup_{0\leq t\leq 1} t^{2-\nu}|g_{k,l}(t)| + |h_{k,l}| \right), \quad 0 < r < 1. \tag{2.17}$$

Case $k = 0, \ldots, k_1$. A solution to (2.14) is given by:

- if $\alpha^\pm_{k,l} \notin \mathbb{R}$

$$\phi_{k,l}(r) = \frac{1}{b}\int_0^r s\left(\frac{s}{r}\right)^{\frac{N-2}{2}} \sin\left(b_k \log\frac{s}{r}\right) g_{k,l}(s)\, ds, \tag{2.18}$$

- if $\alpha^+_{k,l} = \alpha^-_{k,l} = \frac{N-2}{2}$:

$$\phi_{k,l}(r) = \int_0^r s\left(\frac{s}{r}\right)^{\frac{N-2}{2}} \log\left(\frac{s}{r}\right) g_{k,l}(s)\, ds, \tag{2.19}$$

- if $\alpha^\pm_{k,l} \in \mathbb{R}, \alpha^\pm_{k,l} \neq \frac{N-2}{2}$:

$$\phi_{k,l}(r) = \frac{1}{\alpha^+_k - \alpha^-_k}\int_0^r s\left(\left(\frac{s}{r}\right)^{\alpha^+_k} - \left(\frac{s}{r}\right)^{\alpha^-_k}\right) g_{k,l}(s)\, ds. \tag{2.20}$$

In each case, (2.17) holds and (2.16) follows from (2.11).

Concerning uniqueness, suppose that $\phi_{k,l}$ satisfies (2.14) with $g_{k,l} = 0$ and (2.16) with $h_{k,l} = 0$. Then $\phi_{k,l}$ is a linear combination of the functions w^1, w^2 defined in (2.5)–(2.7). By (2.9), (2.13) and (2.17), $\phi_{k,l}$ has to be zero.

Case $k \geq k_1 + 1$. Observe that (2.14) is equivalent to

$$-\Delta\tilde{\phi}_{k,l} + \frac{\lambda_k - c}{|x|^2}\tilde{\phi}_{k,l} = \tilde{g}_{k,l} \quad \text{in } B \setminus \{0\},$$

where $\tilde{\phi}_{k,l}(x) = \phi_{k,l}(|x|)$ and $\tilde{g}_{k,l}(x) = g_{k,l}(|x|)$. Since $\alpha_k^{\pm} \in \mathbb{R}$ we must have $\lambda_k - c \geq -(\frac{N-2}{2})^2$ and hence the equation

$$\begin{cases} -\Delta\tilde{\phi}_{k,l} + \dfrac{\lambda_k - c}{|x|^2}\tilde{\phi}_{k,l} = \tilde{g}_{k,l} & \text{in } B \\ \qquad\qquad\qquad \tilde{\phi}_{k,l} = h_{k,l} & \text{on } \partial B, \end{cases} \tag{2.21}$$

has a unique solution $\tilde{\phi}_{k,l} \in H$, where H is the completion of $C_0^\infty(B)$ with the norm

$$\|\varphi\|_H^2 = \int_B |\nabla\varphi|^2 + \frac{\lambda_k - c}{|x|^2}\varphi^2,$$

see [106].

To show (2.17), observe that for some constant C depending only on N, λ_k and ν,

$$A_{k,l}(r) = r^\nu C\left(\sup_{0<t\leq 1} t^{2-\nu}|\tilde{g}_{k,l}(t)| + |h_{k,l}|\right)$$

is a supersolution to (2.21) and $-A_{k,l}$ is a subsolution. To see this, we emphasize that the condition $-\alpha_k^- > \nu > -(N-2)/2$ implies $\nu^2 + (N-2)\nu + c - \lambda_k < 0$. It follows that $|\tilde{\phi}_{k,l}(x)| \leq A_{k,l}(|x|)$ for $0 < |x| \leq 1$.

We note that $\tilde{\phi}_{k,l}$ is uniquely determined. Indeed, any solution w of (2.21) such that $|w(x) \leq C|x|^\nu$ satisfies, by a scaling argument, $|\nabla w(x)| \leq C|x|^{\nu-1}$ and this together with (2.13) implies $w \in H^1(B)$, which is contained in H. Uniqueness for (2.21) in $H^1(B)$ can then be proved by an improved Hardy inequality (see [20]).

The computations above also yield the necessity of condition (2.11). Indeed, assuming a solution $\phi \in L_\nu^\infty(B)$ exists, since $\phi_{k,l}$ satisfies the ODE (2.14) we see that for $k = 0, \ldots, k_1$ the difference between $\phi_{k,l}$ and one of the particular solutions (2.18), (2.19) or (2.20) can be written in the form $c_{k,l}r^{-\alpha_k^+} + d_{k,l}r^{-\alpha_k^-}$. Since $|\phi_{k,l}(r)| \leq Cr^\nu$ and $\nu > -\alpha_{k_1}^-$ we have $c_{k,l} = d_{k,l} = 0$ and this implies (2.11).

Step 2. Define for $m \geq 1$

$$\mathcal{G}_m = \left\{ g = \sum_{k=0}^m \sum_l g_{k,l}(r)\varphi_{k,l}(\theta) \,:\, |x|^{2-\nu}g(x) \in L^\infty(B) \right\}$$

and

$$\mathcal{H}_m = \left\{ h = \sum_{k=0}^m \sum_l h_{k,l}\varphi_{k,l}(\theta) \,:\, h_{k,l} \in \mathbb{R} \right\}.$$

Let $g_m \in \mathcal{G}_m$, $h_m \in \mathcal{H}_m$ be such that (2.11) holds. Write

$$g_m(x) = \sum_{k=0}^{m} \sum_{l} g_{k,l}(r)\varphi_{k,l}(\theta), \quad h_m(\theta) = \sum_{k=0}^{m} h_{k,l}\varphi_{k,l}(\theta).$$

Let $\phi_{k,l}$ be the unique solution to (2.14), (2.15) and (2.16) associated to $g_{k,l}$, $h_{k,l}$ and define $\phi_m(x) = \sum_{k=0}^{m} \sum_l \phi_{k,l}(r)\varphi_{k,l}(\theta)$. We claim that there exists C independent of m such that

$$|\phi_m(x)| \leq C|x|^\nu \left(\sup_B |y|^{2-\nu}|g_m(y)| + \sup_{\partial B} |h_m| \right), \quad 0 < |x| < 1. \tag{2.22}$$

By the previous step, (2.22) holds for some constant C which may depend on m. In particular, choosing $m = k_1$, we obtain a bound on the first components $\phi_{k,l}$, $k = 0 \ldots k_1$. Hence, it suffices to prove (2.22) in the case $g_{k,l} \equiv 0$ and $h_{k,l} = 0$, $k = 0, \ldots, k_1$. Working as in [101], we argue by contradiction assuming that

$$\|\phi_m|x|^{-\nu}\|_{L^\infty(B)} \geq C_m(\|g_m|x|^{2-\nu}\|_{L^\infty(B)} + \|h_m\|_{L^\infty(\partial B)}),$$

where $C_m \to \infty$ as $m \to \infty$ (this argument also appears in [28]). Replacing ϕ_m by $\phi_m/\|\phi_m|x|^{-\nu}\|_{L^\infty(B)}$ if necessary, we may assume

$$\begin{aligned} &\|\phi_m|x|^{-\nu}\|_{L^\infty(B)} = 1, \\ &\|g_m|x|^{2-\nu}\|_{L^\infty(B)} + \|h_m\|_{L^\infty(\partial B)} \to 0 \quad \text{as } m \to \infty. \end{aligned} \tag{2.23}$$

Let $x_m \in B \setminus \{0\}$ be such that $|\phi_m(x_m)||x_m|^{-\nu} \in [\frac{1}{2}, 1]$. Let us show that $x_m \to 0$ as $m \to \infty$. Otherwise, up to a subsequence $x_m \to x_0 \neq 0$. By standard elliptic regularity, up to another subsequence, $\phi_m \to \phi$ uniformly on compact sets of $\overline{B} \setminus \{0\}$ and hence

$$\begin{cases} -\Delta\phi - \dfrac{c}{|x|^2}\phi = 0 & \text{in } B \setminus \{0\} \\ \phi = 0 & \text{on } \partial B. \end{cases}$$

Moreover ϕ satisfies $|\phi(x_0)||x_0|^{-\nu} \in [\frac{1}{2}, 1]$ and $|\phi(x)| \leq |x|^\nu$ in B. Writing

$$\phi(x) = \sum_{k \geq k_1+1} \sum_l \phi_{k,l}(r)\varphi_{k,l}(\theta),$$

we see that $\phi_{k,l}$ solves (2.3). The growth restriction $|\phi_{k,l}(r)| \leq Cr^\nu$ and the explicit functions w^1, w^2 given by (2.5)–(2.7) rule out the cases $\alpha_k^\pm \notin \mathbb{R}$, $\alpha_k^- = \alpha_k^+$ and force $\phi_{k,l} = a_{k,l}r^{-\alpha_k^-}$. But $\phi_{k,l}(1) = 0$ so we deduce $\phi_{k,l} \equiv 0$ and hence $\phi \equiv 0$, contradicting $|\phi(x_0)||x_0|^{-\nu} \neq 0$.

The above argument shows that $x_m \to 0$. Define $r_m = |x_m|$ and

$$v_m(x) = r_m^{-\nu} \phi_m(r_m x), \quad x \in B_{1/r_m}.$$

Then $|v_m(x)| \le |x|^\nu$ in B_{1/r_m}, $|v_m(\frac{x_m}{r_m})| \in [\frac{1}{2}, 1]$ and

$$-\Delta v_m(x) - \frac{c}{|x|^2} v_m(x) = r_m^{2-\nu} g(r_m x) \quad \text{in } B_{1/r_m} \setminus \{0\}.$$

But

$$r_m^{2-\nu} |g(r_m x)| \le \| g_m(y) |y|^{2-\nu} \|_{L^\infty(B)} |x|^{\nu-2} \to 0, \quad \text{as } m \to \infty$$

by (2.23). Passing to a subsequence, we have that $\frac{x_m}{r_m} \to x_0$ with $|x_0| = 1$, $v_m \to v$ uniformly on compact sets of $\mathbb{R}^N \setminus \{0\}$ and v satisfies

$$-\Delta v - \frac{c}{|x|^2} v = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

Furthermore, $|v(x)| \le |x|^\nu$ in $\mathbb{R}^N \setminus \{0\}$ and $|v(x_0)| \ne 0$. Write

$$v(x) = \sum_{k=0}^{\infty} \sum_{l} v_{k,l}(r) \varphi_{k,l}(\theta).$$

Then $|v_{k,l}(r)| \le C_k r^\nu$ for $r > 0$. But $v_{k,l}$ has to be a linear combination of the functions w^1, w^2 given in (2.5)–(2.7), and none of these is bounded by $C r^\nu$ for all $r > 0$. Thus $v \equiv 0$ yielding a contradiction. This establishes (2.22).

Step 3. Finally, a density argument shows that if h, g satisfy (2.11) then there exists a solution ϕ to (2.10) and satisfies (2.22). From (2.22) if we assume that $g \in C^{0,\alpha}_{\nu-2,0}(B)$ and $h \in C^{2,\alpha}(\partial B)$, using Schauder estimates and a scaling argument it is possible to show that the solution ϕ found above satisfies (2.12). □

COROLLARY 2.2. *Assume* (2.9), (2.10), (2.11) *and that* $\nu \ge 0$. *If* $|x|^2 g$ *is continuous at the origin, then so is* ϕ.

PROOF. Let (α_n) denote an arbitrary sequence of real numbers converging to zero, $\tilde{g}(x) = |x|^2 g(x)$ and $\phi_n(x) = \phi(\alpha_n x)$ for $x \in B_{1/\alpha_n}(0)$. Then ϕ_n solves

$$-\Delta \phi_n - \frac{c}{|x|^2} \phi_n = \frac{\tilde{g}(\alpha_n x)}{|x|^2} \quad \text{in } B_{1/\alpha_n}(0).$$

Also, (ϕ_n) is uniformly bounded so that up to a subsequence, it converges in the topology of $C^{1,\alpha}(\mathbb{R}^N \setminus \{0\})$ to a bounded solution Φ of

$$-\Delta \Phi - \frac{c}{|x|^2} \Phi = \frac{\tilde{g}(0)}{|x|^2} \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

Now $\Phi + \tilde{g}(0)/c$ is bounded and solves (2.2), so it must be identically zero. It follows that the whole sequence (ϕ_n) converges to $-\tilde{g}(0)/c$. Let now (x_n) denote an arbitrary sequence of points in $\mathbb{R}^N$ converging to 0 and $\alpha_n = |x_n|$. Then, $\phi(x_n) = \phi_n(\frac{x_n}{|x_n|})$ and up to a subsequence, $\phi(x_n) \to -\tilde{g}(0)/c$. Again, since the limit of such a subsequence is unique, the whole sequence converges. □

Now we would like to consider a potential which is the inverse square to a point $\xi \in B_{1/2}$, that is, we consider the problem

$$\begin{cases} -\Delta\phi - \dfrac{c}{|x-\xi|^2}\phi = \dfrac{g}{|x-\xi|^2} + \mu_0 \dfrac{g_0}{|x-\xi|^2} + \displaystyle\sum_{k=1}^{k_1}\sum_{l=1}^{m_k} \mu_{k,l} V_{k,l,\xi} & \text{in } B \\ \phi = h & \text{on } \partial B, \end{cases} \tag{2.24}$$

where

$$V_{k,l,\xi}(x) = \eta(|x-\xi|) W_{k,l}\left(\frac{x-\xi}{1-2\varepsilon_0}\right) \quad \text{for } k \geq 1, l = 1, \ldots, m_k, \tag{2.25}$$

$\eta \in C^\infty(\mathbb{R})$ such that $0 \leq \eta \leq 1$, $\eta \not\equiv 0$ and $\operatorname{supp}(\eta) \subset [\frac{1}{4}, \frac{1}{2}]$ and $\varepsilon_0 > 0$ is fixed (suitably small).

We have:

PROPOSITION 2.3. *Assume*

$$\exists k_1 \text{ such that } \quad \alpha_{k_1}^- \in \mathbb{R} \quad \text{and} \quad -\alpha_{k_1}^- < \nu < -\alpha_{k_1+1}^-. \tag{2.26}$$

Then there exists $\varepsilon_0 > 0$ such that if $|\xi| < \varepsilon_0$ and $g_0 \in C^{0,\alpha}_{\nu,\xi}(B)$ satisfies

$$\|g_0 - 1\|_{L^\infty(B)} < \varepsilon_0,$$

then given any $g \in C^{0,\alpha}_{\nu,\xi}(B)$ and $h \in C^{2,\alpha}(\partial B)$, there exist unique $\phi \in C^{2,\alpha}_{\nu,\xi}(B)$ and $\mu_0, \mu_{k,l} \in \mathbb{R}$ ($k = 1, \ldots, k_1$, $l = 1, \ldots, m_k$) solution to (2.24). *Moreover we have for some constant $C > 0$ independent of g and h*

$$\|\phi\|_{2,\alpha,\nu,\xi;B} + |\mu_0| + \sum_{k=1}^{k_1}\sum_{l=1}^{m_k} |\mu_{k,l}| \leq C(\|g\|_{0,\alpha,\nu,\xi;B} + \|h\|_{C^{2,\alpha}(\partial B)}). \tag{2.27}$$

PROOF. We work with $0 < |\xi| < \varepsilon_0$, where $\varepsilon_0 \in (0, 1/2)$ is going to be fixed later on, small enough. Let $R = 1 - 2\varepsilon_0$. This implies in particular that $B_R(\xi) \subset B$.

We define an operator $T_1 : C^{2,\alpha}(\partial B_R(\xi)) \to C^{1,\alpha}(\partial B_R(\xi)) \times \mathbb{R}$ as follows: given $\phi_0 \in C^{2,\alpha}(\partial B_R(\xi))$, find $\phi \in C^{2,\alpha}_{\nu,\xi}(B_R(\xi))$ and $\gamma_0, \gamma_{k,l}$ the unique solution to

$$\begin{cases} -\Delta\phi_1 - \dfrac{c}{|x-\xi|^2}\phi_1 = \gamma_0 \dfrac{g_0}{|x-\xi|^2} + \displaystyle\sum_{k=1}^{k_1}\sum_{l=1}^{m_k} \gamma_{k,l} V_{k,l,\xi} & \text{in } B_R(\xi) \\ \phi_1 = \phi_0 & \text{on } \partial B_R(\xi), \end{cases} \tag{2.28}$$

and set $T_1(\phi_0) = (\frac{\partial\phi_1}{\partial n}, \gamma_0)$. This can be done (see Step 1 below) by adjusting the constants γ_0 and $\gamma_{k,l}$ in such a way that the orthogonality relations (2.11) in Lemma 2.1 are satisfied. Similarly, there is a unique $\tilde\phi_1 \in C^{2,\alpha}_{\nu,\xi}(B_R(\xi))$ and $\tilde\gamma_0$, $\tilde\gamma_{k,l}$ such that

$$\begin{cases} -\Delta\tilde\phi_1 - \dfrac{c}{|x-\xi|^2}\tilde\phi_1 = \dfrac{g}{|x-\xi|^2} + \tilde\gamma_0\dfrac{g_0}{|x-\xi|^2} \\ \qquad\qquad + \displaystyle\sum_{k=1}^{k_1}\sum_{l=1}^{m_k}\tilde\gamma_{k,l}V_{k,l,\xi} & \text{in } B_R(\xi) \\ \tilde\phi_1 = 0 & \text{on } \partial B_R(\xi). \end{cases} \tag{2.29}$$

Given $\tilde\phi_1$, $\tilde\gamma_0$ as in (2.29), we define $\tilde\phi_2$ by

$$\begin{cases} -\Delta\tilde\phi_2 - \dfrac{c}{|x-\xi|^2}\tilde\phi_2 = \dfrac{g}{|x-\xi|^2} + \tilde\gamma_0\dfrac{g_0}{|x-\xi|^2} & \text{in } B\setminus B_R(\xi) \\ \dfrac{\partial\tilde\phi_2}{\partial n} = \dfrac{\partial\tilde\phi_1}{\partial n} & \text{on } \partial B_R(\xi) \\ \tilde\phi_2 = h & \text{on } \partial B. \end{cases} \tag{2.30}$$

We also define an operator $T_2 : C^{1,\alpha}(\partial B_R(\xi)) \times \mathbb{R} \to C^{2,\alpha}(\partial B_R(\xi))$ by

$$T_2(\Psi, \gamma_0) = \phi_2|_{\partial B_R(\xi)},$$

where ϕ_2 is the solution to

$$\begin{cases} -\Delta\phi_2 - \dfrac{c}{|x-\xi|^2}\phi_2 = \gamma_0\dfrac{g_0}{|x-\xi|^2} & \text{in } B\setminus B_R(\xi) \\ \dfrac{\partial\phi_2}{\partial n} = \Psi & \text{on } \partial B_R(\xi) \\ \phi_2 = 0 & \text{on } \partial B. \end{cases} \tag{2.31}$$

As we shall see later (see Step 2), equations (2.30) and (2.31) possess indeed a unique solution if ξ is sufficiently small, because the domain $B \setminus B_R(\xi)$ is small.

We construct a solution ϕ of (2.24) as follows: choose $\phi_0 \in C^{2,\alpha}(\partial B_R(\xi))$, let ϕ_1 be the solution to (2.28) and let ϕ_2 be the solution to (2.31) with $\Psi = \frac{\partial\phi_1}{\partial n}$ and γ_0 from problem (2.28). Then set

$$\phi = \begin{cases} \phi_1 + \tilde\phi_1 & \text{in } B_R(\xi) \\ \phi_2 + \tilde\phi_2 & \text{in } B\setminus B_R(\xi), \end{cases}$$

and $\mu_0 = \gamma_0 + \tilde\gamma_0$, $\mu_{k,l} = \gamma_{k,l} + \tilde\gamma_{k,l}$. If we have in addition

$$\phi_1 + \tilde\phi_1 = \phi_2 + \tilde\phi_2 \qquad \text{on } \partial B_R(\xi), \tag{2.32}$$

then ϕ, μ_0 and $\mu_{k,l}$ form a solution to (2.24).

With this notation, solving equation (2.24) thus reduces to finding $\phi_0 \in C^{2,\alpha}(\partial B_R(\xi))$ such that (2.32) holds i.e.

$$T_2 \circ T_1(\phi_0) + \widetilde{\phi}_2 = \phi_0 \quad \text{in } \partial B_R(\xi).$$

The fact that this equation is uniquely solvable (when ξ is small) will follow once we show that $\|T_2\| \to 0$ as $\varepsilon_0 \to 0$, while $\|T_1\|$ remains bounded.

Step 1. Given $\phi_0 \in C^{2,\alpha}(\partial B_R(\xi))$ there exist γ_0 and $\gamma_{k,l}$ such that (2.28) has a unique solution ϕ_1 in $C^{2,\alpha}_{\nu,\xi}(\partial B_R(\xi))$.

In this step we change variables $y = x - \xi$ and work in $B_R(0)$. Solving for γ_0 in the orthogonality relations (2.11) yields

$$\gamma_0 = \frac{\frac{1}{R}\int_{\partial B_R} \phi_0 \frac{\partial W_{0,0}}{\partial n}(\frac{y}{R})}{\int_{B_R} g_0(y+\xi)|y|^{-2} W_{0,0}(\frac{y}{R})} \tag{2.33}$$

and a computation, using $\|g_0 - 1\|_{L^\infty(B_R)} < \varepsilon_0$ shows that

$$\int_{B_R} g_0(y+\xi)|y|^{-2} W_{0,0}\left(\frac{y}{R}\right) = R^{\nu+N-2} C(N,c) + O(\varepsilon_0),$$

where $C(N,c) \neq 0$. In particular this integral remains bounded away from zero as $R \to 1$ ($R = 1 - 2\varepsilon_0$ and $\varepsilon_0 \to 0$) and hence γ_0 stays bounded.

Regarding $\gamma_{k,l}$ we have

$$\gamma_{k,l} = \frac{\frac{1}{R}\int_{\partial B_R(0)} \phi_0 \frac{\partial W_{k,l}}{\partial n}(\frac{y}{R}) - \gamma_0 \int_{B_R} g_0(y+\xi)|y|^{-2} W_{k,l}(\frac{y}{R})}{\int_{B_R} \eta(|y|) W_{k,l}(\frac{y}{R})^2}, \tag{2.34}$$

and we observe that $\int_{B_R} \eta(|y|) W_{k,l}(\frac{y}{R})^2$ is a positive constant depending on k, l and R (which stays bounded away from zero as $R \to 1$). Using Lemma 2.1, it follows that $\|T_1\|$ remains bounded as $R \to 1$ i.e. when $\varepsilon_0 \to 0$.

Step 2. For ξ small enough equation (2.31) is uniquely solvable and $\|T_2\| \le C|\xi|$. Let $z_0 = 1 - |x|^2$. Then $z_0(|\gamma_0| \sup_{B \setminus B_R(\xi)} \frac{|g_0|}{|x-\xi|^2} + \sup_{\partial B_R(\xi)} |\Psi|))$ is a positive supersolution of (2.31). This shows that this equation is solvable and that for its solution ϕ_2 we have the estimate $|\phi_2| \le C|\xi|(|\gamma_0| + \sup_{\partial B_R(\xi)} |\Psi|)$. This and Schauder estimates yield $\|\phi_2\|_{C^{2,\alpha}(\partial B_R(\xi))} \le C|\xi|(|\gamma_0| + \|\Psi\|_{C^{2,\alpha}(\partial B_R(\xi))})$, which is the desired estimate.

Finally, estimate (2.27) follows from (2.12) and formulas (2.33), (2.34). □

Consider each $\xi \in B_{\varepsilon_0}$ functions $g_0(\cdot,\xi)$, $g(\cdot,\xi) \in C^{0,\alpha}_{\nu,\xi}(B)$ and $h(\cdot,\xi) \in C^{2,\alpha}(\partial B)$. By Proposition 2.3 there is a unique $\phi(\cdot,\xi) \in C^{2,\alpha}_{\nu,\xi}(B)$ solution to (2.24). We want to investigate the differentiability properties of the map $\xi \mapsto \phi(\cdot,\xi)$.

PROPOSITION 2.4. *Assume the following conditions:*

$$\exists k_1 \text{ such that} \quad \alpha^-_{k_1} \in \mathbb{R} \quad \text{and} \quad -\alpha^-_{k_1} < \nu < -\alpha^-_{k_1+1},$$

$$\nu > -\frac{N}{2} + 2 \tag{2.35}$$

and

$$\nu - 1 \neq -\alpha_{k_1}^{-}.$$

Let $\varepsilon_0 > 0$ *and for* $\xi \in B_{\varepsilon_0}$, *let* $g_0(\cdot, \xi)$, $g(\cdot, \xi)$ *be such that*

$$A_0 \equiv \sup_{\xi \in B_{\varepsilon_0}} (\|g_0(\cdot, \xi)\|_{1,\alpha,\nu,\xi;B} + \|D_\xi g_0(\cdot, \xi)\|_{0,\alpha,\nu-1,\xi;B}) < \infty \tag{2.36}$$

and

$$A \equiv \sup_{\xi \in B_{\varepsilon_0}} (\|g(\cdot, \xi)\|_{1,\alpha,\nu,\xi;B} + \|D_\xi g(\cdot, \xi)\|_{0,\alpha,\nu-1,\xi;B}) < \infty.$$

Let $h(\cdot, \xi) \in C^{3,\alpha}(\partial B)$ *with*

$$\sup_{\xi \in B_{\varepsilon_0}} (\|h(\cdot, \xi)\|_{C^3(\partial B)} + \|D_\xi h(\cdot, \xi)\|_{C^{2,\alpha}(\partial B)}) < \infty.$$

Let $\phi(\cdot, \xi)$ *denote the solution to* (2.24). *Then there exists* $\bar{\varepsilon}_0 > 0$ *and a constant* C *such that if* $\varepsilon_0 < \bar{\varepsilon}_0$ *and if* $\|g_0(\cdot, \xi) - 1\|_{L^\infty(B)} < \varepsilon_0$, $|t| < \varepsilon_0$ *and* $\xi_1, \xi_2 \in B_{\varepsilon_0}$ *then*

$$\|\phi(\cdot + \xi_2, \xi_2) - \phi(\cdot + \xi_1, \xi_1)\|_{2,\alpha,\nu-1,0;B_{1/2}} \leq C|\xi_2 - \xi_1|. \tag{2.37}$$

Moreover the map $\xi \in B_{\varepsilon_0} \mapsto \phi(\cdot; \xi)$ *is differentiable in the sense that*

$$D_\xi \phi(x, \xi)\eta = \lim_{\tau \to 0} \frac{1}{\tau}(\phi(x, \xi + \tau\eta) - \phi(x, \xi)) \quad \textit{exists for all } x \in B \setminus \{\xi\} \tag{2.38}$$

and $\eta \in \mathbb{R}^N$. *Furthermore* $D_\xi \phi(\cdot, \xi) \in C^{2,\alpha}_{\nu-1,\xi}(B)$, *the maps* $\xi \in B_{\varepsilon_0} \mapsto \mu_0, \mu_{k,l} \in \mathbb{R}$ *are differentiable and*

$$\begin{aligned} &\|D_\xi \phi(\cdot, \xi)\|_{2,\alpha,\nu-1,\xi;B} + |D_\xi \mu_0| + \sum_{k=1}^{k_1} \sum_{l=1}^{m_k} |D_\xi \mu_{k,l}| \\ &\quad \leq C(\|g(\cdot, \xi)\|_{0,\alpha,\nu,\xi;B} + \|D_\xi g(\cdot, \xi)\|_{0,\alpha,\nu-1,\xi;B} \\ &\qquad + \|h(\cdot, \xi)\|_{C^{2,\alpha}(\partial B)} + \|D_\xi h(\cdot, \xi)\|_{C^{2,\alpha}(\partial B)}). \end{aligned} \tag{2.39}$$

The proof of this result can be found in [43] and we omit it. For simplicity we have stated Proposition 2.4 under the assumption $\nu - 1 \neq -\alpha_{k_1}^{-}$. A similar result also holds if $\nu - 1 = -\alpha_{k_1}^{-}$, but estimate (2.37) has to be replaced by:

$$\|\phi(\cdot + \xi_2, \xi_2) - \phi(\cdot + \xi_1, \xi_1)\|_{2,\alpha,\bar{\nu}-1,0;B_{1/2}} \leq C|\xi_2 - \xi_1|,$$

where $\nu - \delta < \bar{\nu} < \nu$ for some $\delta > 0$ and with the constant C now depending on $\bar{\nu}$. Similarly, (2.39) is replaced by

$$\begin{aligned} &\|D_\xi \phi(\cdot,\xi)\|_{2,\alpha,\bar{\nu}-1,\xi;B} + |D_\xi \mu_0| + \sum_{k=1}^{k_1}\sum_{l=1}^{m_k} |D_\xi \mu_{k,l}| \\ &\quad \le C(\|g(\cdot,\xi)\|_{0,\alpha,\bar{\nu},\xi;B} + \|D_\xi g(\cdot,\xi)\|_{0,\alpha,\bar{\nu}-1,\xi;B} \\ &\qquad + \|h(\cdot,\xi)\|_{C^{2,\alpha}(\partial B)} + \|D_\xi h(\cdot,\xi)\|_{C^{2,\alpha}(\partial B)}). \end{aligned} \tag{2.40}$$

Next we extend Proposition 2.3 to an operator of the form $-\Delta - L_t - \frac{c}{|x-\xi|^2}$, where L_t is a suitably small second-order differential operator. We will take L_t of the form

$$L_t w = a_{ij}(x,t) D_{ij} w + b_i(x,t) D_i w + c(x,t) w. \tag{2.41}$$

LEMMA 2.5. *Suppose that the coefficients of L_t satisfy: $a_{ij}(\cdot,t)$, $b_i(\cdot,t)$, $c_i(\cdot,t)$ are $C^\alpha(\overline{B})$ and for some C it holds*

$$\|a_{ij}(\cdot,t)\|_{C^\alpha(\overline{B})} + \|b_i(\cdot,t)\|_{C^\alpha(\overline{B})} + \|c(\cdot,t)\|_{C^\alpha(\overline{B})} \le C|t|.$$

Assume

$$\exists k_1 \text{ such that } \quad \alpha^-_{k_1} \in \mathbb{R} \quad \text{and} \quad -\alpha^-_{k_1} < \nu < -\alpha^-_{k_1+1}.$$

Then there exists $\varepsilon_0 > 0$ such that if $|\xi| < \varepsilon_0$, $|t| < \varepsilon_0$ and $g_0 \in C^{0,\alpha}_{\nu,\xi}(B)$ satisfies $\|g_0 - 1\|_{L^\infty(B)} < \varepsilon_0$, then given any $g \in C^{0,\alpha}_{\nu,\xi}(B)$ and $h \in C^{2,\alpha}(\partial B)$, there exist unique $\phi \in C^{2,\alpha}_{\nu,\xi}(B)$ and $\mu_0, \mu_{k,l} \in \mathbb{R}$ ($k = 1,\dots,k_1$, $l = 1,\dots,m_k$) solution to

$$\begin{cases} -\Delta\phi - L_t\phi - \dfrac{c}{|x-\xi|^2}\phi = \dfrac{g}{|x-\xi|^2} + \mu_0 \dfrac{g_0}{|x-\xi|^2} \\ \qquad\qquad + \displaystyle\sum_{k=1}^{k_1}\sum_{l=1}^{m_k} \mu_{k,l} V_{k,l,\xi} & \text{in } B \\ \phi = h & \text{on } \partial B. \end{cases} \tag{2.42}$$

Moreover

$$\|\phi\|_{2,\alpha,\nu,\xi;B} + |\mu_0| + \sum_{k=1}^{k_1}\sum_{l=1}^{m_k} |\mu_{k,l}| \le C(\|g\|_{0,\alpha,\nu,\xi;B} + \|h\|_{C^{2,\alpha}(\partial B)}). \tag{2.43}$$

PROOF. Fix $h \in C^{2,\alpha}(\partial B)$ and $|\xi| < \varepsilon_0$, where ε_0 is the constant appearing in Proposition 2.3. For $g \in C^{0,\alpha}_{\nu,\xi}(B)$ let $\phi = T(g/|x-\xi|^2)$ be the solution to (2.24) as defined in Proposition 2.3. Then (2.42) is equivalent to $\phi = T(g/|x-\xi|^2 + L_t\phi)$. Define

$$\widetilde{T}(\phi) = T(g/|x-\xi|^2 + L_t\phi).$$

We apply the fixed point theorem to the operator $\tilde{T}$ in a ball $\mathcal{B}_R$ of the Banach space $C^{2,\alpha}_{\nu,\xi}(B)$ equipped with the norm $\|\cdot\|_{2,\alpha,\nu,\xi;B}$.

Note that by Proposition 2.3 we have $\|T(g/|x-\xi|^2)\|_{2,\alpha,\nu,\xi;B} \le C(\|g\|_{0,\alpha,\nu,\xi;B} + \|h\|_{C^{2,\alpha}(\partial B)})$. Using this inequality, for $\|\phi\|_{2,\alpha,\nu,\xi;B} \le R$ we have

$$\begin{aligned}\|\widetilde{T}(\phi)\|_{2,\alpha,\nu,\xi;B} &\le C(\|g\|_{0,\alpha,\nu,\xi;B} + \|L_t\phi\|_{0,\alpha,\nu-2,\xi;B} + \|h\|_{C^{2,\alpha}(\partial B)}) \\ &\le C(\|g\|_{0,\alpha,\nu,\xi} + |t|R + \|h\|_{C^{2,\alpha}(\partial B)}) \le R,\end{aligned}$$

if we first take t so small that $C|t| \le \frac{1}{2}$, and then choose R so large that $C(\|g\|_{0,\alpha,\nu,\xi;B} + \|h\|_{C^{2,\alpha}(\partial B)}) \le \frac{R}{2}$.

For $\|\phi_1\|_{2,\alpha,\nu,\xi;B} \le R$, $\|\phi_2\|_{2,\alpha,\nu,\xi;B} \le R$ we have

$$\begin{aligned}\|\widetilde{T}(\phi_1) - \widetilde{T}(\phi_2)\|_{2,\alpha,\nu,\xi;B} &\le C\|L_t(\phi_1-\phi_2)\|_{0,\alpha,\nu-2,\xi;B} \\ &\le C|t|\,\|\phi_1-\phi_2\|_{2,\alpha,\nu,\xi;B},\end{aligned}$$

and we see that $\widetilde{T}$ is a contraction on the ball $\mathcal{B}_R$ of $C^{2,\alpha}_{\nu,\xi}(B)$ if t is chosen small enough. □

The previous results on differentiability also hold for perturbed operators of the form $-\Delta - L_t - \frac{c}{|x-\xi|^2}$.

PROPOSITION 2.6. *Assume the following conditions:*

$$\exists k_1 \text{ such that} \quad \alpha^-_{k_1} \in \mathbb{R} \quad \text{and} \quad -\alpha^-_{k_1} < \nu < -\alpha^-_{k_1+1}$$

$$\nu > -\frac{N}{2} + 2,$$

and

$$\nu - 1 \ne -\alpha^-_{k_1}. \tag{2.44}$$

Let $\varepsilon_0 > 0$ *and for* $\xi \in B_{\varepsilon_0}$ *let* $g_0(\cdot,\xi), g(\cdot,\xi) \in C^{1,\alpha}_{\nu,\xi}(B)$ *be such that*

$$A_0 \equiv \sup_{\xi\in B_{\varepsilon_0}} (\|g_0(\cdot,\xi)\|_{1,\alpha,\nu,\xi;B} + \|D_\xi g_0(\cdot,\xi)\|_{0,\alpha,\nu-1,\xi;B}) < \infty$$

and

$$A \equiv \sup_{\xi\in B_{\varepsilon_0}} (\|g(\cdot,\xi)\|_{1,\alpha,\nu,\xi;B} + \|D_\xi g(\cdot,\xi)\|_{0,\alpha,\nu-1,\xi;B}) < \infty.$$

On the operator L_t *we assume*

$$\|a_{ij}(\cdot,t)\|_{C^{1,\alpha}(\overline{B})} + \|b_i(\cdot,t)\|_{C^{1,\alpha}(\overline{B})} + \|c(\cdot,t)\|_{C^{1,\alpha}(\overline{B})} \le C|t|.$$

Let $h(\cdot,\xi) \in C^{3,\alpha}(\partial B)$ *with*

$$\sup_{\xi\in B_{\varepsilon_0}} (\|h(\cdot,\xi)\|_{C^3(\partial B)} + \|D_\xi h(\cdot,\xi)\|_{C^{2,\alpha}(\partial B)}) < \infty$$

and let $\phi(\cdot,\xi)$ *denote the solution to* (2.42). *Then there exist* $\bar{\varepsilon}_0 > 0$, $C > 0$ *such that if* $\varepsilon_0 < \bar{\varepsilon}_0$, $\|g_0(\cdot,\xi) - 1\|_{L^\infty(B)} < \varepsilon_0$, $|t| < \varepsilon_0$ *and* $\xi_1, \xi_2 \in B_{\varepsilon_0}$, *we have*

$$\|\phi(\cdot + \xi_2, \xi_2) - \phi(\cdot + \xi_1, \xi_1)\|_{2,\alpha,\nu-1,0;B_{1/2}} \le C|\xi_2 - \xi_1|. \tag{2.45}$$

Furthermore,

$$D_\xi \phi(x;\xi)\eta = \lim_{t\to 0} \frac{1}{t}(\phi(x;\xi + t\eta) - \phi(x;\xi)) \quad \textit{exists } \forall x \in B \setminus \{\xi\}, \forall \eta \in \mathbb{R}^N,$$

the maps $\xi \in B_{\varepsilon_0}(0) \mapsto \mu_0, \mu_{k,l} \in \mathbb{R}$ *are differentiable and*

$$\begin{aligned} \|D_\xi \phi(x;\xi)\|_{2,\alpha,\nu-1,\xi;B} \le C(&\|g(\cdot,\xi)\|_{0,\alpha,\nu,\xi;B} + \|D_\xi g(\cdot,\xi)\|_{0,\alpha,\nu-1,\xi;B} \\ &+ \|h(\cdot,\xi)\|_{C^{2,\alpha}(\partial B)} + \|D_\xi h(\cdot,\xi)\|_{C^{2,\alpha}(\partial B)}). \end{aligned} \tag{2.46}$$

The argument uses again the fixed point theorem. Details can be found in [43].

2.2. *Perturbation of singular solutions*

Recall that $c^* = 2(N-2)$. Hence, if $N \ge 4$ then $N - 1 < c^* < 2N$ and therefore $\alpha_1^- > 0$, $\alpha_2^- < 0$ (cf. (2.4)). As mentioned before we choose $\nu = 0$. We see that (2.26) holds now with $k_1 = 1$. We may thus apply Proposition 2.3 and Lemma 2.5. In dimension $N \ge 5$, since (2.35) and (2.44) hold, we may also apply Propositions 2.4 and 2.6.

Write

$$V_{\ell,\xi} := V_{1,\ell,\xi} \quad \ell = 1, \dots, N,$$

where $V_{1,\ell,\xi}$ is defined in (2.25), and set

$$\tilde{f}(x,t) = L_t\left(\log \frac{1}{|x-\xi|^2}\right)$$

and note that

$$\|\tilde{f}(x,t)|x-\xi|^2\|_{0,\alpha,-2,\xi} \le C|t|. \tag{2.47}$$

Concerning (1.25) we prove:

LEMMA 2.7. *Write* $c = c^* = 2(N-2)$. *Then there exists* $\varepsilon_0 > 0$ *such that if* $|\xi| < \varepsilon_0$, $|t| < \varepsilon_0$, *there exist* $\phi \in C^{2,\alpha}_{0,\xi}(B)$ *and* $\mu_0, \dots, \mu_N \in \mathbb{R}$ *such that*

$$\begin{cases} -\Delta\phi - L_t\phi - \dfrac{c}{|x-\xi|^2}\phi = \dfrac{c}{|x-\xi|^2}(e^\phi - 1 - \phi) + \mu_0 \dfrac{1}{|x-\xi|^2} e^\phi \\ \qquad\qquad + \tilde{f}(x,t) + \displaystyle\sum_{i=1}^N \mu_i V_{i,\xi} \quad \textit{in } B \\ \phi = -\log \dfrac{1}{|x-\xi|^2} \quad \textit{on } \partial B. \end{cases} \tag{2.48}$$

If $N \geq 5$, we have in addition that:

- *the map $\xi \in B_{\varepsilon_0} \mapsto \phi(\cdot, \xi)$ is differentiable in the sense that*

$$D_\xi \phi(x,\xi)\eta = \lim_{\tau \to 0} \frac{1}{\tau}(\phi(x, \xi + \tau\eta) - \phi(x,\xi)) \quad \textit{exists for all } x \in B \setminus \{\xi\}$$

and $\eta \in \mathbb{R}^N$.

- *for $\bar\nu < 0$ small, $D_\xi \phi(\cdot,\xi) \in C^{2,\alpha}_{\bar\nu - 1,\xi}(B)$, the maps $\xi \in B_{\varepsilon_0} \mapsto \mu_0, \mu_i \in \mathbb{R}$ are differentiable and there exists a constant C independent of ξ such that*

$$\|D_\xi \phi(\cdot,\xi)\|_{2,\alpha,\bar\nu-1,\xi;B} + |D_\xi \mu_0| + \sum_{k=1}^{k_1} \sum_{l=1}^{m_k} |D_\xi \mu_{k,l}| \leq C. \tag{2.49}$$

PROOF. *Case $N \geq 5$.*

Let ε_0 be as in Lemma 2.5. Consider the Banach space X of functions $\phi(x,\xi)$ defined for $x \in B$, $\xi \in B_{\varepsilon_0}$, which are twice continuously differentiable with respect to x and once with respect to ξ for $x \neq \xi$ for which the following norm is finite

$$\|\phi\|_X = \sup_{\xi \in B_{\varepsilon_0}} \|\phi(\cdot,\xi)\|_{2,\alpha,0,\xi;B} + \lambda \|D_\xi \phi(\cdot,\xi)\|_{2,\alpha,\bar\nu-1,\xi;B},$$

where $\lambda > 0$ is a parameter to be fixed later on and $\bar\nu < 0$ is close to zero.

Let $\mathcal{B}_R = \{\phi \in X \mid \|\phi\|_X \leq R\}$. Using Lemma 2.5 we may define a nonlinear map $F : \mathcal{B}_R \to X$ by $F(\psi) = \phi$, where $\phi(\cdot,\xi)$ is the solution to (2.42) with

$$g = c(e^\psi - 1 - \psi) + |x-\xi|^2 \tilde f(x,t), \quad g_0 = e^\psi, \quad h = -\log \frac{1}{|x-\xi|^2}. \tag{2.50}$$

We shall choose later on $R > 0$ small. Observe that in Lemma 2.5 the constants C in (2.43) and ε_0 associated to $g_0 = e^\psi$, stay bounded and bounded away from zero respectively as we make R smaller, since $e^{-R} \leq e^\psi \leq e^R$ for $\psi \in \mathcal{B}_R$.

Let us show that if t is small then one can choose R small and $\lambda > 0$ small so that $F : \mathcal{B}_R \to \mathcal{B}_R$. Indeed, let $\psi \in \mathcal{B}_R$ and $\phi = F(\psi)$. Then by (2.43), (2.47) we have

$$\begin{aligned} \|\phi\|_{2,\alpha,0,\xi;B} &\leq C(\|c(e^\psi - 1 - \psi) + |x-\xi|^2 \tilde f(x,t)\|_{0,\alpha,0,\xi;B} + |\xi|) \\ &\leq C(R^2 + |t| + |\xi|) < \frac{R}{2}, \end{aligned} \tag{2.51}$$

provided R is first taken small enough and then $|t|$ and $|\xi| < \varepsilon_0$ are chosen small. Similarly, recalling (2.40),

$$\begin{aligned} &\|D_\xi \phi\|_{2,\alpha,\bar\nu-1,\xi;B} \\ &\quad \leq C(\|c(e^\psi - 1 - \psi) + |x-\xi|^2 \tilde f(x,t)\|_{0,\alpha,0,\xi;B} \\ &\qquad + \|c D_\xi(e^\psi - 1 - \psi) + D_\xi(|x-\xi|^2 \tilde f(x,t))\|_{0,\alpha,\bar\nu-1,\xi;B} + 1) \\ &\quad \leq C\left(R^2 + t + \frac{R^2}{\lambda} + 1\right) \leq \frac{R}{2\lambda}, \end{aligned}$$

if we choose now λ small enough.

Next we show that F is a contraction on $\mathcal{B}_R$. Let $\psi_1,\ \psi_2 \in \mathcal{B}_R$ and $\phi_\ell = F(\psi_\ell)$, $\ell = 1, 2$. Let $\mu_i^{(\ell)}$, $i = 0, \ldots, N$ be the constants in (2.42) associated with ψ_ℓ. By (2.43) and repeating the calculation in (2.51)

$$\sum_{i=0}^{N} |\mu_i^{(\ell)}| \le R. \tag{2.52}$$

Let $\phi = \phi_1 - \phi_2$. Then ϕ satisfies

$$\begin{cases} -\Delta\phi - L_t\phi - \dfrac{c}{|x-\xi|^2}\phi = c\left(\dfrac{e^{\psi_1} - 1 - \psi_1}{|x-\xi|^2} - \dfrac{e^{\psi_2} - 1 - \psi_2}{|x-\xi|^2}\right) \\ \qquad + \mu_0^{(2)}\dfrac{e^{\psi_1} - e^{\psi_2}}{|x-\xi|^2} + (\mu_0^{(1)} - \mu_0^{(2)})\dfrac{e^{\psi_1}}{|x-\xi|^2} \\ \qquad + \displaystyle\sum_{i=1}^{N}(\mu_i^{(1)} - \mu_i^{(2)})V_{i,\xi} \qquad \text{in } B \\ \phi = 0 \qquad \text{on } \partial B. \end{cases} \tag{2.53}$$

Apply (2.43) with $g_0 = \frac{e^{\psi_1}}{|x-\xi|^2}$, $h = 0$ and

$$g := c\left(\frac{e^{\psi_1} - 1 - \psi_1}{|x-\xi|^2} - \frac{e^{\psi_2} - 1 - \psi_2}{|x-\xi|^2}\right) + \mu_0^{(2)}\frac{e^{\psi_1} - e^{\psi_2}}{|x-\xi|^2}, \tag{2.54}$$

to conclude that

$$\|\phi\|_{2,\alpha,0,\xi} + \sum_{i=0}^{N} |\mu_i^{(1)} - \mu_i^{(2)}| \le C\|g\|_{0,\alpha,0,\xi}. \tag{2.55}$$

Using (2.52), we have in particular that $|\mu_0^{(2)}| \le R$ and it follows from (2.54) and (2.55) that

$$\|\phi_1 - \phi_2\|_{2,\alpha,0,\xi} \le CR\|\psi_1 - \psi_2\|_{2,\alpha,0,\xi}. \tag{2.56}$$

Thanks to (2.46) we also have the bound

$$\begin{aligned} \|D_\xi(\phi_1 - \phi_2)\|_{1,\alpha,\bar{\nu}-1,\xi;B} &\le C\big(\|e^{\psi_1} - \psi_1 - (e^{\psi_1} - \psi_2)\|_{0,\alpha,0,\xi;B} \\ &\qquad + \|D_\xi(e^{\psi_1} - \psi_1 - (e^{\psi_1} - \psi_2))\|_{0,\alpha,\bar{\nu}-1,\xi;B}\big) \\ &\le CR\|\psi_1 - \psi_2\|_{2,\alpha,0,\nu;B} \\ &\qquad + CR\|D_\xi(\psi_1 - \psi_2)\|_{0,\alpha,\bar{\nu}-1,\xi;B}. \end{aligned} \tag{2.57}$$

Combining (2.56), (2.57) we obtain

$$\|F(\psi_1) - F(\psi_2)\|_X \le CR\|\psi_1 - \psi_2\|_X.$$

This shows that F is a contraction if R is taken small enough.

Case $N = 4$. In this case (2.35) fails for $\nu = 0$ and estimates like (2.45) or (2.46) may not hold. So we work with the Banach space X of functions $\phi(x, \xi)$ which are twice continuously differentiable with respect to x and continuous with respect to ξ for $x \neq \xi$, for which the norm

$$\|\phi\|_X = \sup_{\xi \in B_{\varepsilon_0}} \|\phi(\cdot, \xi)\|_{2,\alpha,0,\xi;B}$$

is finite. Working as in the previous case, we easily obtain that F is a contraction on some ball $\mathcal{B}_R$ of X. □

Proof of Theorem 1.11. We define the map $(\xi, t) \mapsto \phi(\xi, t)$ as the small solution to (2.48) constructed in Lemma 2.7 for t, ξ small. We need to show that for t small enough there is a choice of ξ such that $\mu_i = 0$ for $i = 1, \ldots, N$. Let

$$\widehat{V}_j(x; \xi) = W_{1,j}(x - \xi)\eta_1(|x - \xi|), \quad j = 0, \ldots, N, \tag{2.58}$$

where $\eta_1 \in C^\infty(\mathbb{R})$ is a cut-off function such that $0 \le \eta_1 \le 1$,

$$\begin{cases} \eta_1(r) = 0 & \text{for } r \le \frac{1}{8}, \\ \eta_1(r) = 1 & \text{for } r \ge \frac{1}{4}. \end{cases} \tag{2.59}$$

Multiplication of (2.48) by $\widehat{V}_j(x; \xi)$ and integration in B gives

$$\begin{aligned} &\int_B \left(-\Delta \widehat{V}_j(x;\xi) - L_t \widehat{V}_j(x;\xi) - \frac{c}{|x-\xi|^2} \widehat{V}_j(x;\xi) \right) \phi \\ &\quad + \int_{\partial B} \log \frac{1}{|x-\xi|^2} \frac{\partial \widehat{V}_j(x;\xi)}{\partial n} - \int_{\partial B} \frac{\partial \phi}{\partial n} \widehat{V}_j(x;\xi) \\ &\quad = \int_B \frac{c}{|x-\xi|^2} (e^\phi - 1 - \phi) \widehat{V}_j(x;\xi) + \mu_0 \int_B \frac{e^\phi}{|x-\xi|^2} \widehat{V}_j(x;\xi) \\ &\qquad + \int_B \tilde{f}(x,t) \widehat{V}_j(x;\xi) + \sum_{i=1}^N \mu_i \int_B V_{i,\xi} \widehat{V}_j(x;\xi). \end{aligned}$$

When $\xi = 0$ the matrix $A = A(\xi)$ defined by

$$A_{i,j}(\xi) = \int_B V_{i,\xi} \widehat{V}_j(x;\xi) \quad \text{for } i, j = 1 \ldots N$$

is diagonal and invertible and by continuity it is still invertible for small ξ. Thus, we see that $\mu_i = 0$ for $i = 1, \ldots, N$ if and only if

$$H_j(\xi, t) = 0, \quad \forall j = 1, \ldots, N, \tag{2.60}$$

where, given $j = 1, \dots, N$,

$$\begin{aligned} H_j(\xi, t) = &\int_B \frac{c}{|x-\xi|^2}(e^\phi - 1 - \phi)\widehat{V}_j(x;\xi) + \mu_0 \int_B \frac{e^\phi}{|x-\xi|^2}\widehat{V}_j(x;\xi) \\ &+ \int_B \tilde{f}(x,t)\widehat{V}_j(x;\xi) - \int_{\partial B} \log\frac{1}{|x-\xi|^2}\frac{\partial \widehat{V}_j(x;\xi)}{\partial n} \\ &+ \int_{\partial B} \frac{\partial\phi}{\partial n}\widehat{V}_j(x;\xi) - \int_B \Big(-\Delta \widehat{V}_j(x;\xi) - L_t \widehat{V}_j(x;\xi) \\ &- \frac{c}{|x-\xi|^2}\widehat{V}_j(x;\xi)\Big)\phi. \end{aligned}$$

If this holds, then $\mu_1(\xi,t) = \cdots = \mu_N(\xi,t) = 0$ and $\phi(\xi,t)$ is the desired solution to (1.25) (with μ in (1.25) equal to $\mu_0(\xi,t)$).

Observe that

$$\begin{aligned} &\frac{\partial}{\partial \xi_k}\Big[\int_{\partial B} \log\frac{1}{|x-\xi|^2}\frac{\partial \widehat{V}_j(x;\xi)}{\partial n}\Big]_{\xi=0} \\ &\quad = 2\int_{\partial B} x_k \frac{\partial \widehat{V}_j(x;0)}{\partial n} + \int_{\partial B} \log\frac{1}{|x-\xi|^2}\frac{\partial}{\partial \xi_k}\frac{\partial \widehat{V}_j(x;\xi)}{\partial n}\Big|_{\xi=0} \\ &\quad = 2\int_{\partial B} x_k \frac{\partial \widehat{V}_j(x;0)}{\partial n}. \end{aligned} \tag{2.61}$$

For $j = 1, \dots, N$ we have $W_{1,j}(x) = (|x|^{-\alpha_j^+} - |x|^{-\alpha_j^-})\varphi_j(\frac{x}{|x|})$ for $x \in \partial B$, and hence $\frac{\partial W_{1,j}}{\partial n}(x) = (\alpha_j^- - \alpha_j^+)\varphi_j(x) = \frac{\alpha_j^- - \alpha_j^+}{(\int_{S^{N-1}} x_j^2)^{1/2}} x_j$.

Case $N \ge 5$. By Lemma 2.7, $\phi(\cdot,\xi)$ is differentiable with respect to ξ. We may then compute the derivatives of the other terms of H_j. For instance

$$\frac{\partial}{\partial \xi_k}\int_B \frac{c}{|x-\xi|^2}(e^\phi - 1 - \phi)\widehat{V}_j(x;\xi)\Big|_{\xi=0,t=0} = 0$$

because the expression above is quadratic in ϕ and the computation can be justified using estimate (2.49).

Similarly

$$\frac{\partial}{\partial \xi_k}\Big[\mu_0\int_B \frac{e^\phi}{|x-\xi|^2}\widehat{V}_j(x;\xi)\Big]_{\xi=0} = 0.$$

Finally, using that $\phi|_{\xi=0} \equiv 0$ and integration by parts, we find

$$\begin{aligned} &\frac{\partial}{\partial \xi_k}\Big[\int_{\partial B_1} \frac{\partial\phi}{\partial n}\widehat{V}_j - \int_B \Big(-\Delta\widehat{V}_j - L_t\widehat{V}_j - \frac{c}{|x|^2}\widehat{V}_j\Big)\phi\Big]_{\xi=0,t=0} \\ &\qquad = \int_{\partial B} \frac{\partial \widehat{V}_j}{\partial n}\frac{\partial\phi}{\partial \xi_k} - \int_B \Big(-\Delta\frac{\partial\phi}{\partial \xi_k} - \frac{c}{|x|^2}\frac{\partial\phi}{\partial \xi_k}\Big)\widehat{V}_j. \end{aligned} \tag{2.62}$$

But when $\xi = 0$, $\frac{\partial \phi}{\partial \xi_k}$ satisfies

$$\begin{cases} -\Delta \dfrac{\partial \phi}{\partial \xi_k} - \dfrac{c}{|x|^2} \dfrac{\partial \phi}{\partial \xi_k} = \dfrac{\partial \mu_0}{\partial \xi_k} \dfrac{1}{|x|^2} + \displaystyle\sum_{i=1}^{N} \dfrac{\partial \mu_i}{\partial \xi_k} V_{i,0} & \text{in } B \\ \dfrac{\partial \phi}{\partial \xi_k} = 2x_k & \text{on } \partial B \end{cases} \tag{2.63}$$

since at $\xi = 0$, $\phi = 0$ and $\mu_i = 0$ for $0 \le i \le N$. By the conditions (2.11) we find $\frac{\partial \mu_0}{\partial \xi_k} = 0$ and

$$\frac{\partial \mu_i}{\partial \xi_k} = 2 \frac{\int_{\partial B} x_k \frac{\partial W_{1,i}}{\partial \nu}}{\int_B V_{i,0} W_{1,i}}, \quad 1 \le i \le N. \tag{2.64}$$

The integral above is zero whenever $i \neq k$ and thus, using (2.63), (2.64) in (2.62) we obtain

$$\frac{\partial}{\partial \xi_k} \left[\int_{\partial B_1} \frac{\partial \phi}{\partial n} \widehat{V}_j - \int_B \left(-\Delta \widehat{V}_j - L_t \widehat{V}_j - \frac{c}{|x|^2} \widehat{V}_j \right) \phi \right]_{\xi=0, t=0}$$
$$= 2 \int_{\partial B} x_k \frac{\partial \widehat{V}_j}{\partial \nu} - 2 \frac{\int_{\partial B} x_k \frac{\partial W_{1,k}}{\partial \nu}}{\int_B V_{k,0} W_{1,k}} \int_B V_{k,0} \widehat{V}_j = 0$$

thanks to (2.59). This and (2.61) imply that the matrix $\left(\frac{\partial H_j}{\partial \xi_k}(0,0) \right)_{ij}$ is invertible.

We may then apply the Implicit Function Theorem, to conclude that there exists a differentiable curve $t \to \xi(t)$ defined for $|t|$ small, such that (2.60) holds for $\xi = \xi(t)$. Letting $v(x) = \log \frac{1}{|x - \xi(t)|^2} + \phi(x, \xi(t))$ for $x \in B$ and $u(y) = v(y + t\tilde{\psi}(y))$ for $y \in \Omega_t$, we conclude that u is the desired solution of (1.22).

Case $N = 4$. We use the Brouwer Fixed Point Theorem as follows. Define $H = (H_1, \ldots, H_N)$ and

$$B(\xi) = (B_1, \ldots, B_N) \quad \text{with} \quad B_j(\xi) = \int_{\partial B} \log \frac{1}{|x - \xi|^2} \frac{\partial W_{j,\xi}}{\partial n}.$$

By (2.61), B is differentiable and $DB(0)$ is invertible. (2.60) is then equivalent to

$$\xi = G(\xi),$$

where

$$G(\xi) = DB(0)^{-1} \left(DB(0)\xi - H(\xi, t) \right).$$

To apply the Brouwer Fixed Point Theorem it suffices to prove that for t, ρ small, G is a continuous function of ξ and $G : \overline{B_\rho} \to \overline{B_\rho}$. The following two lemmas are proved in [43].

LEMMA 2.8. *G is continuous for t, ξ small.*

LEMMA 2.9. *If $\rho > 0$ and $|t|$ are small enough then $G : \overline{B_\rho} \to \overline{B_\rho}$.*

3. Reaction on the boundary

3.1. *Characterization and uniqueness of the extremal solution*

In this section we are interested in the characterization of the extremal solution presented in Lemma 1.16. As mentioned in Section 1.4 we shall prove this characterization under the assumptions that g satisfies (1.35) and (1.36), since the argument is simpler and works in the case that Γ_1 and Γ_2 form an angle. Later on in Section 3.2 we shall prove the uniqueness of the extremal solution for the problem with reaction on the boundary, which is the analog of Theorem 1.8 for g satisfying (1.2) and (1.3), assuming that $\partial\Omega$ is smooth.

LEMMA 3.1. *Suppose that* $u \in H^1(\Omega)$ *is a weak solution to* (1.34). *Then for any* $0 < \lambda < \lambda^*$ (1.34) *has a bounded solution.*

PROOF. Let u be an energy solution to (1.34). We basically use the truncation method of [19]. For this the first step is to show that if $\Phi : [0, \infty) \to [0, \infty)$ is a concave C^2 function such that $\Phi' \in L^\infty$ then $\Phi(u)$ is a supersolution, in the sense that

$$\int_\Omega \nabla\Phi(u)\nabla\varphi \geq \lambda \int_{\Gamma_1} \Phi'(u)g(u)\varphi \quad \forall\varphi \in C^1(\overline{\Omega}), \varphi \geq 0. \tag{3.1}$$

Indeed, let $h = \lambda g(u)$ and for $m > 0$ let

$$h_m = \begin{cases} h_m = h & \text{if } |h| \leq m \\ h_m = -m & \text{if } h < -m \\ h_m = m & \text{if } h > m. \end{cases}$$

Let u_m denote the H^1 solution of

$$\begin{cases} \Delta u_m = 0 & \text{in } \Omega \\ \dfrac{\partial u_m}{\partial \nu} = h_m & \text{on } \Gamma_1 \\ u_m = 0 & \text{on } \Gamma_2. \end{cases}$$

Note that $u_m \to u$ in $H^1(\Omega)$ and in $L^1(\Gamma_1)$. Let $\varphi \in C^1(\overline{\Omega})$, $\varphi \geq 0$. Using $\Phi'(u_m)\varphi$ as a test function we find that

$$\int_\Omega \nabla u_m\big(\Phi''(u_m)\nabla u_m\varphi + \Phi'(u_m)\nabla\varphi\big)\,dx - \int_{\Gamma_1} \Phi'(u_m)h_m\varphi = 0.$$

Using that $\Phi'' \leq 0$ and $\varphi \geq 0$ we have

$$\int_\Omega \nabla(\Phi(u_m))\nabla\varphi\,dx \geq \int_{\Gamma_1} h_m\Phi'(u_m)\varphi\,dx.$$

Now we let $m \to \infty$. Since $\Phi' \in L^\infty$ it is not difficult to verify that

$$\int_\Omega \nabla(\Phi(u_m))\nabla\varphi\,dx \to \int_\Omega \nabla(\Phi(u))\nabla\varphi\,dx$$

and

$$\int_{\Gamma_1} h_m \Phi'(u_m)\varphi\, dx \to \int_{\Gamma_1} h\Phi'(u)\varphi\, dx$$

since we have convergence a.e. for a subsequence and

$$|h_m \Phi'(u_m)\varphi| \le \|\Phi'\|_\infty \|\varphi\|_{L^\infty} |h| \in L^1(\Gamma_1)$$

since $h = \lambda g(u) \in L^1(\Gamma)$.

Now note that under (1.36) we have $g(t) \ge ct^\alpha$ for some $\alpha > 1$ and $c > 0$ and hence

$$\int_0^\infty \frac{ds}{g(s)} < +\infty. \tag{3.2}$$

Let $0 < \lambda' < \lambda$ and define

$$\Phi(t) = H^{-1}(\lambda' H(u)/\lambda), \tag{3.3}$$

where

$$H(u) = \int_0^u \frac{ds}{g(s)}. \tag{3.4}$$

Then it is possible to verify that Φ is a C^2 concave function with bounded derivative. Since $\lambda\Phi'(u)g(u) = \lambda' g(\Phi(u))$ it follows from (3.1) that $v = \Phi(u)$ satisfies

$$\int_\Omega \nabla v \nabla\varphi \ge \lambda' \int_{\Gamma_1} g(v)\varphi \quad \forall\varphi \in C^1(\overline{\Omega}), \quad \varphi \ge 0$$

and is thus a supersolution to (1.34) with parameter λ'. Now, condition (3.2) implies that $v = \Phi(u)$ is bounded. By the method of sub and supersolutions (1.34) with parameter λ' has a bounded solution. □

PROOF OF LEMMA 1.16. Under hypothesis (1.36) the argument to prove Lemma 1.16 is similar to that of Theorem 1.9 but simpler because we can immediately say that $u^* \in H^1(\Omega)$ and we do not need to rely on a uniqueness result for u^* similar to Theorem 1.8. By Lemma 3.1 $\lambda \le \lambda^*$. Now, if $\lambda < \lambda^*$ then exactly the same argument as in Theorem 1.9 leads to a contradiction. Thus $\lambda = \lambda^*$. We wish to show that $v = u^*$. Since v is a supersolution to (1.34) we see that $u_\lambda \le v$ for all $0 < \lambda < \lambda^*$ and taking $\lambda \to \lambda^*$ we conclude $u^* \le v$. For the opposite inequality observe that by density (1.39) holds for $\varphi \in H^1(\Omega)$ such that $\varphi = 0$ on Γ_2. By hypothesis $v \in H^1(\Omega)$ and since g satisfies (1.36) we have $u^* \in H^1(\Omega)$. Thus we may choose $\varphi = v - u^*$. We obtain

$$\int_{\Gamma_1} (g(u^*) - (g(v) + g'(v)(u^* - v)))(v - u^*) \le 0.$$

But the integrand is nonnegative since $v \ge u^*$ a.e. and g is convex. This implies

$$g(u^*) = g(v) + g'(v)(u^* - v) \quad \text{a.e. on } \Gamma_1.$$

It follows that g is linear in intervals of the form $[u^*(x), v(x)]$ for a.e. $x \in \Gamma_1$. The union of such intervals is an interval of the form $[a, \infty)$ for some $a \ge 0$. Assuming this property for a moment we reach a contradiction with (1.36).

To prove the claim above we follow the argument of Dupaigne and Nedev [51]. First we show that $u^*(\Gamma_1)$ is dense in $[\operatorname{ess\,inf}_{\Gamma_1} u^*, \operatorname{ess\,sup}_{\Gamma_1} u^*]$. Indeed, if not, then there exists a nontrivial interval (a, b) such that $\{x \in \Gamma_1 : u^*(x) \le a\}$ and $\{x \in \Gamma_1 : u^*(x) \ge b\}$ both have positive measure in Γ_1. Hence there is a smooth function $\eta : \mathbb{R} \to \mathbb{R}$ with $0 \le \eta \le 1$ such that $\eta(u^*)$, is either 0 or 1, but such that $\{x \in \Gamma_1 : \eta(u^*(x)) = 0\}$ and $\{x \in \Gamma_1 : \eta(u^*(x)) = 1\}$ have positive measure. Since $u^* \in H^1(\Omega)$ we have $\eta(u^*) \in H^1(\Omega)$ and therefore $\eta(u^*) \in H^{1/2}(\Gamma_1)$ and has values 0 and 1. But it is known, see for instance Bourgain *et al.* [16], that a function in $W^{s,p}(\Gamma_1; \mathbb{Z})$ with $sp \ge 1$ is constant. This contradiction shows that indeed $u^*(\Gamma_1)$ is dense in $[\operatorname{ess\,inf}_{\Gamma_1} u^*, \operatorname{ess\,sup}_{\Gamma_1} u^*]$. Let $S \subset \Gamma_1$ by a compact set with $dist(S, I) > 0$. By the strong maximum principle $\operatorname{ess\,inf}_S(v - u^*) > 0$. It follows that $\cup_{x \in S}[u^*(x), v(x)] \supseteq \cup_{x \in S}[u^*(x), u^*(x) + \varepsilon]$ and hence is an interval $[a, \infty)$, because $\operatorname{ess\,sup}_{\Gamma_1} u^* = +\infty$ as u^* is unbounded. □

3.2. *Weak solutions and uniqueness of the extremal solution*

Throughout this section we will assume that g satisfies (1.2) and (1.3).

An important tool in the proofs in [19,87] is Hopf's lemma, so before adapting their arguments we need to find a suitable statement that replaces this lemma for problems with mixed boundary condition. Let us recall a form of Hopf's lemma combined with the strong maximum principle which will be our model. Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain. If u satisfies

$$\begin{cases} -\Delta u = h & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.5}$$

with $h \in L^\infty(\Omega)$, $h \ge 0$, $h \not\equiv 0$ then there exists $c_1 > 0$ such that

$$c_1 \delta \le u \quad \text{in } \Omega, \tag{3.6}$$

where

$$\delta(x) = dist(x, \partial\Omega).$$

The bound is sharp in the sense that $u \le c_2 \delta$ for some $c_2 > 0$ by Schauder's estimates. The constant c_1 in the lower bound of (3.6) above can be made more precise in its dependence on h

$$c\delta(x)\Big(\int_\Omega \delta h\Big) \le u(x) \quad \forall x \in \Omega,$$

where $c > 0$ depends only on Ω. This estimate was proved by Morel and Oswald (unpublished) and can also be found in [18].

Let us consider the following linear problem with mixed boundary condition

$$\begin{cases} \Delta u = h_1 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = h_2 & \text{on } \Gamma_1 \\ u = 0 & \text{on } \Gamma_2, \end{cases} \tag{3.7}$$

where h_1, h_2 are smooth functions defined on $\overline{\Omega}$ and Γ_1 respectively. Here Γ_1, Γ_2 is a partition of $\partial\Omega$ into surfaces separated by a smooth interface. More precisely $\Gamma_1, \Gamma_2 \subset \partial\Omega$ are smooth $N-1$-dimensional manifolds with a common boundary $\Gamma_1 \cap \Gamma_2 = I$ which is a smooth $N-2$-dimensional manifold.

We shall define next a function which will play the role of δ for (3.5). The definition is motivated by the fact that the function

$$v = Im(z^{1/2}) = \frac{1}{\sqrt{2}}\sqrt{\sqrt{x^2+y^2}-x}, \qquad z = x + iy$$

is harmonic in the upper half of the complex plane $\{z \in \mathbb{C} \mid Re(z) > 0\}$, and satisfies the mixed boundary condition

$$v(x,0) = 0 \quad x > 0, \qquad \frac{\partial v}{\partial y}(x,0) = 0 \quad x < 0.$$

For x in a small fixed neighborhood of $\partial\Omega$ we write $\hat{x}$ for the projection of x on $\partial\Omega$, that is, $\hat{x}$ is the point in Ω closest to x. We let $\nu(x)$ denote the outer unit normal vector to $\partial\Omega$ at $\hat{x}$. Given $x \in \partial\Omega$ in a fixed small neighborhood of I we write $I(x)$ for the point in I with smallest geodesic distance on $\partial\Omega$ to x. Then there exists a neighborhood $\mathcal{U}$ of I in Ω and $r > 0$ such that

$$x \in \mathcal{U} \to (I(\hat{x}), d_I(\hat{x}), \delta(x)) \in I \times (-r, r) \times (0, r) \tag{3.8}$$

is a diffeomorphism, where $d_I(\hat{x})$ denotes the signed geodesic distance on $\partial\Omega$ from $\hat{x}$ to $I(\hat{x})$ with the sign such that

$$d_I(\hat{x}) \le 0 \quad \text{if } \hat{x} \in \Gamma_1, \qquad d_I(\hat{x}) \ge 0 \quad \text{if } \hat{x} \in \Gamma_2.$$

We define $\zeta(x)$ for $x \in \mathcal{U}$ as:

$$\zeta(x) = \sqrt{\sqrt{s^2+t^2}-s}, \quad \text{where } t = \delta(x),\ s = d_I(\hat{x}),$$

and we extend ζ to $\Omega \setminus \mathcal{U}$ as a smooth function such that

$$\inf_{\Gamma_1 \setminus \mathcal{U}} \zeta > 0 \quad \text{and} \quad \zeta = 0 \quad \text{on } \Gamma_2 \setminus \mathcal{U}.$$

The next result is the analog of (3.6) for (3.5).

PROPOSITION 3.2. *Let $h_1 \in L^\infty(\Omega)$, $h_1 \geq 0$ and $h_2 \in L^\infty(\Gamma_1)$, $h_2 \geq 0$ and assume $h_1 \not\equiv 0$ or $h_2 \not\equiv 0$. Let u be the solution to (3.7). Then there exist constants $c_1, c_2 > 0$ such that*

$$c_1\zeta \leq u \leq c_2\zeta \quad in\ \Omega. \tag{3.9}$$

PROOF. For convenience we write $\mathcal{U}_r$ as the neighborhood of I in Ω introduced in (3.8). We will show, using suitable barriers, that (3.9) holds in $\mathcal{U}_r$ for some $r > 0$ small. Using then the strong maximum principle and the usual Hopf's lemma we will establish the desired inequality in Ω.

Recall that $t = \delta(x)$ and $s = d_I(\hat{x})$ are well-defined smooth functions on $\mathcal{U}_r$. For a function $v(s,t)$ its Laplacian can be expressed as

$$\Delta v = \frac{\partial^2 v}{\partial s^2} + \frac{\partial^2 v}{\partial t^2} + O(t+|s|)|D^2 v| + O(1)|Dv|, \tag{3.10}$$

where $O(t+|s|)$ denotes a function bounded by $t+|s|$ in $\mathcal{U}_r$, $O(1)$ a bounded function, $|D^2v|$ and $|Dv|$ are the norms of the Hessian and gradient of v respectively. Indeed, let us consider a smooth change of variables of a neighborhood of x_0 in I onto an open set in $\mathbb{R}^{N-2}$, that is $\phi : B_r(x_0) \cap I \to V \subset \mathbb{R}^{N-2}$. Define the map

$$\psi(x) = (\phi(I(\hat{x})), d_I(\hat{x}), \delta(x)) = (z,s,t) \in V \times (-r,r) \times (0,r) \subset \mathbb{R}^N.$$

We shall write $y = (z,s,t)$, that is $z = (y_1, \ldots, y_{N-2})$, $s = y_{N-1}$, $t = y_N$. Then $y_1, \ldots, y_N$ are local coordinates of a neighborhood of x_0, and

$$\Delta v = \frac{1}{\sqrt{g}} \partial_{y_k} \left(\sqrt{g} g^{kl} \partial_{y_l} v \right), \tag{3.11}$$

where $g_{ij} = \langle \frac{\partial \psi^{-1}}{\partial y_i}, \frac{\partial \psi^{-1}}{\partial y_j} \rangle$ is the Euclidean metric tensor in the coordinates $y_1, \ldots, y_N$, $g = \det(g_{i,j})$ and g^{kl} is the inverse matrix of g_{ij}. By construction of ψ, when $t = s = 0$ (which corresponds to the interface I) the coefficients g_{ij} are 0 whenever $i = N-1, N$ or $j = N-1, N$, since at I $D\psi$ maps the normal vector $\nu = \frac{\partial}{\partial t}$ to the vector e_N, the vector $\frac{\partial}{\partial s}$ perpendicular to I and tangent to $\partial\Omega$ to e_{N-1} and vectors in the tangent space to I to vectors of $\mathbb{R}^N$ with the last two components equal to 0. Hence if $k = N-1, N$ or $l = N-1, N$ we have $g^{kl} = O(t+|s|)$ and formula (3.10) follows from (3.11).

Let us introduce polar coordinates for s, t:

$$s = r\cos(\theta), \qquad t = r\sin(\theta).$$

As a first term for the subsolution we take

$$u_1 = r^{1/2}\sin(\theta/2).$$

Then according to (3.10) and since $|D^2u_1| = O(r^{-3/2})$, $|Du_1| = O(r^{-1/2})$ we have

$$\Delta u_1 = O(r^{-1/2}). \tag{3.12}$$

Let $1/2 < \gamma < \alpha < 1$, $b > 0$ and define

$$u_2 = r^\alpha(\sin(\gamma\theta) + b\theta^2).$$

Using (3.10) again we find

$$\begin{aligned}\Delta u_2 &= r^{\alpha-2}\Big((\alpha^2 - \gamma^2)\sin(\gamma\theta) + \alpha^2 b\theta^2 + 2b\Big) + O(r^{\alpha-1})\\ &\geq cr^{\alpha-2} + O(r^{\alpha-1}),\end{aligned} \tag{3.13}$$

for some positive constant c. Set

$$\underline{u} = u_1 + u_2.$$

By (3.12) and (3.13) there exists $r_0 > 0$ but small such that

$$\Delta\underline{u} \geq cr^{2-\alpha} \quad \text{in the region } r < r_0$$

for some $c > 0$. Let us compute the normal derivative:

$$\frac{\partial\underline{u}}{\partial\nu} = -\frac{\partial\underline{u}}{\partial t}\Big|_{t=0} = \frac{1}{r}\frac{\partial\underline{u}}{\partial\theta}\Big|_{\theta=\pi} = r^{\alpha-1}(\gamma\cos(\gamma\pi) + 2b\pi) \leq -cr^{\alpha-1},$$

where $c > 0$, if b is taken sufficiently small.

We use the maximum principle in the region D contained in $\mathcal{U}_{r_0}$, which in terms of the polar coordinates is given by

$$D = \{r < r_0, 0 < \theta < \pi\}.$$

The boundary of D consists of

$$\partial D = M_0 \cup M_1 \cup M_2,$$

where

$$\begin{aligned} M_1 &= \{0 \leq r \leq r_0, \theta = \pi\} = \partial D \cap \Gamma_1\\ M_2 &= \{0 \leq r \leq r_0, \theta = 0\} = \partial D \cap \Gamma_2\\ M_3 &= \{r = r_0, 0 < \theta < \pi\} = \partial D \cap \Omega.\end{aligned}$$

We have

$$\begin{aligned} &\Delta u \leq 0, \quad \Delta\underline{u} > 0 \quad \text{in } D\\ &\frac{\partial u}{\partial\nu} \geq 0, \quad \frac{\partial\underline{u}}{\partial\nu} < 0 \quad \text{on } M_1\\ &u = 0, \quad \underline{u} = 0 \quad \text{on } M_2\end{aligned}$$

and

$$u \geq c\underline{u} \quad \text{on } M_3,$$

for some $c > 0$. This follows from the standard strong maximum principle and Hopf's lemma applied to u, since the distance from M_3 to the interface I is strictly positive. It follows that

$$u \geq c\underline{u} \quad \text{in } D.$$

This yields the lower bound for u.

To obtain the upper bound for u in (3.9) choose

$$\overline{u} = u_1 - u_2,$$

where α, γ, b are as before, that is $1/2 < \gamma < \alpha < 1$, $b > 0$. By (3.12) and (3.13)

$$\Delta\overline{u} \le -cr^{\alpha-2} + O(r^{\alpha-1}) + O(r^{-1/2}) \le -cr^{\alpha-2} \tag{3.14}$$

for small r for some positive fixed c. Similarly

$$\frac{\partial\overline{u}}{\partial\nu} = -\frac{\partial\overline{u}}{\partial t}\Big|_{t=0} = \frac{1}{r}\frac{\partial\overline{u}}{\partial\theta}\Big|_{\theta=\pi} = -r^{\alpha-1}(\gamma\cos(\gamma\pi) + 2b\pi) \ge cr^{\alpha-1}, \tag{3.15}$$

where $c > 0$, if b is taken sufficiently small. Applying the maximum principle in the same region D as before we find $u \le C\overline{u}$ in D. □

One consequence of (3.9) is that even if h_1, h_2 are smooth the solution u to (3.7) is in general not smooth, having at worst a behavior of the form $u(x) \sim dist(x, I)^{1/2}$ and $|\nabla u(x)| \sim dist(x, I)^{-1/2}$.

We need to define the notion of weak solution to (1.4), and before this, we need to define what we understand as weak solution to a linear problem. Define the space $L^1_\zeta(\Gamma_1)$ as the space of measurable functions $h : \Gamma_1 \to \mathbb{R}$ such that $\int_{\Gamma_1} |h|\zeta < +\infty$. We define the class of test functions $\mathcal{T}$ as the collection of $\varphi \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $\varphi = 0$ on Γ_2, $\Delta\varphi$ can be extended to a continuous function in $\overline{\Omega}$, for any $x \in \partial\Omega \setminus I$ there is $r > 0$ such that $\nabla\varphi$ admits a continuous extension to $\overline{\Omega} \cap B_r(x)$ and $\frac{\partial\varphi}{\partial\nu}$, which is now well defined in $\Gamma_1 \setminus I$ and can be extended as a continuous function on Γ_1. In particular, given $\eta_1 \in C(\overline{\Omega})$, $\eta_2 \in C(\Gamma_1)$ the solution φ to

$$\begin{cases} -\Delta\varphi = \eta_1 & \text{in } \Omega \\ \dfrac{\partial\varphi}{\partial\nu} = \eta_2 & \text{on } \Gamma_1 \\ \varphi = 0 & \text{on } \Gamma_2, \end{cases} \tag{3.16}$$

is in $\mathcal{T}$. Moreover by Proposition 3.2 we see that φ satisfies

$$|\varphi| \le C\zeta \quad \text{in } \Omega. \tag{3.17}$$

LEMMA 3.3. *Given $h \in L^1_\zeta(\Gamma_1)$ there is a unique $u_1 \in L^1(\Omega)$, $u_2 \in L^1(\Gamma_1)$ such that*

$$\int_\Omega u_1(-\Delta\varphi) + \int_{\Gamma_1}\left(h\varphi - u_2\frac{\partial\varphi}{\partial\nu}\right) = 0 \quad \forall\varphi \in \mathcal{T}. \tag{3.18}$$

Moreover

$$\|u_1\|_{L^1(\Omega)} + \|u_2\|_{L^1(\Gamma_1)} \le C\|h\|_{L^1_\zeta(\Gamma_1)}, \tag{3.19}$$

and if $h \ge 0$ then $u_1, u_2 \ge 0$.

PROOF. We deal with uniqueness first. Suppose $u_1 \in L^1(\Omega)$, $u_2 \in L^1(\Gamma_1)$ satisfy (3.18) with $h = 0$. Given $\eta \in C_0^\infty(\Gamma_1)$ let φ be the solution to (3.16) with $\eta_1 = 0$, $\eta_2 = \eta$. Then $\varphi \in \mathcal{T}$ and by (3.18)

$$\int_{\Gamma_1} u_2 \eta = 0.$$

Hence $u_2 \equiv 0$. Then given $\eta \in C_0^\infty(\Omega)$, setting φ as the solution to

$$\begin{cases} -\Delta\varphi = \eta & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega \end{cases} \tag{3.20}$$

we deduce $\int_\Omega u_1 \eta = 0$. It follows that $u_1 = 0$.

We prove (3.19) in the case $u_1 \geq 0$, $u_2 \geq 0$. For this we may take $\eta_1 = 1$ and $\eta_2 = 1$ in (3.16). Then from (3.18) and (3.17) we see that (3.19) holds.

For the existence part we take $h \in L^1_\zeta(\Gamma_1)$, $h \geq 0$ and let $h_m = \min(m, h)$. Then

$$\begin{cases} \Delta u_m = 0 & \text{in } \Omega \\ \dfrac{\partial u_m}{\partial \nu} = h_m & \text{on } \Gamma_1 \\ u_m = 0 & \text{on } \Gamma_2 \end{cases}$$

has a solution $u_m \in H^1(\Omega)$ and we have the bound

$$\|u_m - u_n\|_{L^1(\Omega)} + \|u_m - u_n\|_{L^1(\Gamma_1)} \leq \|h_n - h_m\|_{L^1_\zeta(\Gamma_1)}.$$

Thus $u_m \to u_1$ in $L^1(\Omega)$ and $u_m|_{\Gamma_1} \to u_2$ in $L^1(\Gamma_1)$. For $\varphi \in \mathcal{T}$ we have

$$\int_\Omega u_m(-\Delta\varphi) + \int_{\Gamma_1} u_m \frac{\partial \varphi}{\partial \nu} - h_m \varphi = 0.$$

Passing to the limit shows that u_1, u_2 satisfies condition (3.18). We see also that $u_1 \geq 0$, $u_2 \geq 0$. For general h we may rewrite it as the difference of two nonnegative functions. □

If h is smooth then we may find a solution $u \in \mathcal{T}$ to

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = h & \text{on } \Gamma_1 \\ u = 0 & \text{on } \Gamma_2 \end{cases} \tag{3.21}$$

and u_1, u_2 in Lemma 3.3 correspond to u restricted to Ω and Γ_1 respectively.

DEFINITION 3.4. We say that $u_1 \in L^1(\Omega)$, $u_2 \in L^1(\Gamma_1)$ is a weak solution to (3.21) if they satisfy (3.18). In the sequel, when referring to a weak solution $u_1 \in L^1(\Omega)$, $u_2 \in L^1(\Gamma)$ to (3.21) we will identify u_1 and u_2 as just u, and according to the context we write $u \in L^1(\Omega)$ or $u \in L^1(\Gamma_1)$.

Weak supersolutions are defined as:

DEFINITION 3.5. We say that $u \in L^1(\Gamma_1)$ is a weak supersolution to (3.21) if

$$\int_\Omega u_1(-\Delta\varphi) + \int_{\Gamma_1} \left(h\varphi - u_2 \frac{\partial\varphi}{\partial\nu} \right) \geq 0 \quad \forall\varphi \in \mathcal{T}, \varphi \geq 0. \tag{3.22}$$

We consider the problem (1.4), that is,

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = \lambda g(u) & \text{on } \Gamma_1 \\ u = 0 & \text{on } \Gamma_2. \end{cases} \tag{3.23}$$

DEFINITION 3.6. We say that $u \in L^1(\Gamma_1)$ is a weak solution to (3.23) if $g(u) \in L^1_\zeta(\Gamma_1)$ and (3.23) holds in the sense of Definition 3.4 with $h = \lambda g(u)$.

Let us remark that only with hypotheses (1.2) and (1.3) the extremal solution u^* is a weak solution in the sense of Definition 3.6. Indeed, the same calculations as in (1.6) and (1.7) with $\varphi_1 > 0$ the first eigenfunction for

$$\begin{cases} \Delta\varphi_1 = 0 & \text{in } \Omega \\ \dfrac{\partial\varphi_1}{\partial\nu} = \lambda_1\varphi_1 & \text{on } \Gamma_1 \\ \varphi_1 = 0 & \text{on } \Gamma_2 \end{cases}$$

show that

$$\int_{\Gamma_1} g(u_\lambda)\varphi_1 \leq C$$

with C independent of λ. Note that by Proposition 3.2 we have $\zeta \leq C\varphi_1$ and it follows that

$$\int_{\Gamma_1} g(u_\lambda)\zeta \leq C. \tag{3.24}$$

To show that $u^* \in L^1(\Omega)$ let χ solve

$$\begin{cases} -\Delta\chi = 1 & \text{in } \Omega \\ \dfrac{\partial\chi}{\partial\nu} = 0 & \text{on } \Gamma_1 \\ \chi = 0 & \text{on } \Gamma_2. \end{cases}$$

By Proposition 3.2 we have $\chi \leq C\zeta$ and hence, after multiplying (3.23) by χ and integrating by parts we have

$$\int_\Omega u_\lambda = \lambda \int_{\Gamma_1} u_\lambda \chi \leq C \int_{\Gamma_1} g(u_\lambda)\zeta \leq C$$

by (3.24). Hence $u^* \in L^1(\Omega)$, $g(u^*) \in L^1_\zeta(\Gamma_1)$ and it is not difficult to verify that it satisfies Definition 3.6.

Our next result is an adaptation of a result of Nedev [96] for (1.1), that shows that u^* is bounded in dimensions $N \le 3$ for that problem. It also provides some estimates of the form $g(u^*)$ in L^p for some $p > 1$ in any dimension. The argument is the same as in [96] except that some of the exponents change slightly.

THEOREM 3.7. *Assume g satisfies* (1.2) *and* (1.3). *Then if $N \le 2$ we have $u^* \in L^\infty(\Omega)$. If $N \ge 3$ then $g(u^*) \in L^p(\Gamma_1)$ for $1 \le p < \frac{N-1}{2(N-2)}$ and $u^* \in L^p(\Gamma_1)$ for $1 \le p < \frac{N-1}{N-3}$.*

PROOF. We estimate the minimal solution u_λ for $0 < \lambda < \lambda^*$. Let

$$\psi(t) = \int_0^t g'(s)^2\,ds$$

and multiply (3.23) by $\psi(u_\lambda)$ to obtain

$$\int_\Omega g'(u_\lambda)^2 |\nabla u_\lambda|^2 = \lambda \int_{\Gamma_1} g(u_\lambda)\psi(u_\lambda). \tag{3.25}$$

We shall use the notation $\tilde{g}(u) = g(u) - g(0)$. Using the weak stability of u_λ with $\tilde{g}(u_\lambda)$ we have

$$\lambda \int_{\Gamma_1} g'(u_\lambda)\tilde{g}(u_\lambda)^2 \le \int_\Omega g'(u_\lambda)^2 |\nabla u_\lambda|^2.$$

Hence, by (3.25) we have

$$\int_{\Gamma_1} g'(u_\lambda)\tilde{g}(u_\lambda)^2 = \int_{\Gamma_1} g(u_\lambda)\psi(u_\lambda) = \int_{\Gamma_1} \tilde{g}(u_\lambda)\psi(u_\lambda) + g(0)\int_{\Gamma_1} \psi(u_\lambda). \tag{3.26}$$

As in [96] let

$$h(t) = \int_0^t g'(s)(g'(t) - g'(s))\,ds.$$

Then from (3.26) we have

$$\int_{\Gamma_1} \tilde{g}(u_\lambda)h(u_\lambda) \le g(0)\int_{\Gamma_1} \psi(u_\lambda). \tag{3.27}$$

But

$$\lim_{t\to+\infty} \frac{h(t)}{g'(t)} = +\infty. \tag{3.28}$$

Indeed, for any $M > 0$, by the convexity of g we have

$$\begin{aligned} h(t) \ge \int_0^M g'(s)(g'(t) - g'(s))\,ds &\ge \int_0^M g'(s)(g'(t) - g'(M))\,ds \\ &= (g(M) - g(0))(g'(t) - g'(M)). \end{aligned}$$

Dividing by $g'(t)$ we have

$$\liminf_{t\to+\infty} \frac{h(t)}{g'(t)} \geq (g(M) - g(0))$$

(by (1.3) $\lim_{t\to+\infty} g'(t) = +\infty$). Since M is arbitrary we deduce (3.28).

On the other hand

$$\psi(t) = \int_0^t g'(s)^2\, ds \leq g'(t)\tilde{g}(t). \tag{3.29}$$

Thus, by (3.27), (3.28) and (3.29) we find

$$\int_{\Gamma_1} \tilde{g}(u_\lambda)h(u_\lambda) \leq C \quad \text{and} \quad \int_{\Gamma_1} \psi(u_\lambda) \leq C$$

with C independent of λ and also

$$\int_{\Gamma_1} \tilde{g}(u_\lambda)g'(u_\lambda) \leq C.$$

The convexity of g implies $g'(t) \geq \tilde{g}(t)/t$, and hence

$$\int_{\Gamma_1} \frac{\tilde{g}(u_\lambda)^2}{u_\lambda} \leq C. \tag{3.30}$$

It follows that $g(u_\lambda) \in L^1(\Gamma_1)$ since, one needs to control $\int g(u_\lambda)$ in the region where $u_\lambda \geq M$, and there, $\frac{\tilde{g}(u_\lambda)^2}{u_\lambda} \geq u_\lambda$ if M is large enough. By regularity theory

$$\|u_\lambda\|_{L^p(\Gamma_1)} \leq C \quad \text{for } 1 \leq p < \tfrac{N-1}{N-2} \text{ (any } p < \infty \text{ if } N = 2).$$

Let $0 < \alpha < 1$ and

$$A = \{x \in \Gamma_1 : \tilde{g}(u_\lambda) < u_\lambda^{1/\alpha}\}, \qquad B = \{x \in \Gamma_1 : \tilde{g}(u_\lambda)^2/u_\lambda \geq \tilde{g}(u_\lambda)^{2-\alpha}\}.$$

Then A, B cover all Γ_1. By (3.30)

$$\int_B \tilde{g}(u_\lambda)^{2-\alpha} \leq C$$

and

$$\int_A \tilde{g}(u_\lambda)^p \leq \int_A \tilde{u}_\lambda^{p/\alpha} \leq C$$

if $p/\alpha < \frac{N-2}{N-1}$. Choosing $\alpha = \frac{2(N-2)}{2N-3}$ we see that

$$\|g(u_\lambda)\|_{L^p(\Gamma_1)} \leq C \quad \text{for } 1 \leq p < \tfrac{2(N-1)}{2N-3}.$$

Repeating this process yields the desired conclusion. □

Next we show, following the argument of Brezis, Cazenave, Martel and Ramiandrisoa, that there are no weak solutions for $\lambda > \lambda^*$.

THEOREM 3.8. *Assume g satisfies (1.2) and (1.3). Then, for $\lambda > \lambda^*$, problem (3.23) has no weak solutions.*

For the proof we need the following:

LEMMA 3.9. *Let $h \in L^1_\zeta(\Gamma_1)$ and $u \in L^1(\Gamma_1)$ be weak solutions of (3.21). Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a C^2 concave function with $\Phi' \in L^\infty$ and $\Phi(0) = 0$. Then $\Phi(u)$ is a weak supersolution to (3.21) with h replaced by $\Phi'(u)h$.*

PROOF. For $m > 0$ let $h_m = h$ if $|h| \le m$, $h_m = -m$ if $h < -m$ and $h_m = m$ if $h > m$, and let u_m denote the H^1 solution of (3.21) with h replaced by h_m. Note that $u_m \to u$ in $L^1(\Omega)$ and in $L^1(\Gamma_1)$ by (3.19). Let $\varphi \in \mathcal{T}$, $\varphi \ge 0$. Using $\Phi'(u_m)\varphi$ as a test function we find that

$$\int_\Omega \nabla u_m \big(\Phi''(u_m)\nabla u_m \varphi + \Phi'(u_m)\nabla\varphi\big)\, dx - \int_{\Gamma_1} \Phi'(u_m) h_m \varphi = 0.$$

Using that $\Phi'' \le 0$ and $\varphi \ge 0$ we have

$$\int_\Omega \nabla(\Phi(u_m))\nabla\varphi\, dx - \int_{\Gamma_1} h_m \Phi'(u_m)\varphi\, dx \ge 0 \tag{3.31}$$

and integrating by parts

$$\int_\Omega \Phi(u_m)(-\Delta\varphi) + \int_{\Gamma_1} \Phi(u_m)\frac{\partial\varphi}{\partial\nu} - h_m \Phi'(u_m)\varphi \ge 0.$$

Now we let $m \to \infty$. We have

$$\int_\Omega |\Phi(u_m) - \Phi(u)||\Delta\varphi|\, dx \le \|\Delta\varphi\|_\infty \|\Phi'\|_\infty \int_\Omega |u_m - u|\, dx \to 0$$

$$\int_{\Gamma_1} |\Phi(u_m) - \Phi(u)| \left|\frac{\partial\varphi}{\partial\nu}\right| dx \le \left\|\frac{\partial\varphi}{\partial\nu}\right\|_\infty \|\Phi'\|_\infty \int_{\Gamma_1} |u_m - u|\, dx \to 0$$

and

$$\int_{\Gamma_1} h_m \Phi'(u_m)\varphi\, dx \to \int_{\Gamma_1} h\Phi'(u)\varphi\, dx$$

since we have convergence a.e. (at least for a subsequence) and

$$|h_m \Phi'(u_m)\varphi| \le \|\Phi'\|_\infty |h|\zeta \in L^1(\Gamma_1)$$

by the assumption $h \in L^1_\zeta(\Gamma_1)$. □

LEMMA 3.10. *If (3.23) has a weak supersolution $w \ge 0$ then it has a weak solution.*

PROOF. The proof is by the standard iteration method: set $u_0 = 0$ and u_{k+1} as the solution to

$$\begin{cases} \Delta u_{k+1} = 0 & \text{in } \Omega \\ \dfrac{\partial u_{k+1}}{\partial \nu} = \lambda g(u_k) & \text{on } \Gamma_1 \\ u_{k+1} = 0 & \text{on } \Gamma_2. \end{cases}$$

The u_k is an increasing sequence bounded above by w which belongs to $L^1(\Omega)$ and $L^1(\Gamma_1)$, and $g(u_k)$ is increasing, bounded above by $g(w) \in L^1_\zeta(\Gamma_1)$. The limit $u = \lim_{k\to+\infty} u_k$ thus exists and is a weak solution. □

PROOF OF THEOREM 3.8. Assume that (λ, u) is a weak supersolution to (3.23). Let $0 < \lambda' < \lambda$ and Φ be defined as in (3.3), (3.4). By Lemma 3.9 we see that $\Phi(u)$ is a supersolution to (3.23) with parameter λ'. Suppose first that g satisfies $\int_0^\infty ds/g(s) < +\infty$. Then $\Phi(u)$ is also bounded and hence (3.23) with parameter λ' has a bounded solution.

Next we consider the case $\int_0^\infty ds/g(s) = +\infty$. As in [19] let $\varepsilon > 0$ be small and let $\lambda' = (1-\varepsilon)\lambda$. Let $v_1 = \Phi(u)$. Then $0 \le v_1 \le w$. But H is concave, so

$$H(u) \le H(v_1) + (u - v_1)H'(v_1) = H(v_1) + \frac{u - v_1}{g(v_1)}.$$

Recall that by definition of Φ and H (3.3), (3.4) we have $H(v_1) = (1-\varepsilon)H(u)$. Hence

$$\varepsilon H(u) \le \frac{u - v_1}{g(v_1)}$$

and therefore

$$g(v_1) \le C\frac{u}{H(u)} \le C(1+u) \in L^1(\Gamma_1). \tag{3.32}$$

Then by Lemma 3.10 there exists a weak solution u_1 to (3.23) with parameter $(1-\varepsilon)\lambda$ such that $u_1 \le v_1$ and by (3.32) we have $g(u_1) \in L^1(\Gamma_1)$. Thus $u_1 \in L^p(\Gamma_1)$ for any $p < \frac{N-1}{N-2}$ ($p < \infty$ if $N = 2$). Repeating this process, we define $v_2 = \Phi(u_1)$ and as before obtain $g(v_2) \le C(1+u_1) \in L^p(\Gamma_1)$ for any $p < \frac{N-1}{N-2}$ ($p < \infty$ if $N = 2$). Then there is a solution $u_2 \le v_2$ to (3.23) with parameter $(1-\varepsilon)^2\lambda$ and it satisfies $g(u_2) \in L^p(\Gamma_1)$ for $p < \frac{N-1}{N-2}$. By induction there is a solution u_k to (3.23) with parameter $(1-\varepsilon)^k\lambda$ and satisfying $g(u_k) \in L^p(\Gamma_1)$ for any $\frac{1}{p} > 1 - \frac{k-1}{N-1}$ provided $1 - \frac{k-1}{N-1} > 0$. For $k > N$ we find $u_k \in L^\infty(\Gamma_1)$. □

Finally this is the uniqueness result of [87] in the context of problem (3.23).

THEOREM 3.11. *Suppose that g satisfies* (1.2) *and* (1.3). *Then for $\lambda = \lambda^*$, problem* (3.23) *has a unique weak solution.*

PROOF. Let u_1, u_2 be different solutions to (3.23), and without loss of generality we may assume that $u_1 = u^*$ is the minimal one, so that $u_2 > u_1$ in Ω.

First we show that (3.23) has a strict supersolution v. For this we note that any convex combination $v_t = tu_1 + (1-t)u_2$, $t \in (0,1)$ is a supersolution, by the convexity of g. Suppose v_t is still a solution for all $0 < t < 1$. Then

$$g\,(tu_1(x) + (1-t)u_2(x)) = tg(u_1(x)) + (1-t)g(u_2(x)) \quad \text{a.e. on } \Gamma_1$$

and for all $t \in (0,1)$. Then there is a set E of full measure in Γ_1 such that $g\,(tu_1(x) + (1-t)u_2(x)) = tg(u_1(x)) + (1-t)g(u_2(x))$ holds for $t \in (0,1) \cap \mathbb{Q}$ and $x \in E$. This means g is linear in $[u_1(x), u_2(x)]$ for a.e. $x \in \Gamma_1$. The union of the intervals $[u_1(x), u_2(x)]$ with x in a set of full measure in Γ_1 is an interval. The argument is the same as in the end of the proof of Lemma 1.16, with the only difference that in this case, we do not have the information that $u_1 = u^*$ is in $H^1(\Omega)$. But now, thanks to Theorem 3.7 we know that $g(u^*) \in L^p(\Gamma_1)$ for some $p > 1$. Then by L^p theory [2,3] we also have $\nabla u^* \in L^p(\Gamma_1)$ for some $p > 1$ and therefore $u^* \in W^{1,p}(\Gamma)$. As in the proof of Lemma 1.16, this is sufficient to guarantee that $u^*(\Gamma_1)$ is dense in $[\operatorname{ess\,inf}_{\Gamma_1} u^*, \operatorname{ess\,sup}_{\Gamma_1} u^*]$. The conclusion from the previous argument is that u_1, u_2 solve a problem with a linear g, say $g(t) = a + bt$. By a bootstrap argument, u_1, u_2 are bounded solutions. Recall that by the implicit function theorem the first eigenvalue of the linearized operator at u^* is zero. Let $\varphi_1 > 0$ denote the first eigenfunction of the linearized operator, that is,

$$\begin{cases} \Delta\varphi_1 = 0 & \text{in } \Omega \\ \dfrac{\partial\varphi_1}{\partial\nu} = \lambda^* b\varphi_1 & \text{on } \Gamma_1 \\ \varphi_1 = 0 & \text{on } \Gamma_2. \end{cases}$$

Since u^* solves (3.23) with $g(t) = a + bt$, multiplying that equation by φ_1 and integrating by parts yields

$$\int_{\Gamma_1} \lambda^*(a + bu)\varphi_1 = \int_{\Gamma_1} \lambda^* bu\varphi_1.$$

Then $a = 0$ and we reach a contradiction.

We claim that there is some $\varepsilon > 0$ such that

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial\nu} = \lambda^* g(u) + \varepsilon & \text{on } \Gamma_1 \\ u = 0 & \text{on } \Gamma_2 \end{cases} \tag{3.33}$$

has a weak supersolution. Indeed, there is a strict supersolution v to (3.23). Let V be the solution of

$$\begin{cases} \Delta V = 0 & \text{in } \Omega \\ \dfrac{\partial V}{\partial\nu} = \lambda^* g(v) & \text{on } \Gamma_1 \\ V = 0 & \text{on } \Gamma_2 \end{cases}$$

and let χ solve

$$\begin{cases} \Delta\chi = 0 & \text{in } \Omega \\ \dfrac{\partial\chi}{\partial\nu} = 1 & \text{on } \Gamma_1 \\ \chi = 0 & \text{on } \Gamma_2. \end{cases} \tag{3.34}$$

By Proposition 3.2 there is a constant $\varepsilon > 0$ so that $v - V \geq \varepsilon\chi$. Hence $w = V + \varepsilon\chi \leq v$ and

$$\frac{\partial w}{\partial\nu} = \lambda^* g(v) + \varepsilon \geq \lambda^* g(w) + \varepsilon$$

and thus w is the desired supersolution.

Let $0 < \varepsilon_1 < \varepsilon$. Then there exists a bounded supersolution to

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial\nu} = \lambda^* g(u) + \varepsilon_1 & \text{on } \Gamma_1 \\ u = 0 & \text{on } \Gamma_2. \end{cases} \tag{3.35}$$

To see this, define $\Phi : [0, \infty) \to [0, \infty)$ so that

$$\int_0^{\Phi(t)} \frac{ds}{\lambda^* g(s) + \varepsilon_1} = \int_0^t \frac{ds}{\lambda^* g(s) + \varepsilon} \quad \text{for all } u \geq 0.$$

A calculation as in [19] shows that Φ satisfies the hypothesis of Lemma 3.9. If $\int_0^\infty ds/g(s) < +\infty$ then Φ is bounded. Let w be a supersolution of (3.33). Then by Lemma 3.9 $\Phi(w)$ is a bounded supersolution for (3.35). By the method of sub and supersolutions there is a bounded solution to (3.35).

If $\int_0^\infty ds/g(s) = +\infty$ then an iteration with Φ as in the proof of Theorem 3.8 still yields a bounded solution to (3.35). In fact, let

$$H_\varepsilon(t) = \int_0^t \frac{ds}{\lambda^* g(s) + \varepsilon}$$

and let $0 < \varepsilon_1 < \varepsilon$. Then we may restate the definition of Φ as $\Phi = H_{\varepsilon_1}^{-1} \circ H_\varepsilon$ or

$$H_{\varepsilon_1}(\Phi(t)) = H_\varepsilon(t) \quad \text{for all } t \geq 0.$$

Denoting by w the supersolution to (3.33) and $v = \Phi(w)$ we thus have

$$H_{\varepsilon_1}(v) = H_\varepsilon(w). \tag{3.36}$$

The function H_ε is concave and $v \leq w$, so

$$\frac{H_\varepsilon(w) - H_\varepsilon(v)}{w - v} \leq H_{\varepsilon_1}(v) = \frac{1}{\lambda^* g(v) + \varepsilon}. \tag{3.37}$$

But, thanks to (3.36)

$$H_\varepsilon(w) - H_\varepsilon(v) = H_{\varepsilon_1}(v) - H_\varepsilon(v) = \int_0^v \left(\frac{1}{\lambda^* g(s) + \varepsilon_1} - \frac{1}{\lambda^* g(s) + \varepsilon} \right) ds$$
$$\geq (\varepsilon - \varepsilon_1) \int_0^v \frac{1}{(\lambda^* g(s))^2}\, ds. \tag{3.38}$$

From (3.37) and (3.38) we see that

$$g(v) \leq \frac{C(1+w)}{\varepsilon - \varepsilon_1}.$$

The rest of the argument proceeds as in the proof of Theorem 3.8.

Since (3.35) has a bounded supersolution it also has a bounded solution w. Let $\lambda' > \lambda^*$ to be chosen later, and set

$$W = \frac{\lambda'}{\lambda^*} w - \varepsilon_1 \chi,$$

where χ is the solution of (3.34). Then observe that

$$\frac{\partial W}{\partial \nu} = \lambda' g(w) + \frac{\lambda'}{\lambda^*} \varepsilon_1 - \varepsilon_1 \geq \lambda' g(w). \tag{3.39}$$

We now choose λ'/λ^* close to 1, so that

$$\left(\frac{\lambda'}{\lambda^*} - 1 \right) w \leq \varepsilon_1 \chi,$$

and therefore

$$w \geq W. \tag{3.40}$$

This is possible because $w \in L^\infty$ and therefore $w \leq C\chi$ for some constant $C > 0$, by Proposition 3.2. Then (3.39) combined with (3.40) implies that W is a supersolution of (3.23) with λ^* replaced by λ'. This is in contradiction with λ^* being the maximal parameter for (3.23). □

3.3. *Kato's inequality*

In this section we will prove

THEOREM 3.12. *Let $B = B_1(0)$ be the unit ball in $\mathbb{R}^N$, $N \geq 3$. Then for any $1 \leq q < 2$ there exists $c = c(N, q) > 0$ such that*

$$\int_{\mathbb{R}^N_+ \cap B} |\nabla \varphi|^2 \geq H_N \int_{\partial \mathbb{R}^N_+ \cap B} \frac{\varphi^2}{|x|} + c \|\varphi\|^2_{W^{1,q}(\mathbb{R}^N_+ \cap B)}, \quad \forall \varphi \in C_0^\infty(\overline{\mathbb{R}^N_+} \cap B).$$

REMARK 3.13. (a) The singular weight $\frac{1}{|x|}$ on the right-hand side of (1.43) is optimal, in the sense that it may not be replaced by $\frac{1}{|x|^\alpha}$ with $\alpha > 1$. This can be easily seen by choosing $\varphi \in H^1(\mathbb{R}^N_+)$ such that $\varphi(x) = |x|^{-\frac{N-2}{2}+\frac{\alpha-1}{2}}$ in a neighborhood of the origin. Moreover, the infimum in (1.44) is not achieved.
(b) In dimension $N = 2$ the infimum (1.44) is zero.
(c) Using Stirling's formula it is possible to verify that

$$H_N = \frac{N-3}{2} + O\left(\frac{1}{N}\right) \qquad \text{as } N \to \infty. \tag{3.41}$$

Let us turn our attention to the proof of Theorem 3.12. Following an idea of Brezis and Vázquez (equation (4.6) on page 453 of [20]) it turns out to be useful to replace φ in (1.43) by $v = \varphi/w$, where $w = w_\alpha$ with $\alpha = \frac{N-2}{2}$ as defined in (1.47). Observe that $C(N, \frac{N-2}{2}) = H_N$ by (3.57) and hence w is harmonic in the half space $\mathbb{R}^N_+$ and satisfies

$$\frac{\partial w}{\partial \nu} = H_N \frac{w}{|x|} \quad \text{on } \partial\mathbb{R}^N_+.$$

PROOF OF THEOREM 3.12. When $N \geq 3$, $C_0^\infty(\mathbb{R}^N_+ \setminus \{0\})$ is dense in $H^1(\mathbb{R}^N_+)$. So it suffices to prove (1.43) for $\varphi \in C_0^\infty(\mathbb{R}^N_+ \setminus \{0\})$. Fix such a $\varphi \not\equiv 0$ and let w be the function defined by (1.47). Notice that, on supp φ, w is smooth and bounded from above and from below by some positive constants. Hence $v := \frac{\varphi}{w} \in C_0^\infty(\overline{\mathbb{R}^N_+})$ is well defined. Now, $\varphi = vw$, $\nabla\varphi = v\nabla w + w\nabla v$ and

$$|\nabla\varphi|^2 = v^2|\nabla w|^2 + w^2|\nabla v|^2 + 2vw\nabla v\nabla w.$$

Integrating

$$\int_{\mathbb{R}^N_+} |\nabla\varphi|^2 = \int_{\mathbb{R}^N_+} v^2|\nabla w|^2 + \int_{\mathbb{R}^N_+} w^2|\nabla v|^2 + 2\int_{\mathbb{R}^N_+} vw\nabla v\nabla w$$

and by Green's formula

$$\int_{\mathbb{R}^N_+} v^2|\nabla w|^2 = \int_{\partial\mathbb{R}^N_+} v^2 w\frac{\partial w}{\partial \nu} - \int_{\mathbb{R}^N_+} w\nabla(v^2\nabla w)$$

$$= \int_{\partial\mathbb{R}^N_+} v^2 w\frac{\partial w}{\partial \nu} - 2\int_{\mathbb{R}^N_+} wv\nabla w\nabla v,$$

since w is harmonic in $\mathbb{R}^N_+$. Thus,

$$\begin{aligned}\int_{\mathbb{R}^N_+} |\nabla\varphi|^2 &= \int_{\mathbb{R}^N_+} w^2|\nabla v|^2 + \int_{\partial\mathbb{R}^N_+} v^2 w\frac{\partial w}{\partial \nu} \\ &= \int_{\mathbb{R}^N_+} w^2|\nabla v|^2 + \int_{\partial\mathbb{R}^N_+} \frac{\varphi^2}{w}\frac{\partial w}{\partial \nu}.\end{aligned} \tag{3.42}$$

But by (3.57) $\frac{\frac{\partial w}{\partial \nu}(x)}{w(x)} = \frac{H_N}{|x|}$ for $x \in \partial\mathbb{R}^N_+$ and hence,

$$\int_{\mathbb{R}^N_+} |\nabla\varphi|^2 \geq H_N \int_{\partial\mathbb{R}^N_+} \frac{\varphi^2}{|x|} + \int_{\mathbb{R}^N_+} w^2|\nabla v|^2 \quad \forall\varphi \in H^1(\mathbb{R}^N_+). \tag{3.43}$$

The second term on the right-hand side of the above inequality yields the improvement of Kato's inequality when φ has support in the unit ball.

Now we assume $\varphi \in C_0^\infty(\overline{\mathbb{R}^N_+} \setminus \{0\} \cap B)$ and, as before, set $v = \frac{\varphi}{w}$. Our aim is to prove that given $1 \leq q < 2$ there exists $C > 0$ such that

$$I := \int_{\mathbb{R}^N_+} w^2|\nabla v|^2 \geq \frac{1}{C}\|\varphi\|_{W^{1,q}}. \tag{3.44}$$

In spherical coordinates

$$I = \int_0^1 r^{N-1} \int_{S_1^+} w^2(r\theta)|\nabla v(r\theta)|^2 d\theta\, dr,$$

where $S_1^+ = S_1 \cap \mathbb{R}^N_+$ and $S_1 = \{x \in \mathbb{R}^N \,/\, |x| = 1\}$ is the sphere of radius 1. From (3.56) we have $w(x) \geq \frac{1}{C}|x|^{-\frac{N-2}{2}}$ for some $C > 0$ and all $x \in B \cap \mathbb{R}^N_+$. Hence

$$I \geq \frac{1}{C}\int_0^1 r \int_{S_1^+} |\nabla v(r\theta)|^2 d\theta\, dr.$$

Let us compute the Sobolev norm of φ:

$$\begin{aligned}
\|\varphi\|^q_{W^{1,q}} &= \int_{\mathbb{R}^N_+ \cap B} |\nabla\varphi|^q dx = \int_0^1 r^{N-1} \int_{S_1^+} |\nabla\varphi(r\theta)|^q d\theta\, dr \\
&= \int_0^1 r^{N-1} \int_{S_1^+} |\nabla v(r\theta)\, w(r\theta) + \nabla w(r\theta)\, v(r\theta)|^q d\theta\, dr \\
&\leq C_q \int_0^1 r^{N-1} \int_{S_1^+} |\nabla v(r\theta)|^q |w(r\theta)|^q + |\nabla w(r\theta)|^q |v(r\theta)|^q d\theta\, dr.
\end{aligned}$$

Define

$$\begin{aligned}
I_1 &:= \int_0^1 r^{N-1} \int_{S_1^+} |\nabla v(r\theta)|^q |w(r\theta)|^q d\theta\, dr \\
I_2 &:= \int_0^1 r^{N-1} \int_{S_1^+} |\nabla w(r\theta)|^q |v(r\theta)|^q d\theta\, dr.
\end{aligned}$$

Since $w(x) \leq C|x|^{-\frac{N-2}{2}}$ we have by Hölder's inequality

$$\begin{aligned}
I_1 &\leq C \int_0^1 r^{N-1-\frac{(N-2)q}{2}} \int_{S_1^+} |\nabla v(r\theta)|^q d\theta\, dr \\
&\leq C\left[\int_0^1 r \int_{S_1^+} |\nabla v(r\theta)|^2 d\theta dr\right]^{\frac{q}{2}} \left[\int_0^1 r^{(N-1-\frac{Nq}{2}+\frac{q}{2})\frac{2}{2-q}} dr\right]^{\frac{2-q}{2}} = C I^{\frac{q}{2}},
\end{aligned} \tag{3.45}$$

since $q < 2$.

Using $|\nabla w(x)| \le C|x|^{-\frac{N}{2}}$ we estimate I_2:

$$I_2 \le C \int_{S_1^+} \int_0^1 r^{N-1-\frac{Nq}{2}} |v(r\theta)|^q \, dr \, d\theta.$$

From the classical Hardy inequality

$$\int_0^1 r^\gamma |f(r)|^p \, dr \le \left(\frac{p}{\gamma+1} \right)^p \int_0^1 r^{\gamma+p} |f'(r)|^p \, dr$$

($p \ge 1, \gamma > -1, f \in C_0^\infty(0,1)$) we deduce

$$\int_0^1 r^{N-1-\frac{Nq}{2}} |v(r\theta)|^q \, dr \le C \int_0^1 r^{N-1-\frac{Nq}{2}+q} |\nabla v(r\theta)|^q \, dr$$

and therefore

$$I_2 \le C \int_{S_1^+} \int_0^1 r^{N-1-\frac{Nq}{2}+q} |\nabla v(r\theta)|^q \, dr \, d\theta.$$

Hölder's inequality yields

$$\begin{aligned} I_2 &\le C \left[\int_{S_1^+} \int_0^1 r |\nabla v(r\theta)|^2 \, dr \, d\theta \right]^{\frac{q}{2}} \left[\int_{S_1^+} \int_0^1 r^{(N-1-\frac{Nq}{2}+\frac{q}{2})\frac{2}{2-q}} \, dr \, d\theta \right]^{1-\frac{q}{2}} \\ &= C I^{\frac{q}{2}}, \end{aligned} \tag{3.46}$$

where we have used $q < 2$. Gathering (3.45) and (3.46) we conclude that (3.44) holds. □

3.4. *Boundedness of the extremal solution in the exponential case*

In this section we shall give a proof of Theorem 1.20. We proceed by contradiction, assuming that u^* is unbounded. A central point in the argument is to obtain some information of the singularity that u^* should have at the origin. More precisely, we claim that for any $0 < \sigma < 1$ there exists $r > 0$ such that

$$u^*(x) \ge (1-\sigma) \log \frac{1}{|x|} \quad \forall x \in \Gamma_1, |x| \le r. \tag{3.47}$$

Observe first that for all $0 < \lambda < \lambda^*$ the minimal solution u_λ is symmetric in the variables $x_1, \ldots, x_{N-1}$ by uniqueness of the minimal solution and the symmetry of Ω. Moreover, using the symmetry and convexity assumptions on Ω combined with the moving-plane method (see Proposition 5.2 in [32]) we deduce that u_λ achieves its maximum at the origin.

Assume by contradiction that (3.47) is false. Then there exists $\sigma > 0$ and a sequence $x_k \in \Gamma_1$ with $x_k \to 0$ such that

$$u^*(x_k) < (1-\sigma)\log\frac{1}{|x_k|}. \tag{3.48}$$

Let $s_k = |x_k|$ and choose $0 < \lambda_k < \lambda^*$ such that

$$\max_{\overline{\Omega}} u_{\lambda_k} = u_{\lambda_k}(0) = \log\frac{1}{s_k}. \tag{3.49}$$

Note that $\lambda_k \to \lambda^*$, otherwise u_{λ_k} would remain bounded. Let

$$v_k(x) = \frac{u_{\lambda_k}(s_k x)}{\log\frac{1}{s_k}} \qquad x \in \Omega_k \equiv \frac{1}{s_k}\Omega.$$

Then $0 \le v_k \le 1$, $v_k(0) = 1$, $\Delta v_k = 0$ in Ω_k and

$$\begin{aligned}\frac{\partial v_k}{\partial \nu}(x) &= \frac{1}{\log\frac{1}{s_k}} s_k\lambda_k \exp(u_{\lambda_k}(s_k x))\\ &\le \frac{\lambda_k}{\log\frac{1}{s_k}} \to 0,\end{aligned}$$

by (3.49). By elliptic regularity $v_k \to v$ uniformly on compact sets of $\overline{\mathbb{R}^N_+}$ to a function v satisfying $0 \le v \le 1$, $v(0) = 1$, $\Delta v = 0$ in $\mathbb{R}^N_+$, $\frac{\partial v}{\partial \nu} = 0$ on $\partial\mathbb{R}^N_+$. Extending v evenly to $\mathbb{R}^N$ we deduce that $v \equiv 1$. Since $|x_k| = s_k$ we deduce that

$$\frac{u_{\lambda_k}(x_k)}{\log\frac{1}{s_k}} \to 1,$$

which contradicts (3.48).

Now we use (3.47) to obtain a contradiction with the stability property of u^*. Let $\phi(x) = \int_{\partial\mathbb{R}^N_+} K(x,y)|y|^{2-N+\varepsilon}dy$ and $\psi(x) = \int_{\partial\mathbb{R}^N_+} K(x,y)|y|^{\frac{2-N+\varepsilon}{2}}dy$. Then,

$$\frac{\partial\phi}{\partial\nu} = K_\phi|x|^{1-N+\varepsilon} \qquad \frac{\partial\psi}{\partial\nu} = K_\psi|x|^{\frac{-N+\varepsilon}{2}}, \tag{3.50}$$

where the constants K_ϕ, K_ψ are given by

$$K_\phi = \lambda_{0,N}\varepsilon + O(\varepsilon^2) \quad \text{and} \quad K_\psi = H_N + O(\varepsilon).$$

Indeed, since u_0 and ϕ are harmonic in Ω,

$$\int_{\partial\Omega} u_0\frac{\partial\phi}{\partial\nu} = \int_{\partial\Omega}\phi\frac{\partial u_0}{\partial\nu}.$$

Clearly, $\int_{\Gamma_2} \left|\phi \frac{\partial u_0}{\partial \nu}\right| \le C$, for some constant C independent of ε. So

$$K_\phi \int_0^1 \log\left(\frac{1}{r}\right) \frac{1}{r} r^{2-N+\varepsilon} r^{N-2} dr = \lambda_{0,N} \int_0^1 \frac{1}{r} r^{2-N+\varepsilon} r^{N-2} dr + O(1)$$
$$= \frac{\lambda_{0,N}}{\varepsilon} + O(1).$$

Now, $\int_0^1 \log \frac{1}{r} r^{-1+\varepsilon} dr = \frac{1}{\varepsilon^2}$ so we end up with

$$K_\phi = \lambda_{0,N}\varepsilon + O(\varepsilon^2).$$

Similarly, since ψ and w (defined in (1.47)) are harmonic in Ω, we have

$$\int_{\partial\Omega} w \frac{\partial \psi}{\partial \nu} = \int_{\partial\Omega} \psi \frac{\partial w}{\partial \nu}.$$

As before the boundary terms on Γ_2 are bounded independently of ε so

$$K_\psi \int_0^1 r^{-1+\varepsilon} dr = H_N \int_0^1 r^{-1+\varepsilon} dr + O(1).$$

Hence,

$$K_\psi = H_N + O(\varepsilon).$$

Multiplying (1.34) by ϕ and integrating by parts twice yields

$$\int_{\partial\Omega} u_\lambda \frac{\partial \phi}{\partial \nu} = \lambda \int_{\partial\Omega} \phi \frac{\partial u_\lambda}{\partial \nu} = \lambda \int_{\Gamma_1} \phi e^{u_\lambda} + \lambda \int_{\Gamma_2} \phi \frac{\partial u_\lambda}{\partial \nu} \le \lambda \int_{\Gamma_1} \phi e^{u_\lambda}. \tag{3.51}$$

Let $\eta \in C^\infty(\mathbb{R}^N)$ be such that $\eta \equiv 1$ in $B_R(0)$, where $R > 0$ is small and fixed, and $\eta = 0$ on Γ_2. Using the stability condition (1.37) with $\eta\psi$ yields

$$\lambda \int_{\Gamma_1 \cap B_R(0)} e^{u_\lambda} \psi^2 \le \int_\Omega |\nabla(\eta\psi)|^2 = \int_{\partial\Omega} \frac{\partial}{\partial \nu}(\eta\psi)(\eta\psi) - \int_\Omega (\eta\psi)\Delta(\eta\psi)$$
$$\le \int_{\Gamma_1 \cap B_R(0)} \frac{\partial \psi}{\partial \nu} \psi + C, \tag{3.52}$$

where the constant C does not depend on ε and λ. Since $\psi^2 = \phi$ on $\partial\mathbb{R}^N_+$ combining (3.51) and (3.52) we obtain

$$\int_{\partial\Omega} u_\lambda \frac{\partial \phi}{\partial \nu} \le \int_{\Gamma_1 \cap B_R(0)} \frac{\partial \psi}{\partial \nu} \psi + C$$

and letting $\lambda \nearrow \lambda^*$ we find

$$\int_{\partial\Omega} u^* \frac{\partial \phi}{\partial \nu} \le \int_{\Gamma_1 \cap B_R(0)} \frac{\partial \psi}{\partial \nu} \psi + C.$$

Using (3.50) we arrive at

$$K_\phi \int_{\Gamma_1 \cap B_R(0)} u^* |x|^{1-N+\varepsilon} \le K_\psi \int_{\Gamma_1 \cap B_R(0)} |x|^{1-N+\varepsilon} + C$$

and thus

$$\int_{\Gamma_1 \cap B_R(0)} u^* |x|^{1-N+\varepsilon} \le \frac{1}{\varepsilon^2} \omega_{N-1} \frac{H_N}{\lambda_{0,N}} + O\left(\frac{1}{\varepsilon}\right),$$

where ω_{N-1} is the area of the $N-1$-dimensional sphere. We rewrite this inequality as

$$\int_0^R r^{-1+\varepsilon} u^*(r)\, dr \le \frac{1}{\varepsilon^2} \frac{H_N}{\lambda_{0,N}} + O\left(\frac{1}{\varepsilon}\right). \tag{3.53}$$

Let $\sigma > 0$ and $r(\sigma) > 0$ be such that (3.47) holds for $|x| \le r(\sigma)$. Then using (3.47) and (3.53) we find

$$(1-\sigma) \int_0^{r(\sigma)} \log \frac{1}{r} r^{\varepsilon - 1}\, dr \le \frac{1}{\varepsilon} \frac{K_\psi}{K_\phi} + C = \frac{1}{\varepsilon^2} \frac{H_N}{\lambda_{0,N}} + O\left(\frac{1}{\varepsilon}\right). \tag{3.54}$$

Integrating

$$(1-\sigma)\left(\frac{1}{\varepsilon^2} r(\sigma)^\varepsilon + \frac{1}{\varepsilon} r(\sigma)^\varepsilon \log \frac{1}{r(\sigma)}\right) \le \frac{1}{\varepsilon^2} \frac{H_N}{\lambda_{0,N}} + O\left(\frac{1}{\varepsilon}\right).$$

Letting $\varepsilon \to 0$ yields

$$(1-\sigma) \le \frac{H_N}{\lambda_{0,N}}.$$

As σ is arbitrarily small we deduce $H_N \ge \lambda_{0,N}$. But

$$H_N \ge \lambda_{0,N} \quad \text{if and only if} \quad N \ge 10 \tag{3.55}$$

(see [45]). This proves the theorem. □

3.5. *Auxiliary computations*

PROOF OF LEMMA 1.21. We write $x = (x', x_N) \in \mathbb{R}^N_+$ with $x' \in \mathbb{R}^{N-1}$, $x_N > 0$. It follows from (1.47) and a simple change of variables that

$$w_\alpha(x', x_N) = w_\alpha(e(x'), x_N) \quad \text{for all rotations } e \in \mathcal{O}(N-1),$$

and similarly

$$w_\alpha(Rx', Rx_N) = R^{-\alpha} w_\alpha(x', x_N). \tag{3.56}$$

Differentiating with respect to x_N yields

$$\frac{\partial w_\alpha}{\partial x_N}(Rx', Rx_N) = R^{-\alpha-1}\frac{\partial w_\alpha}{\partial x_N}(x', x_N).$$

Let $x \in \partial\mathbb{R}^N_+$, $x = (x', 0)$ and plug $R = \frac{1}{|x|} = \frac{1}{|x'|}$ in the previous formula to find

$$\frac{\partial w_\alpha}{\partial \nu}(x) = -\frac{\partial w_\alpha}{\partial x_N}(x', 0) = |x|^{-\alpha-1}\left(-\frac{\partial w_\alpha}{\partial x_N}\left(\frac{x'}{|x'|}, 0\right)\right).$$

Define

$$C(N,\alpha) = -\frac{\partial w_\alpha}{\partial x_N}\left(\frac{x'}{|x'|}, 0\right) \tag{3.57}$$

and observe that it is independent of $x' \in \mathbb{R}^{N-1}$.

Using (3.56) and the radial symmetry of w in the variables x', there exists a function $v : [0,\infty) \to \mathbb{R}$ such that

$$w_\alpha(x', x_N) = |x'|^{-\alpha} w_\alpha\left(\frac{x'}{|x'|}, \frac{x_N}{|x'|}\right) = |x'|^{-\alpha} v\left(\frac{x_N}{|x'|}\right). \tag{3.58}$$

Writing $r = |x'|$, $t = \frac{x_N}{|x'|}$, we have

$$r^{-\alpha}v(t) = w_\alpha(x', rt), \quad \forall x' \in \mathbb{R}^{N-1}, \ |x'| = r.$$

The equation $\Delta w = 0$ is equivalent to

$$(1+t^2)v''(t) + (2\alpha+4-N)tv'(t) + \alpha(\alpha-N+3)v(t) = 0, \quad t > 0, \tag{3.59}$$

while (1.48) implies

$$v(0) = 1.$$

The initial condition for v' is related to (3.57)

$$v'(0) = -C(N,\alpha).$$

In addition to these initial conditions we remark that w_α is a smooth function in $\mathbb{R}^N_+$ and this together with (3.58) implies that

$$\lim_{t\to\infty} v(t)t^\alpha \text{ exists.} \tag{3.60}$$

Using the change of variables $z = it$ with i the imaginary unit and defining the new unknown $h(z) := v(-iz)$ equation (3.59) becomes

$$(1-z^2)h''(z) - (2\alpha+4-N)zh'(z) - \alpha(\alpha-N+3)h(z) = 0, \tag{3.61}$$

with initial conditions

$$\lim_{t>0,t\to 0} h(it) = 1, \qquad \lim_{t>0,t\to 0} h'(it) = iC(N,\alpha). \tag{3.62}$$

On the other hand (3.60) implies

$$\lim_{t\in\mathbb{R},t\to\infty} h(it)t^{\alpha} \text{ exists.} \tag{3.63}$$

The substitution

$$g(z) = (1-z^2)^{\frac{\alpha}{2}+\frac{1}{2}-\frac{N}{4}} h(z) \tag{3.64}$$

transforms equation (3.61) into

$$(1-z^2)g''(z) - 2zg'(z) + \left(\nu(\nu+1) - \frac{\mu^2}{1-z^2}\right) g(z) = 0, \tag{3.65}$$

with

$$\mu = \alpha + \frac{2-N}{2}, \qquad \nu = \frac{N-4}{2}. \tag{3.66}$$

The general solution to (3.65) is well known. Indeed, equation (3.65) belongs to the class of Legendre's equations. Two linearly independent solutions of (3.65) are given by the Legendre functions $P_\nu^\mu(z)$, $Q_\nu^\mu(z)$ (see [1]), which are defined in $\mathbb{C} \setminus \{-1, 1\}$ and analytic in $\mathbb{C} \setminus (-\infty, 1]$ (see [1, Formulas 8.1.2 – 8.1.6]). Moreover the limits of $P_\nu^\mu(z)$, $Q_\nu^\mu(z)$ on both sides of $(-1, 1)$ exist and we shall use the notation

$$\begin{aligned} P_\nu^\mu(x+i0) &= \lim_{z\to x,\mathrm{Im}(z)>0} P_\nu^\mu(z), \quad -1 < x < 1, \\ P_\nu^\mu(x-i0) &= \lim_{z\to x,\mathrm{Im}(z)<0} P_\nu^\mu(z), \quad -1 < x < 1, \end{aligned} \tag{3.67}$$

and a similar notation for Q_ν^μ.

The solution g of (3.65) is therefore given by

$$g(z) = c_1 P_\nu^\mu(z) + c_2 Q_\nu^\mu(z),$$

for appropriate constants c_1, c_2. These constants are determined by the initial conditions (3.62), which imply:

$$c_1 P_\nu^\mu(0+i0) + c_2 Q_\nu^\mu(0+i0) = 1, \tag{3.68}$$

$$c_1 \frac{d}{dz} P_\nu^\mu(0+i0) + c_2 \frac{d}{dz} Q_\nu^\mu(0+i0) = iC(N,\alpha). \tag{3.69}$$

In order to evaluate $C(N,\alpha)$, we also use condition (3.63), which is equivalent to

$$\lim_{t\to\infty,\ t\in\mathbb{R}} (c_1 P_\nu^\mu(it) + c_2 Q_\nu^\mu(it)) t^{\frac{N}{2}-1} \text{ exists.} \tag{3.70}$$

But according to [1, Formulas 8.1.3, 8.1.5]

$$P_\nu^\mu(z) \sim z^\nu \quad \text{as } |z| \to \infty$$
$$Q_\nu^\mu(z) \sim z^{-\nu-1} \quad \text{as } |z| \to \infty.$$

This and (3.64), (3.70) imply that $c_1 = 0$ and we obtain from (3.68), (3.69)

$$C(N, \alpha) = -i \frac{\frac{d}{dz} Q_\nu^\mu(0 + i0)}{Q_\nu^\mu(0 + i0)}. \tag{3.71}$$

From the properties and formulas in [1] the following values can be deduced:

$$Q_\nu^\mu(0 + i0) = -i 2^{\mu-1} \pi^{\frac{1}{2}} e^{i\mu\pi - i\nu\frac{\pi}{2}} \frac{\Gamma(\frac{\nu}{2} + \frac{\mu}{2} + \frac{1}{2})}{\Gamma(\frac{\nu}{2} - \frac{\mu}{2} + 1)} \tag{3.72}$$

$$\frac{d}{dz} Q_\nu^\mu(0 + i0) = 2^\mu \pi^{\frac{1}{2}} e^{i\mu\pi - i\nu\frac{\pi}{2}} \frac{\Gamma(\frac{\nu}{2} + \frac{\mu}{2} + 1)}{\Gamma(\frac{\nu}{2} - \frac{\mu}{2} + \frac{1}{2})}. \tag{3.73}$$

The relations (3.71), (3.72), (3.73) and the values (3.66) yield formula (1.49). □

4. A fourth-order variant of the Gelfand problem

4.1. *Comparison principles*

LEMMA 4.1 (Boggio's principle, [15]). *If $u \in C^4(\overline{B}_R)$ satisfies*

$$\begin{cases} \Delta^2 u \geq 0 & \text{in } B_R \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial B_R \end{cases}$$

then $u \geq 0$ in B_R.

LEMMA 4.2. *Let $u \in L^1(B_R)$ and suppose that*

$$\int_{B_R} u \Delta^2 \varphi \geq 0$$

for all $\varphi \in C^4(\overline{B}_R)$ such that $\varphi \geq 0$ in B_R, $\varphi|_{\partial B_R} = 0 = \frac{\partial \varphi}{\partial n}|_{\partial B_R}$. Then $u \geq 0$ in B_R.

PROOF. Let $\zeta \in C_0^\infty(\Omega)$, $\zeta \geq 0$ and solve

$$\begin{cases} \Delta^2 \varphi = \zeta & \text{in } B_R \\ \varphi = \dfrac{\partial \varphi}{\partial n} = 0 & \text{on } \partial B_R. \end{cases}$$

By Boggio's principle $\varphi \geq 0$ in B_R and we deduce that $\int_{B_R} u\zeta \geq 0$. Since $\zeta \in C_0^\infty(\Omega)$, $\zeta \geq 0$ is arbitrary we deduce $u \geq 0$. □

LEMMA 4.3. *If $u \in H^2(B_R)$ is radial, $\Delta^2 u \geq 0$ in B_R in the weak sense, that is*

$$\int_{B_R} \Delta u \Delta \varphi \geq 0 \quad \forall \varphi \in C_0^\infty(B_R),\ \varphi \geq 0$$

and $u|_{\partial B_R} \geq 0$, $\frac{\partial u}{\partial n}|_{\partial B_R} \leq 0$ then $u \geq 0$ in B_R.

PROOF. We only deal with the case $R = 1$ for simplicity. Solve

$$\begin{cases} \Delta^2 u_1 = \Delta^2 u & \text{in } B_1 \\ u_1 = \dfrac{\partial u_1}{\partial n} = 0 & \text{on } \partial B_1 \end{cases}$$

in the sense $u_1 \in H_0^2(B_1)$ and $\int_{B_1} \Delta u_1 \Delta \varphi = \int_{B_1} \Delta u \Delta \varphi$ for all $\varphi \in C_0^\infty(B_1)$. Then $u_1 \geq 0$ in B_1 by Lemma 4.2.

Let $u_2 = u - u_1$ so that $\Delta^2 u_2 = 0$ in B_1. Define $f = \Delta u_2$. Then $\Delta f = 0$ in B_1 and since f is radial we find that f is constant. It follows that $u_2 = ar^2 + b$. Using the boundary conditions we deduce $a + b \geq 0$ and $a \leq 0$, which imply $u_2 \geq 0$. □

Similarly we have

LEMMA 4.4. *If $u \in H^2(B_R)$ and $\Delta^2 u \geq 0$ in B_R in the weak sense, that is*

$$\int_{B_R} \Delta u \Delta \varphi \geq 0 \quad \forall \varphi \in C_0^\infty(B_R), \varphi \geq 0$$

and $u|_{\partial B_R} = 0$, $\frac{\partial u}{\partial n}|_{\partial B_R} \leq 0$ then $u \geq 0$ in B_R.

The next lemma is a consequence of a decomposition lemma of Moreau [95]. For a proof see [67] or [68].

LEMMA 4.5. *Let $u \in H_0^2(B_R)$. Then there exist unique $w, v \in H_0^2(B_R)$ such that $u = w + v$, $w \geq 0$, $\Delta^2 v \leq 0$ in B_R and $\int_{B_R} \Delta w \Delta v = 0$.*

PROOF OF LEMMA 1.29. (a) Let $u = u_1 - u_2$. By Lemma 4.5 there exist $w, v \in H_0^2(B_R)$ such that $u = w + v$, $w \geq 0$ and $\Delta^2 v \leq 0$. Observe that $v \leq 0$ so $w \geq u_1 - u_2$. By hypothesis we have

$$\int_{B_R} \Delta(u_1 - u_2)\Delta\varphi \leq \lambda \int_{B_R} (e^{u_1} - e^{u_2})\varphi \quad \forall \varphi \in C_0^\infty(B_R), \varphi \geq 0,$$

and by density this holds also for w:

$$\int_{B_R} (\Delta w)^2 = \int_{B_R} \Delta(u_1 - u_2)\Delta w \leq \lambda \int_{B_R} (e^{u_1} - e^{u_2}) w, \tag{4.1}$$

where the equality holds because $\int_{B_R} \Delta w \Delta v = 0$. By density we deduce from (1.39)

$$\lambda \int_{B_R} e^{u_1} w^2 \leq \int_{B_R} (\Delta w)^2. \tag{4.2}$$

Combining (4.1) and (4.2) we obtain

$$\int_{B_R} e^{u_1} w^2 \le \int_{B_R} (e^{u_1} - e^{u_2}) w.$$

Since $u_1 - u_2 \le w$ the previous inequality implies

$$0 \le \int_{B_R} (e^{u_1} - e^{u_2} - e^{u_1}(u_1 - u_2)) w. \tag{4.3}$$

But by convexity of the exponential function $e^{u_1} - e^{u_2} - e^{u_1}(u_1 - u_2) \le 0$ and we deduce from (4.3) that $(e^{u_1} - e^{u_2} - e^{u_1}(u_1 - u_2))w = 0$. Recalling that $u_1 - u_2 \le w$ we deduce that $u_1 \le u_2$.

(b) We solve for $\tilde{u} \in H_0^2(B_R)$ such that

$$\int_{B_R} \Delta\tilde{u}\Delta\varphi = \int_{B_R} \Delta(u_1 - u_2)\Delta\varphi \quad \forall\varphi \in C_0^\infty(B_R).$$

By Lemma 4.3 it follows that $\tilde{u} \ge u_1 - u_2$. Next we apply the decomposition of Lemma 4.5 to $\tilde{u}$, that is $\tilde{u} = w + v$ with $w, v \in H_0^2(B_R)$, $w \ge 0$, $\Delta^2 v \le 0$ in B_R and $\int_{B_R} \Delta w \Delta v = 0$. Then the argument follows that of Lemma 1.29. □

4.2. *Uniqueness of the extremal solution*

PROOF OF THEOREM 1.24. Suppose that $v \in H^2(B)$ satisfies (1.52), (1.53) and $v \not\equiv u^*$. Notice that we do not need v to be radial.

The idea of the proof is as follows:

Step 1. The function

$$u_0 = \frac{1}{2}(u^* + v)$$

is a supersolution to the following problem

$$\begin{cases} \Delta^2 u = \lambda^* e^u + \mu\eta e^u & \text{in } B \\ u = a & \text{on } \partial B \\ \dfrac{\partial u}{\partial n} = b & \text{on } \partial B \end{cases} \tag{4.4}$$

for some $\mu = \mu_0 > 0$, where $\eta \in C_0^\infty(B)$, $0 \le \eta \le 1$ is a fixed radial cut-off function such that

$$\eta(x) = 1 \quad \text{for } |x| \le \tfrac{1}{2}, \qquad \eta(x) = 0 \quad \text{for } |x| \ge \tfrac{3}{4}.$$

Step 2. Using a solution to (4.4) we construct, for some $\lambda > \lambda^*$, a supersolution to (1.50). This provides a solution u_λ for some $\lambda > \lambda^*$, which is a contradiction.

Proof of Step 1. Observe that given $0 < R < 1$ we must have for some $c_0 = c_0(R) > 0$

$$v(x) \geq u^*(x) + c_0 \quad |x| \leq R. \tag{4.5}$$

To prove this we recall the Green's function for Δ^2 with Dirichlet boundary conditions

$$\begin{cases} \Delta_x^2 G(x,y) = \delta_y & x \in B \\ G(x,y) = 0 & x \in \partial B \\ \dfrac{\partial G}{\partial n}(x,y) = 0 & x \in \partial B, \end{cases}$$

where δ_y is the Dirac mass at $y \in B$. Boggio gave an explicit formula for $G(x, y)$ which was used in [71] to prove that in dimension $N \geq 5$ (the case $1 \leq N \leq 4$ can be treated similarly)

$$G(x,y) \sim |x-y|^{4-N} \min\left(1, \frac{d(x)^2 d(y)^2}{|x-y|^4}\right), \tag{4.6}$$

where

$$d(x) = dist(x, \partial B) = 1 - |x|,$$

and $a \sim b$ means that for some constant $C > 0$ we have $C^{-1}a \leq b \leq Ca$ (uniformly for $x, y \in B$). Formula (4.6) yields

$$G(x,y) \geq c d(x)^2 d(y)^2 \tag{4.7}$$

for some $c > 0$ and this in turn implies that for smooth functions $\tilde{v}$ and $\tilde{u}$ such that $\tilde{v} - \tilde{u} \in H_0^2(B)$ and $\Delta^2(\tilde{v} - \tilde{u}) \geq 0$,

$$\begin{aligned} \tilde{v}(y) - \tilde{u}(y) &= \int_{\partial B} \left(\frac{\partial \Delta_x G}{\partial n_x}(x,y)(\tilde{v} - \tilde{u}) - \Delta_x G(x,y) \frac{\partial(\tilde{v} - \tilde{u})}{\partial n} \right) dx \\ &\quad + \int_B G(x,y) \Delta^2 (\tilde{v} - \tilde{u})\, dx \\ &\geq c d(y)^2 \int_B (\Delta^2 \tilde{v} - \Delta^2 \tilde{u}) d(x)^2\, dx. \end{aligned}$$

Using a standard approximation procedure, we conclude that

$$v(y) - u^*(y) \geq c d(y)^2 \lambda^* \int_B (e^v - e^{u^*}) d(x)^2\, dx.$$

Since $v \geq u^*$, $v \not\equiv u^*$ we deduce (4.5).

Let $u_0 = (u^* + v)/2$. Then by Taylor's theorem

$$e^v = e^{u_0} + (v - u_0)e^{u_0} + \frac{1}{2}(v - u_0)^2 e^{u_0} + \frac{1}{6}(v - u_0)^3 e^{u_0} + \frac{1}{24}(v - u_0)^4 e^{\xi_2} \tag{4.8}$$

for some $u_0 \le \xi_2 \le v$ and

$$e^{u^*} = e^{u_0} + (u^* - u_0)e^{u_0} + \frac{1}{2}(u^* - u_0)^2 e^{u_0} + \frac{1}{6}(u^* - u_0)^3 e^{u_0} + \frac{1}{24}(u^* - u_0)^4 e^{\xi_1} \tag{4.9}$$

for some $u^* \le \xi_1 \le u_0$. Adding (4.8) and (4.9) yields

$$\frac{1}{2}(e^v + e^{u^*}) \ge e^{u_0} + \frac{1}{8}(v - u^*)^2 e^{u_0}. \tag{4.10}$$

From (4.5) with $R = 3/4$ and (4.10) we see that $u_0 = (u^* + v)/2$ is a supersolution of (4.4) with $\mu_0 := c_0/8$.

Proof of Step 2. Let us now show how to obtain a weak supersolution of (1.50) for some $\lambda > \lambda^*$. Given $\mu > 0$, let u denote the minimal solution to (4.4). Define φ_1 as the solution to

$$\begin{cases} \Delta^2 \varphi_1 = \mu \eta e^u & \text{in } B \\ \varphi_1 = 0 & \text{on } \partial B \\ \dfrac{\partial \varphi_1}{\partial n} = 0 & \text{on } \partial B, \end{cases}$$

and φ_2 be the solution of

$$\begin{cases} \Delta^2 \varphi_2 = 0 & \text{in } B \\ \varphi_2 = a & \text{on } \partial B \\ \dfrac{\partial \varphi_2}{\partial n} = b & \text{on } \partial B. \end{cases}$$

If $N \ge 5$ (the case $1 \le N \le 4$ can be treated similarly), relation (4.7) yields

$$\varphi_1(x) \ge c_1 d(x)^2 \quad \text{for all } x \in B, \tag{4.11}$$

for some $c_1 > 0$. But u is a radial solution of (4.4) and therefore it is smooth in $B \setminus B_{1/4}$. Thus

$$u(x) \le M\varphi_1 + \varphi_2 \quad \text{for all } x \in B_{1/2}, \tag{4.12}$$

for some $M > 0$. Therefore, from (4.11) and (4.12), for $\lambda > \lambda^*$ with $\lambda - \lambda^*$ sufficiently small we have

$$\left(\frac{\lambda}{\lambda^*} - 1\right) u \le \varphi_1 + \left(\frac{\lambda}{\lambda^*} - 1\right) \varphi_2 \quad \text{in } B.$$

Let $w = \frac{\lambda}{\lambda^*} u - \varphi_1 - (\frac{\lambda}{\lambda^*} - 1)\varphi_2$. The inequality just stated guarantees that $w \le u$. Moreover

$$\Delta^2 w = \lambda e^u + \frac{\lambda \mu}{\lambda^*} \eta e^u - \mu \eta e^u \ge \lambda e^u \ge \lambda e^w \quad \text{in } B$$

and

$$w = a \quad \frac{\partial w}{\partial n} = b \quad \text{on } \partial B.$$

Therefore w is a supersolution to (1.50) for λ. By the method of sub and supersolutions a solution to (1.50) exists for some $\lambda > \lambda^*$, which is a contradiction. □

PROOF OF COROLLARY 1.26. Let u denote the extremal solution of (1.50) with $b \geq -4$. We may also assume that $a = 0$. If u is smooth, then the result is trivial. So we restrict to the case where u is singular. By Theorem 1.25 we have in particular that $N \geq 13$. If $b = -4$ by Theorem 1.24 we know that if $N \geq 13$ then $u = -4 \log |x|$ so that the desired conclusion holds. Hence we assume $b > -4$ in this section.

For $\rho > 0$ define

$$u_\rho(r) = u(\rho r) + 4 \log \rho,$$

so that

$$\Delta^2 u_\rho = \lambda^* e^{u_\rho} \quad \text{in } B_{1/\rho}.$$

Then

$$\left.\frac{du_\rho}{d\rho}\right|_{\rho=1, r=1} = u'(1) + 4 > 0.$$

Hence, there is $\delta > 0$ such that

$$u_\rho(r) < u(r) \quad \text{for all } 1 - \delta < r \leq 1, 1 - \delta < \rho \leq 1.$$

This implies

$$u_\rho(r) < u(r) \quad \text{for all } 0 < r \leq 1, 1 - \delta < \rho \leq 1. \tag{4.13}$$

Otherwise set

$$r_0 = \sup\{0 < r < 1 | u_\rho(r) \geq u(r)\}.$$

This definition yields

$$u_\rho(r_0) = u(r_0) \quad \text{and} \quad u'_\rho(r_0) \leq u'(r_0). \tag{4.14}$$

Write $\alpha = u(r_0)$, $\beta = u'(r_0)$. Then u satisfies

$$\begin{cases} \Delta^2 u = \lambda e^u & \text{on } B_{r_0} \\ u(r_0) = \alpha \\ u'(r_0) = \beta \end{cases}$$

while u_ρ is a supersolution to this problem, since $u'_\rho(r_0) \leq \beta$ by (4.14). But this problem does not have a strict supersolution, and we conclude that

$$u(r) = u_\rho(r) \quad \text{for all } 0 < r \leq r_0,$$

which in turn implies by standard ODE theory that

$$u(r) = u_\rho(r) \quad \text{for all } 0 < r \leq 1,$$

a contradiction. This proves estimate (4.13).

From (4.13) we see that

$$\left.\frac{du_\rho}{d\rho}\right|_{\rho=1}(r) \geq 0 \quad \text{for all } 0 < r \leq 1. \tag{4.15}$$

But

$$\left.\frac{du_\rho}{d\rho}\right|_{\rho=1}(r) = u'(r)r + 4 \quad \text{for all } 0 < r \leq 1$$

and this together with (4.15) implies

$$\frac{du_\rho}{d\rho}(r) = \frac{1}{\rho}(u'(\rho r)\rho r + 4) \geq 0 \quad \text{for all } 0 < r \leq \frac{1}{\rho}, 0 < \rho \leq 1, \tag{4.16}$$

which means that $u_\rho(r)$ is nondecreasing in ρ. We wish to show that $\lim_{\rho\to 0} u_\rho(r)$ exists for all $0 < r \leq 1$. For this we shall show

$$u_\rho(r) \geq -4\log(r) + \log\left(\frac{8(N-2)(N-4)}{\lambda^*}\right) \quad \text{for all } 0 < r \leq \frac{1}{\rho}, 0 < \rho \leq 1. \tag{4.17}$$

Set

$$u_0(r) = -4\log(r) + \log\left(\frac{8(N-2)(N-4)}{\lambda^*}\right)$$

and suppose that (4.17) is not true for some $0 < \rho < 1$. Let

$$r_1 = \sup\{0 < r < 1/\rho | u_\rho(r) < u_0(r)\}.$$

Observe that $\lambda^* > 8(N-2)(N-4)$. Otherwise $w = -4\log r$ would be a strict supersolution of the equation satisfied by u, which is not possible by Theorem 1.24. In particular, $r_1 < 1/\rho$ and

$$u_\rho(r_1) = u_0(r_1) \quad \text{and} \quad u'_\rho(r_1) \geq u'_0(r_1).$$

It follows that u_0 is a supersolution of

$$\begin{cases} \Delta^2 u = \lambda^* e^u & \text{in } B_{r_1} \\ u = A & \text{on } \partial B_{r_1} \\ \dfrac{\partial u}{\partial n} = B & \text{on } \partial B_{r_1}, \end{cases} \tag{4.18}$$

with $A = u_\rho(r_1)$ and $B = u'_\rho(r_1)$. Since u_ρ is a singular stable solution of (4.18), it is the extremal solution of the problem by Proposition 1.28. By Theorem 1.24, there is no strict supersolution of (4.18) and we conclude that $u_\rho \equiv u_0$ first for $0 < r < r_1$ and then for $0 < r \leq 1/\rho$. This is impossible for $\rho > 0$ because $u_\rho(1/\rho) = 0$ and $u_0(1/\rho) < 0$. This proves (4.17).

By (4.16) and (4.17) we see that

$$v(r) = \lim_{\rho\to 0} u_\rho(r) \quad \text{exists for all } 0 < r < +\infty,$$

where the convergence is uniform (even in C^k for any k) on compact sets of $\mathbb{R}^N \setminus \{0\}$. Moreover v satisfies

$$\Delta^2 v = \lambda^* e^v \quad \text{in } \mathbb{R}^N \setminus \{0\}. \tag{4.19}$$

Then for any $r > 0$

$$v(r) = \lim_{\rho\to 0} u_\rho(r) = \lim_{\rho\to 0} u(\rho r) + 4\log(\rho r) - 4\log(r) = v(1) - 4\log(r).$$

Hence, using equation (4.19) we obtain

$$v(r) = -4\log r + \log\left(\frac{8(N-2)(N-4)}{\lambda^*}\right) = u_0(r).$$

But then

$$u'_\rho(r) = u'(\rho r)\rho \to -4, \quad \text{as } \rho \to 0,$$

and therefore, with $r = 1$

$$\rho u'(\rho) \to -4 \quad \text{as } \rho \to 0. \quad \square \tag{4.20}$$

PROOF OF PROPOSITION 1.28. Let $u \in H^2(B)$, $\lambda > 0$ be a weak unbounded solution of (1.50). If $\lambda < \lambda^*$ from Lemma 1.29 we find that $u \le u_\lambda$, where u_λ is the minimal solution. This is impossible because u_λ is smooth and u unbounded. If $\lambda = \lambda^*$ then necessarily $u = u^*$ by Theorem 1.24. $\square$

4.3. *A computer-assisted proof for dimensions* $13 \le N \le 31$

Throughout this section we assume $a = b = 0$. As was mentioned before, the proof of Theorem 1.27 relies on precise estimates for u^* and λ^*. We first present some conditions under which it is possible to find these estimates. Later we show how to meet such conditions with a computer-assisted verification.

The first lemma is analogous to Lemma 1.30.

LEMMA 4.6. *Suppose there exist* $\varepsilon > 0$, $\lambda > 0$ *and a radial function* $u \in H^2(B) \cap W^{4,\infty}_{\text{loc}}(B \setminus \{0\})$ *such that*

$$\Delta^2 u \le \lambda e^u \quad \textit{for all } 0 < r < 1$$

$$|u(1)| \le \varepsilon, \qquad \left|\frac{\partial u}{\partial n}(1)\right| \le \varepsilon$$

$$u \notin L^\infty(B)$$
$$\lambda e^\varepsilon \int_B e^u \varphi^2 \le \int_B (\Delta\varphi)^2 \quad \textit{for all } \varphi \in C_0^\infty(B). \tag{4.21}$$

Then

$$\lambda^* \le \lambda e^{2\varepsilon}.$$

PROOF. Let

$$\psi(r) = \varepsilon r^2 - 2\varepsilon \tag{4.22}$$

so that

$$\Delta^2\psi \equiv 0, \qquad \psi(1) = -\varepsilon, \qquad \psi'(1) = 2\varepsilon$$

and

$$-2\varepsilon \le \psi(r) \le -\varepsilon \quad \text{for all } 0 \le r \le 1.$$

It follows that

$$\Delta^2(u+\psi) \le \lambda e^u = \lambda e^{-\psi} e^{u+\psi} \le \lambda e^{2\varepsilon} e^{u+\psi}.$$

On the boundary we have $u(1) + \psi(1) \le 0$, $u'(1) + \psi'(1) \ge 0$. Thus $u + \psi$ is a singular subsolution to the equation with parameter $\lambda e^{2\varepsilon}$. Moreover, since $\psi \le -\varepsilon$ we have $\lambda e^{2\varepsilon} e^{u+\psi} \le \lambda e^\varepsilon e^u$ and hence, from (4.21) we see that $u+\psi$ is stable for the problem with parameter $\lambda e^{2\varepsilon}$. If $\lambda e^{2\varepsilon} < \lambda^*$ then the minimal solution associated with the parameter $\lambda e^{2\varepsilon}$ would be above $u + \psi$, which is impossible because u is singular. □

LEMMA 4.7. *Suppose we can find $\varepsilon > 0$, $\lambda > 0$ and $u \in H^2(B) \cap W^{4,\infty}_{\mathrm{loc}}(B \setminus \{0\})$ such that*

$$\Delta^2 u \ge \lambda e^u \quad \textit{for all } 0 < r < 1$$
$$|u(1)| \le \varepsilon, \qquad \left|\frac{\partial u}{\partial n}(1)\right| \le \varepsilon.$$

Then

$$\lambda e^{-2\varepsilon} \le \lambda^*.$$

PROOF. Let ψ be given by (4.22). Then $u - \psi$ is a supersolution to the problem with parameter $\lambda e^{-2\varepsilon}$. □

The next result is the main tool to guarantee that u^* is singular. The proof, as in (1.61), is based on an upper estimate of u^* by a stable singular subsolution.

LEMMA 4.8. *Suppose there exist* $\varepsilon_0, \varepsilon > 0$, $\lambda_a > 0$ *and a radial function* $u \in H^2(B) \cap W^{4,\infty}_{\text{loc}}(B \setminus \{0\})$ *such that*

$$\Delta^2 u \le (\lambda_a + \varepsilon_0) e^u \quad \text{for all } 0 < r < 1 \tag{4.23}$$

$$\Delta^2 u \ge (\lambda_a - \varepsilon_0) e^u \quad \text{for all } 0 < r < 1 \tag{4.24}$$

$$|u(1)| \le \varepsilon, \quad \left|\frac{\partial u}{\partial n}(1)\right| \le \varepsilon \tag{4.25}$$

$$u \notin L^\infty(B) \tag{4.26}$$

$$\beta_0 \int_B e^u \varphi^2 \le \int_B (\Delta\varphi)^2 \quad \text{for all } \varphi \in C_0^\infty(B), \tag{4.27}$$

where

$$\beta_0 = \frac{(\lambda_a + \varepsilon_0)^3}{(\lambda_a - \varepsilon_0)^2} e^{9\varepsilon}. \tag{4.28}$$

Then u^* *is singular and*

$$(\lambda_a - \varepsilon_0) e^{-2\varepsilon} \le \lambda^* \le (\lambda_a + \varepsilon_0) e^{2\varepsilon}. \tag{4.29}$$

PROOF. By Lemmas 4.6 and 4.7 we have (4.29). Let

$$\delta = \log\left(\frac{\lambda_a + \varepsilon_0}{\lambda_a - \varepsilon_0}\right) + 3\varepsilon,$$

and define

$$\varphi(r) = -\frac{\delta}{4} r^4 + 2\delta.$$

We claim that

$$u^* \le u + \varphi \quad \text{in } B_1. \tag{4.30}$$

To prove this, we shall show that for $\lambda < \lambda^*$

$$u_\lambda \le u + \varphi \quad \text{in } B_1. \tag{4.31}$$

Indeed, we have

$$\Delta^2 \varphi = -\delta 2N(N+2)$$
$$\varphi(r) \ge \delta \quad \text{for all } 0 \le r \le 1$$
$$\varphi(1) \ge \delta \ge \varepsilon, \quad \varphi'(1) = -\delta \le -\varepsilon$$

and therefore

$$\begin{aligned}\Delta^2(u + \varphi) &\le (\lambda_a + \varepsilon_0) e^u + \Delta^2\varphi \le (\lambda_a + \varepsilon_0) e^u = (\lambda_a + \varepsilon_0) e^{-\varphi} e^{u+\varphi} \\ &\le (\lambda_a + \varepsilon_0) e^{-\delta} e^{u+\varphi}.\end{aligned} \tag{4.32}$$

By (4.29) and the choice of δ

$$(\lambda_a + \varepsilon_0)e^{-\delta} = (\lambda_a - \varepsilon_0)e^{-3\varepsilon} < \lambda^*. \tag{4.33}$$

To prove (4.31) it suffices to consider λ in the interval $(\lambda_a - \varepsilon_0)e^{-3\varepsilon} < \lambda < \lambda^*$. Fix such λ and assume that (4.31) is not true. Write

$$\bar{u} = u + \varphi$$

and let

$$R_1 = \sup\{0 \le R \le 1 \mid u_\lambda(R) = \bar{u}(R)\}.$$

Then $0 < R_1 < 1$ and $u_\lambda(R_1) = \bar{u}(R_1)$. Since $u_\lambda'(1) = 0$ and $\bar{u}'(1) < 0$ we must have $u_\lambda'(R_1) \le \bar{u}'(R_1)$. Then u_λ is a solution to the problem

$$\begin{cases} \Delta^2 u = \lambda e^u & \text{in } B_{R_1} \\ u = u_\lambda(R_1) & \text{on } \partial B_{R_1} \\ \dfrac{\partial u}{\partial n} = u_\lambda'(R_1) & \text{on } \partial B_{R_1} \end{cases}$$

while, thanks to (4.32) and (4.33), $\bar{u}$ is a subsolution to the same problem. Moreover $\bar{u}$ is stable thanks to (4.27) since, by Lemma 4.6,

$$\lambda < \lambda^* \le (\lambda_a + \varepsilon_0)e^{2\varepsilon} \tag{4.34}$$

and hence

$$\lambda e^{\bar{u}} \le (\lambda_a + \varepsilon_0)e^{2\varepsilon}e^{2\delta}e^u \le \beta_0 e^u.$$

We deduce $\bar{u} \le u_\lambda$ in B_{R_1} which is impossible, since $\bar{u}$ is singular while u_λ is smooth. This establishes (4.30).

From (4.30) and (4.34) we have

$$\lambda^* e^{u^*} \le \beta_0 e^{-\varepsilon} e^u$$

and therefore

$$\inf_{\varphi \in C_0^\infty(B)} \frac{\int_B (\Delta\varphi)^2 - \lambda^* e^{u^*}\varphi^2}{\int_B \varphi^2} > 0.$$

This is not possible if u^* is a smooth solution. □

For each dimension $13 \le N \le 31$ we construct u satisfying (4.23) to (4.27) of the form

$$u(r) = \begin{cases} -4\log r + \log\left(\frac{8(N-2)(N-4)}{\lambda}\right) & \text{for } 0 < r < r_0 \\ \tilde{u}(r) & \text{for } r_0 \le r \le 1, \end{cases} \tag{4.35}$$

where $\tilde{u}$ is explicitly given. Thus u satisfies (4.26) automatically.

Numerically it is better to work with the change of variables

$$w(s) = u(e^s) + 4s, \quad -\infty < s < 0$$

which transforms the equation $\Delta^2 u = \lambda e^u$ into

$$Lw + 8(N-2)(N-4) = \lambda e^w, \quad -\infty < s < 0,$$

where

$$\begin{aligned} Lw = {} & \frac{d^4 w}{ds^4} + 2(N-4)\frac{d^3 w}{ds^3} + (N^2 - 10N + 20)\frac{d^2 w}{ds^2} \\ & - 2(N-2)(N-4)\frac{dw}{ds}. \end{aligned}$$

The boundary conditions $u(1) = 0$, $u'(1) = 0$ then yield

$$w(0) = 0, \quad w'(0) = 4.$$

Regarding the behavior of w as $s \to -\infty$ observe that

$$u(r) = -4 \log r + \log\left(\frac{8(N-2)(N-4)}{\lambda}\right) \quad \text{for } r < r_0$$

if and only if

$$w(s) = \log \frac{8(N-2)(N-4)}{\lambda} \quad \text{for all } s < \log r_0.$$

The steps we perform are the following:

(1) We fix $x_0 < 0$ and using numerical software we follow a branch of solutions to

$$\begin{cases} L\hat{w} + 8(N-2)(N-4) = \lambda e^{\hat{w}}, & x_0 < s < 0 \\ \hat{w}(0) = 0, \quad \hat{w}'(0) = t \\ \hat{w}(x_0) = \log \dfrac{8(N-2)(N-4)}{\lambda}, & \dfrac{d^2 \hat{w}}{ds^2}(x_0) = 0, \quad \dfrac{d^3 \hat{w}}{ds^3}(x_0) = 0 \end{cases}$$

as t increases from 0 to 4. The numerical solution $(\hat{w}, \hat{\lambda})$ we are interested in corresponds to the case $t = 4$. The five boundary conditions are due to the fact that we are solving a fourth-order equation with an unknown parameter λ.

(2) Based on $\hat{w}$, $\hat{\lambda}$ we construct a C^3 function w which is constant for $s \le x_0$ and piecewise polynomial for $x_0 \le s \le 0$. More precisely, we first divide the interval $[x_0, 0]$ into smaller intervals of length h. Then we generate a cubic spline approximation g_{fl} with floating point coefficients of $\frac{d^4 \hat{w}}{ds^4}$. From g_{fl} we generate a piecewise cubic polynomial g_{ra} which uses rational coefficients and we integrate it four times to obtain w, where the constants of integration are such that $\frac{d^j w}{ds^j}(x_0) = 0$, $1 \le j \le 3$ and $w(x_0)$ is a rational approximation of $\log(8(N-2)(N-4)/\lambda)$. Thus w is a piecewise

polynomial function that in each interval is of degree 7 with rational coefficients, and which is globally C^3. We also let λ be a rational approximation of $\hat{\lambda}$. With these choices note that $Lw+8(N-2)(N-4)-\lambda e^w$ is a small constant (not necessarily zero) for $s \le x_0$.

(3) The conditions (4.23) and (4.24) we need to check for u are equivalent to the following inequalities for w

$$Lw + 8(N-2)(N-4) - (\lambda+\varepsilon_0)e^w \le 0, \quad -\infty < s < 0 \tag{4.36}$$

$$Lw + 8(N-2)(N-4) - (\lambda-\varepsilon_0)e^w \ge 0, \quad -\infty < s < 0. \tag{4.37}$$

Using a program in Maple we verify that w satisfies (4.36) and (4.37). This is done evaluating a second-order Taylor approximation of $Lw + 8(N-2)(N-4) - (\lambda+\varepsilon_0)e^w$ at sufficiently close mesh points. All arithmetic computations are done with rational numbers, thus obtaining exact results. The exponential function is approximated by a Taylor polynomial of degree 14 and the difference with the real value is controlled.

(4) We show that the operator $\Delta^2 - \beta e^u$, where $u(r) = w(\log r) - 4\log r$, satisfies condition (4.27) for some $\beta \ge \beta_0$, where β_0 is given by (4.28).

We refer the interested reader to [44], but we shall justify here that, although βe^u is singular, the operator $\Delta^2 - \beta e^u$ has indeed a positive eigenfunction in $H_0^2(B)$ with finite eigenvalue if β is not too large, if $N \ge 13$. The reason is that near the origin

$$\beta e^u = \frac{c}{|x|^4},$$

where c is a number close to $8(N-2)(N-4)\beta/\lambda$. If β is not too large compared to λ then $c < N^2(N-4)^2/16$ and hence, using (1.57), $\Delta^2 - \beta e^u$ is coercive in $H_0^2(B_{r_0})$.

The full information on the Maple files and data used can be found at:
http://www.lamfa.u-picardie.fr/dupaigne/
http://www.ime.unicamp.br/˜msm/bilaplace-computations/bilaplace-computations.html

Acknowledgment

This work has been partly supported by grants Fondecyt 1050725 and FONDAP, Chile.

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CHAPTER 3

Positive Solutions to Semi-Linear and Quasi-Linear Elliptic Equations on Unbounded Domains

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Abstract

In this survey we describe recent results on the existence and nonexistence of positive solution to semi-linear and quasi-linear second-order elliptic equations. A typical example is the equation $-\Delta u = |x|^{-\sigma} u^q$ in an exterior of the ball in $\mathbb{R}^N$ or in a cone-like domain in $\mathbb{R}^N$. The equations of this type exhibit a phenomenon of presence of critical exponents in the range of the parameter $q \in \mathbb{R}$, which separate the zones of the existence from the zones of the nonexistence. The values of the critical exponents depends on the geometry of the domain, the type of the operator in the main part (divergent or nondivergent), the behaviour of the coefficients in lower order terms at infinity. We investigate these dependencies mostly in the cases of the exterior and cone-like domains. The proofs are often based on the explicit construction of appropriate barriers and involve the analysis of asymptotic behavior of super-harmonic functions associated to the corresponding second-order elliptic operator, comparison principles and Hardy's inequality in exterior domains. To construct the barriers in the cases of equations with non-smooth coefficients we obtain detailed estimates at infinity of small and large solutions to the corresponding linear equations. Some of the results for the equations with first order term are new and have not been published before. In discussions we list some open problems in this area.

Keywords: Second-order semi-linear elliptic equations, Quasi-linear equations, positive solutions, Exterior domain, Cone-like domains, Critical exponents

Contents

HANDBOOK OF DIFFERENTIAL EQUATIONS
Stationary Partial Differential Equations, volume 6
Edited by M. Chipot

1. Introduction and model results

1.1. *Motivation, statement of the problem and historical remarks*

Let $G \subset \mathbb{R}^N$ be an unbounded domain (connected open set). In this article we study the existence and nonexistence of positive (super-) solutions to the equation

$$\mathcal{L}u = W u^q, \tag{1.1}$$

where $\mathcal{L}$ is a second-order elliptic operator (possibly nonlinear), W is a potential in front of the nonlinearity u^q, and q is a real number. To fix the sign in (1.1) let us think of $\mathcal{L} = -\Delta$ as a model case. In this paper we confine ourselves to the case of polynomial behavior at infinity of the potential W in front of the nonlinear (nonhomogeneous) term u^q. So $W(x) \sim |x|^{-\sigma}$ for σ a real number (and in the main body of the paper we always assume that $0 \notin G$). Mathematically such problems are interesting due to the lack of compactness, and they require different techniques compared to the case of bounded domains. Rich mathematical structure and presence of various critical phenomena make the qualitative theory of equations of type (1.1) an attractive field for mathematicians to attack, as they represent a prototype model for general quasi-linear elliptic equations.

On the other hand, many such equations appear in various applications. The mathematical theory of equations of type (1.1) seems to originate from the classical works of Emden and Fowler at the beginning of the XX century ((1.1) is often referred to as the generalized Emden–Fowler or Lane–Emden equation). They quickly found applications in models in astrophysics (Matukuma proposed (1.1) to improve Eddington's model) and are still used in the analytic theory of polytropic ball model of stellar structures (Lane–Ritter–Emden theory). Equation (1.1) for the whole range of the parameter $q \in \mathbb{R}$ has numerous applications in natural sciences, e.g. in scalar field theory, in phase transition theory, in combustion theory ($q > 1$) [75], population dynamics ($0 < q < 1$) [60], pseudo-plastic fluids ($q < 0$) [32,46,61], ecology models [65]. Not least to mention that equation (1.1) describes stationary states for reaction-diffusion equations which are ubiquitous in applications. Positive solutions are of special interest in the variety of applications in which the unknown quantities involved are positive. The absence of positive solutions to the elliptic equations means also that the existing solutions oscillate, which is also important information in applications.

When considered in the whole of $\mathbb{R}^N$ equation (1.1) has important applications in geometry [22,63,64], and of course the nonexistence results have direct relevance to these geometric applications [34].

It has been known at least since earlier works by Serrin [78] (cf. also the references in [80]) and the celebrated paper by Gidas and Spruck [27] that equations of type (1.1) on unbounded domains admit positive (super-) solutions only for specific values of $(q, \sigma) \in \mathbb{R}^2$ exhibiting so-called *critical exponents*. The nonexistence part of such results on (1.1) can be viewed as the nonlinear generalization of the Liouville theorem on semi-bounded real harmonic functions. We refer the reader to the excellent recent paper [80] on the account of the extensions of Liouville-type theorems, modern methods and historical notes.

One of the first results exhibiting critical exponents in the existence of positive solutions was a deep and beautiful result of Gidas and Spruck.

THEOREM 1.1 (Gidas and Spruck [27]). *Let $N \geq 2$. Let $u \geq 0$ be a classical solution to*

$$-\Delta u = u^q \quad \textit{in } \mathbb{R}^N \quad \textit{with } 1 < q < \frac{N+2}{N-2}. \tag{1.2}$$

Then $u = 0$.

The result is sharp, as it fails for any $q \geq \frac{N+2}{N-2}$.

On the other hand, the above result is extremely unstable to any changes in the statement of the problem. By considering the equation $-\Delta u = c(|x|)u^q$ in $\mathbb{R}^N$ with $c - 1$ being a small (and even compactly supported) perturbation one can produce positive solutions for $\frac{N}{N-2} < q < \frac{N+2}{N-2}$ (see [64]). The exponent $\frac{N}{N-2}$, which is the critical exponent to (1.2) when considered in a punctured $\mathbb{R}^N$, or for supersolutions instead of solutions, is much more stable.

The following theorem appeared as a special case simultaneously in [6,11].

THEOREM 1.2. *Let $N \geq 3$. Let $u \geq 0$ be a weak supersolution to*

$$-\Delta u = u^q \quad \textit{in the exterior of a ball in } \mathbb{R}^N \textit{ with } 1 < q \leq \frac{N}{N-2}. \tag{1.3}$$

Then $u = 0$.

The *critical exponent* $q^* = \frac{N}{N-2}$ is sharp in the sense that it separates the zones of existence and nonexistence, i.e. for $q > q^*$ equation (1.3) has positive (super-) solutions outside a ball. This result has been extended in several directions (see, e.g. [7,9,13–15,17, 36,21,44,45,87,23,38,42,47,80,83,71,87,93,94] (see also [35]), and references therein, this list is by no means complete).

The critical exponent q^* *depends* on the structure of the operator, presence of strong lower-order terms, geometry of the domain G. In [33] Kalton and Verbitsky gave necessary and sufficient conditions for existence of positive solutions to $\mathcal{L}u = Wu^q + f$, $u = 0$ on ∂G, for smooth domains G and $q > 1$, in terms of the Green's function Γ_G of $\mathcal{L}$ on G with Dirichlet boundary conditions. The criterion says that $\Gamma_G[W(\Gamma_G f)^q] \leq C\Gamma_G f$ with some $C > 0$. In applications this inequality presents a separate problem to verify. Although some results stated in this paper can be probably recovered from this criterion, we will not pursue this route. We will study the problem of the existence and nonexistence of positive supersolutions to (1.1) in relation to the explicit conditions describing the behavior of the coefficients of $\mathcal{L}$ at infinity. The results we discuss in this paper are not dependent on the boundary conditions and do not require any smoothness of the domain G, and they will include both the superlinear ($q > 1$) and the sublinear ($q < 1$) cases. Much of this will be precisely stated and proved in the main body of these notes. Here we only briefly outline some recent history. In [38] it was shown that the critical exponent $q^* = \frac{N}{N-2}$ is stable with respect to the change of the Laplacian by a second-order uniformly elliptic divergence-type operator with measurable coefficients $-\sum \partial_i(a_{ij}\partial_j)$, perturbed by a potential, for a sufficiently wide class of potentials. For instance, for $\epsilon > 0$ the equation

$$-\sum_{i,j=1}^{N} \partial_i(a_{ij}\partial_j)u - \frac{B}{|x|^{2+\epsilon}}u = u^q \tag{1.4}$$

in an exterior domain in $\mathbb{R}^N$ ($N \geq 3$) has the same critical exponent as (1.3) [38, Theorem 1.2]. On the other hand it is easy to see that if $\epsilon < 0$ and $B > 0$ then (1.4) has no positive supersolutions for any $q \in \mathbb{R}$, while if $\epsilon < 0$ and $B < 0$ then (1.4) admits positive solutions for all $q \in \mathbb{R}$ ($q \neq 1$). In the borderline case $\epsilon = 0$ the critical exponent q^* becomes dependent on the parameter B. In case of the Laplacian as $\mathcal{L}$ this phenomenon and its relation with Hardy-type inequalities has been recently observed on a ball and/or exterior domains in [15,23,86] in the case $q > 1$.

The equation with first-order term

$$-\Delta u - \frac{A}{|x|^{2+\epsilon}} x \cdot \nabla u = u^q \tag{1.5}$$

in the exterior of a ball in $\mathbb{R}^N$ ($N \geq 3$) represents another type of behavior. If $\epsilon > 0$ then (1.5) has the same critical exponent $q^* = \frac{N}{N-2}$ as (1.3), and q^* is stable with respect to the change of the Laplacian by a second-order uniformly elliptic divergence-type operator [42, Theorem 1.8]. On the other hand, it is easy to see that if $\epsilon < 0$ and $A > 0$ then (1.5) has no positive supersolutions if and only if $q \leq 1$, while if $\epsilon < 0$ and $A < 0$ then (1.5) has no positive supersolutions if and only if $q \geq 1$. In the borderline case $\epsilon = 0$ the critical exponent q^* explicitly depends on the parameter A (see [71,83] for the case $q > 1$).

The geometry of the domain makes a strong impact on the qualitative picture for equation (1.1). We will see at the end of this section that cylinder-type domains present no interest, whereas cone-like domains give interesting changes in the criticality features. When considered on cone-like domains, the nonexistence zone depends in addition on the principal Dirichlet eigenvalue of the cross-section of the cone. In the superlinear case $q > 1$ the equation

$$-\Delta u = u^q \quad \text{in } \mathcal{C}_\Omega^1 \tag{1.6}$$

has been considered in [6,7,9,24] (see also [14,45] for systems and [40] for uniformly elliptic equations with measurable coefficients). A new nonexistence phenomenon for the sublinear case $q < 1$ has been recently revealed in [41]. Particularly, it was discovered that equation (1.6) in a proper cone-like domain has two critical exponents, the second one appearing in the sublinear case, so that (1.6) has no positive supersolutions if and only if $q_* \leq q \leq q^*$, where $q_* < 1$ and $q^* > 1$. In [40] for $q > 1$ it was shown that if the Laplacian is replaced by a second-order uniformly elliptic divergence-type operator $-\sum \partial_i(a_{ij}\partial_j)$ then the value of the critical exponents on the cone depends on the coefficients of the matrix $(a_{ij}(x))$ as well as on the geometry of the cross-section. The existence and nonexistence of positive supersolutions to a singular semi-linear elliptic equation $-\nabla \cdot (|x|^A \nabla u) - B|x|^{A-2}u = c_o|x|^{A-\sigma}u^q$ in cone-like domains of $\mathbb{R}^N$ ($N \geq 2$), for the full range of parameters $A, B, \sigma, q \in \mathbb{R}$ and $c_0 > 0$ was studied in [48]. The same problem for p-Laplacian perturbed by a Hardy-type potential is considered in [49]. Our approach to the problem in this paper is the development of the method introduced in [37, 38]. It is based on the explicit construction of appropriate barriers and involves the analysis of the asymptotic behavior of superharmonic functions associated to the Laplace operator with critical potentials, Phragmèn–Lindelöf type comparison arguments and an improved version of Hardy's inequality in cone-like domains. The advantages of our approach are its transparency and flexibility. Particularly it is applicable to the case of nondivergence-type operators $\mathcal{L} = -\sum_{i,j=1}^{N} a_{i,j}\partial_{x_i}\partial_{x_j}$ for a general (non-smooth) uniformly elliptic matrix $(a_{i,j})$, which was studied recently in [39]. The approach seems promising for applications to general quasi-linear operators, though currently only partial results are available [50].

In this survey we describe in detail all the cases mentioned above and include an account of new results for the divergence-type operator $\mathcal{L}$ containing all the lower-order terms, thus extending the results from [38,42]. Due to the methods involved our notion of the solution is more restrictive than in [15,16] where distributional solutions are studied, whereas we confine ourselves to weak solutions from local Sobolev spaces. The results on the nonexistence of positive supersolutions to equation (1.1) for certain pairs (q, σ) are given alongside with the results on the existence of positive supersolutions for the complementary values of q and σ, which by the method of sub- and supersolutions leads to the existence of positive solutions.

The purpose of this survey is twofold. On one hand, we would like to introduce young researchers to some ideas and methods of the qualitative theory of elliptic second-order PDEs. For that purpose we tried to make our exposition as self-contained as possible and in many cases did not try to present the results in the greatest possible generality. On the other hand, we present here a number of new results, which will hopefully make this article interesting to the experts as well. At the end of every section we discuss open problems and possible future directions of research, which hopefully will attract attention of researchers who wish to enter this area.

Outline of these notes. We continue this section with a detailed analysis of the model example of the equation $-\Delta u = c_0|x|^{-\sigma}u^q$ on the exterior of the unit ball in $\mathbb{R}^N$, $N \geq 3$. Here we demonstrate the main ideas which will be further developed in subsequent sections. Section 2 contains some background material and an account of general facts concerning linear and nonlinear equations. In Section 3 we study positive supersolutions to (1.1) in exterior domains for $\mathcal{L}$ being a second-order divergence-type linear operator with lower-order terms. In every subsection of this section we first prepare the necessary linear theory for the equation $\mathcal{L}u = 0$ and then pass to the semi-linear equation. First, in Section 3.1 we study equation (1.1) with $\mathcal{L} = -\Delta + \frac{c}{|x|^2}$, where the influence of the potential is clearly seen and the complete qualitative picture on the existence of positive supersolutions to (1.1) is presented. In Section 3.2 we study (1.1) with $\mathcal{L}$ being a uniformly elliptic operator with all lower-order terms. Here we obtain new estimates for small and large solutions to the linear equation $\mathcal{L}u = 0$ and apply them to the semi-linear equation. In Section 3.3 we discuss the case of nonuniformly elliptic linear operators $\mathcal{L}$. Section 3 ends with a discussion of open problems and possible extensions of the results. Section 4 contains an account of results on positive supersolutions to $\mathcal{L}u = 0$ and (1.1) on cone-like domains. In Section 5 we describe the result on positive supersolutions to (1.1) in exterior domains for the case of nondivergence-type linear $\mathcal{L}$. New critical phenomena occur in this case. Section 6 contains a brief account of the available results on the existence of positive supersolutions to (1.1) for quasi-linear operators $\mathcal{L}$. Appendices contain some auxiliary information on the extended Dirichlet spaces and improvements of the Hardy inequality.

1.2. *Model example*

We start our discussion from a model problem.

Let $N \geq 3$, $G = \bar{B}_1^c := \mathbb{R}^N \setminus \bar{B}_1 := \{x \in \mathbb{R}^N \ : \ |x| > 1\}$. Let u be a weak nonnegative supersolution to the equation (see the definition below)

$$-\Delta u = c_0 |x|^{-\sigma} u^q. \tag{1.7}$$

First, we establish the existence of a positive supersolution to (1.7) in the form $u = cr^\lambda$, $r = |x|$ with some positive c. Simple computation then shows that $-\lambda(\lambda + N - 2)r^{\lambda-2} \geq c^{q-1} r^{-\sigma+\lambda q}$. So, for such a solution to exist we must have $\lambda - 2 \geq -\sigma + \lambda q$ and $-\lambda(\lambda + N - 2) > 0$. So

$$\lambda(\lambda + N - 2) < 0 \quad \text{and} \quad \lambda(q - 1) \leq \sigma - 2. \tag{1.8}$$

This immediately implies that $q > 1 + \frac{2-\sigma}{N-2}$, if $q > 1$, and $\sigma > 2$ for $q < 1$. Let us introduce the set on the plane (q, σ)

$$\mathcal{N} = \{(q, \sigma) \in \mathbb{R}^2 \setminus \{1, 2\} : \textit{equation (1.7) has no positive supersolutions}\}.$$

For $q = 1$ equation (1.7) becomes linear, and for $\sigma = 2$ (and only for this value of σ) the existence of positive solutions depends on the value of the constant c_0. As we will see in the main body of the paper, the answer is directly related to the Hardy inequality. The answer is this: positive solutions to (1.7) in $\bar{B}_1^c$ exist if and only if $c_0 \leq \left(\frac{N-2}{2}\right)^2$.

From the above calculation we conclude that

$$\mathcal{N} \subset \{(q, \sigma) \ : \ \sigma \leq \min\{2, 2 + (2 - N)q\}.$$

It is remarkable that in fact we have the equality of these sets, so that the conditions (1.8) identify the existence zone precisely. The result is formulated in the next theorem and illustrated by Figure 2.

Let us remind the reader of the precise definition of weak (super-) solutions.

We say that $0 \leq u \in H^1_{\text{loc}}(G)$ is a nonnegative *supersolution* to (1.7) in a domain $G \subseteq \mathbb{R}^N$ if

$$\int_G \nabla u \cdot \nabla \varphi \, dx \ \geq c_0 \int_G u^q \varphi |x|^{-\sigma} \, dx \tag{1.9}$$

for all $0 \leq \varphi \in H^1_c(G) \cap L^\infty_c(G)$ (here and below subindex c stands for elements with compact support). The notions of a nonnegative subsolution and solution are defined similarly by replacing "$\geq$" with "$\leq$" and "$=$" respectively. Here and below $H^1_c(G)$ and $L^\infty_c(G)$ stands for functions from $H^1(G)$, from $L^\infty(G)$ respectively, with compact support in G. (The complete notation for the paper is given at the beginning of Section 2.)

THEOREM 1.3. $\mathcal{N} = \{(q, \sigma) \ : \ \sigma \leq \min\{2, (2 - N)(q - 1) + 2\}\}$.

In order to prove Theorem 1.3 we only have to prove the nonexistence part. And for this we have to distinguish two cases: superlinear $q \geq 1$ and sublinear $q < 1$, as the techniques are different for them.

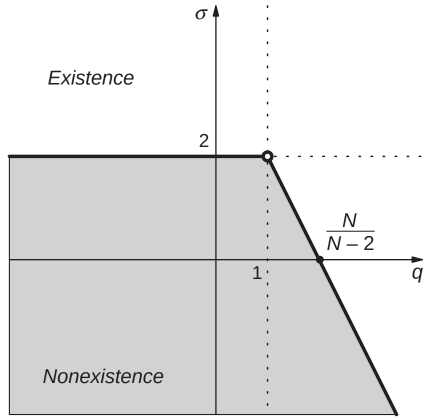

Fig. 2. Existence and nonexistence zones for equation (1.7).

1.2.1. *Superlinear case $q > 1$*

We divide the proof of nonexistence into several steps.

Step 1. Initial estimate for supersolutions. First, notice that $u \geq 0$ in Ω by the maximum principle (see, e.g. Theorem 2.4 below) implies that either $u = 0$ in G, or $u > 0$ in G. So without loss we can assume that $\inf_{|x|=1} u(x) > 0$ (otherwise change G to $\mathbb{R}^N \setminus \bar{B}_2^c$). Set $v(x) = c|x|^{2-N}$ with $c < \inf_{|x|=1} u(x)$. Then by the weak maximum principle we obtain that $u(x) \geq v(x)$ almost everywhere (a.e.), i.e.

$$u(x) \geq c|x|^{2-N} \quad \text{for } |x| > 1.$$

Step 2. Linearization. Write (1.7) in the form $-\Delta u = Vu$ with $V = c_0|x|^{-\sigma}u^{q-1}$. From Step 1 we conclude that u satisfies the inequality

$$-\Delta u \geq c|x|^{-\sigma}|x|^{(q-1)(2-N)}u \quad \text{for } |x| > 1. \tag{1.10}$$

Step 3. Proof of nonexistence in the subcritical case $1 < q < \frac{N-\sigma}{N-2}$. If $q < \frac{N-\sigma}{N-2}$ then (1.10) implies that for some $\varepsilon > 0$ we have

$$-\Delta u \geq c|x|^{-2+\varepsilon}u \quad \text{for } |x| > 1. \tag{1.11}$$

Now we prove that there is no positive u which satisfies (1.11). For this we use a scaling argument. As u satisfies (1.11) in the exterior of the unit ball, we can choose any $R > 1$ and use (1.11) in any annulus $A_{R,2R} := B_{2R} \setminus \bar{B}_R$. Let $y = Rx$, and set $v(y) = u(Rx)$. Then in new variables we have

$$-\Delta v \geq cR^{\varepsilon}v, \qquad 1 < |y| < 2.$$

Note that $\inf_{1<|y|<2} v(y) > 0$, so that $v^{-1} \in L^{\infty}_{\text{loc}} \cap H^1_{\text{loc}}$. Take $\theta \in C_0^{\infty}(A_{1,2})$. Then $\frac{\theta^2}{v} \in H^1_c$, and we can use it as a test function for (1.11), plug it into (1.9) in place of φ. Then we have

$$\int_{1<|y|<2} \nabla v \cdot \nabla\left(\frac{\theta^2}{v}\right) dy \geq cR^{\varepsilon}\|\theta\|_2^2.$$

By the Cauchy–Schwarz inequality this implies

$$\|\nabla\theta\|_2^2 \geq cR^\varepsilon \|\theta\|_2^2, \quad \text{or} \quad \frac{\|\nabla\theta\|_2^2}{\|\theta\|_2^2} \geq cR^\varepsilon.$$

This is a contradiction, as the left-hand side of the last inequality is finite and does not depend on R.
Step 4. Critical case $q = \frac{N-\sigma}{N-2}$. In this case (1.10) implies the inequality

$$-\Delta u \geq \frac{c}{|x|^2} u \quad \text{for } |x| > 1.$$

Choose $\mu > 0$ so that $\mu < \min\left\{\left(\frac{N-2}{2}\right)^2, c\right\}$. Let v be the smaller solutions to the problem

$$-\Delta v - \frac{\mu}{|x|^2} v = 0 \quad \text{for } |x| > 1, \quad v|_{|x|=1} = 1.$$

It is straightforward that $v(x) = |x|^{-\frac{N-2}{2} - \sqrt{\left(\frac{N-2}{2}\right)^2 - \mu}}$, so $v(x) = |x|^{2-N+\delta}$ with some $\delta > 0$. Also $\nabla v \in L^2(B_1^c)$, therefore one can apply a weak maximum principle in order to compare u and v (this point will be explained in detail later in the next section, see Theorem 2.4). Hence we obtained an improved lower estimate for u. Namely, we have that $u(x) \geq c|x|^{2-N+\delta}$ with some positive δ. Using this estimate in (1.10) we repeat Step 3, thus completing the proof. □

1.2.2. *Sublinear case* $q < 1$

We start with the following a priori estimate. This is the bound of Keller–Osserman-type for supersolutions (see, e.g. [89] for the corresponding estimates for subsolutions).

LEMMA 1.4. *Let* $u > 0$ *be a supersolution to* (1.7) *in* G. *Then there exists a constant* $C > 0$ *such that*

$$u(x) \geq C|x|^{\frac{2-\sigma}{1-q}}, \quad |x| > 2.$$

PROOF. By the maximum principle $u > 0$. Moreover, by the weak Harnack inequality (see, e.g. [28, Theorem 8.18], also in the next section) $u^{-1} \in L^\infty_{\text{loc}}(G)$. Take $\theta \in C_0^\infty(G)$ such that

$$\mathbb{1}_{A_{R,4R}} \leq \theta \leq \mathbb{1}_{A_{R/2,9/2R}} \quad \text{and} \quad |\nabla\theta| \leq 3/R.$$

Then we can test (1.7) by $\frac{\theta^2}{u}$ as it is from $H_c^1(G)$. Integrating by parts and using Cauchy–Schwarz we obtain

$$\int |\nabla\theta|^2 dx \geq c_0 \int \theta^2 |x|^{-\sigma} u^{q-1} dx.$$

For the left-hand side (LHS) from the definition of θ we have

$$\int |\nabla\theta|^2 dx \leq CR^{N-2}.$$

For the right-hand side (RHS) by the weak Harnack inequality

$$\int \theta^2 |x|^{-\sigma} u^{q-1} dx \geq CR^{N-\sigma} \left(\inf_{A_{2R,3R}} u(x) \right)^{q-1}.$$

Now the assertion follows. □

The above lemma implies that if a positive supersolution to (1.7) exists, then it grows at infinity at least polynomially. Next argument will show that this is impossible, which will lead to a contradiction.

We can always assume that u is strictly positive on the unit sphere (otherwise we start the argument from the sphere of radius 2). Note that $v(x) = C_0(1 - |x|^{2-N})$ is a harmonic function on G, which is zero on the unit sphere ∂B_1 and tends to the constant C_0 at infinity. Choosing C_0 sufficiently large (e.g. greater than $\inf_{|x|=2} u(x)$) we can make sure that u "cuts a hat" from v, that is there will be a *bounded* region where $v > u$, and on the boundary of which $v = u$. This is contradictory to the maximum principle, as $\Delta(u - v) \leq 0$. Of course, the argument is based on the maximum principle for classical solutions, but it will be made precise for the weak solution in the subsequent sections (by means of Theorem 2.4 below).

1.3. *Discussion*

In the subsequent sections we are going to extend the result of the model example in several directions. First, staying in the same exterior domain we are going to study the same problem with a second-order elliptic differential operator $\mathcal{L}$ in place of the Laplacian. We investigate the influence of the structure of the operator and the behavior of lower-order terms at infinity on the existence of positive solutions to the semi-linear and then quasi-linear equation.

Analysis of the proof of the nonexistence in the model example shows that for the superlinear case $q > 1$ a *minimal solution* (see more on this in the next section) to the Laplace equation $-\Delta u = 0$ played a crucial role, where it was first seen in the initial estimate for supersolutions. In the exterior domain G the fundamental solution of the free Laplacian with the pole at zero can be taken as a minimal solution. So it is clear that the estimates of the fundamental solutions of the second-order elliptic operators are a key technical tool in our method of proving nonexistence of positive supersolutions to the corresponding superlinear equations. For the sublinear case $q < 1$ the key point was the comparison with a *large solution* (a constant in case of the Laplacian). So for future extensions we need to know the behavior of large solutions at infinity.

Another avenue of the investigation is the dependence of the nonexistence phenomenon from the geometry of the underlying domain G. To clarify this point let us consider the existence of positive supersolutions to $-\Delta u \geq u^q$ on a cylinder (we dropped the polynomial-type potential in front of the nonlinear term for simplicity).

Let $\Omega \subset \mathbb{R}^{N-1}$ be a bounded domain. Let $G = \{x \in \mathbb{R}^N : x' = (x_1, \ldots, x_{N-1}) \in \Omega\}$. Then the equation

$$-\Delta u = u^q \tag{1.12}$$

has positive supersolutions for every $q \in \mathbb{R}$.

Indeed, first consider the superlinear case. Let $\phi > 0$ be the eigenfunction of the Dirichlet Laplacian $-\Delta_\Omega$ on Ω corresponding to the first eigenvalue $\lambda_1 > 0$. Then due to the boundedness of ϕ one can choose $\tau > 0$ small enough so that $(\tau\phi)$ is a supersolution to (1.12). Using the method of sub- and supersolutions one can also find a solution to (1.12). Let $\psi \in H_0^1(\Omega)$ be the solution to the equation $-\Delta_\Omega \psi = 1$. Then $0 < \psi \leq \frac{1}{\lambda_1}$. Then it is readily seen that $\tau(1+\psi)$ is a supersolution to (1.12) for a sufficiently small $\tau > 0$, while 1 is an obvious subsolution.

For $0 \leq q < 1$ let ψ be the solution to the problem

$$-\Delta_\Omega \psi = 1, \quad \psi \in H_0^1(\Omega).$$

Then again, for an appropriate choice of $\tau > 0$, we see that $\tau\psi$ is a supersolution to (1.12). For $q < 0$ one can show that the problem

$$-\Delta_\Omega \psi = (\psi + 1)^q, \quad \psi \in H_0^1(\Omega),$$

has a positive supersolution ψ_0. Then $\psi_0 + 1$ is a positive supersolution to (1.12).

As a conclusion, our problem becomes trivial in the case of cylindrical domains, and therefore for domains that can be embedded in a cylinder, i.e. domains with finite inradius (including bounded domains). As an example of a domain with infinite inradius we study cone-like domains where in the case of the Laplacian explicit values of the critical exponents are available. We also study divergence-type operators with non-smooth coefficients (in place of the Laplacian) in cone-like domains, where only qualitative analysis is possible.

The ideas outlined in the model example turn out to be applicable to the case of the nondivergence-type operators. We discuss the critical phenomena for this case in one of the subsequent sections. If one does not assume that the coefficients of the nondivergence-type linear part of the equation stabilize at infinity, then only qualitative results on the number and rough position of the critical exponents are available.

Existence and nonexistence of positive (super-)solutions to quasi-linear equations is a vast area of research with many open problems. In this article we only briefly discuss the results on p-Laplacian perturbed by a Hardy-type potential and also a few results on general quasi-linear equation.

2. Background, framework and auxiliary facts

Notation. Let $G \subseteq \mathbb{R}^N$ be a domain in the Euclidean space $\mathbb{R}^N$, $N \geq 2$. We write $x = (x_1, \ldots, x_N)$, $y = (y_1, \ldots, y_N)$ etc. for points in $\mathbb{R}^N$, and $x \cdot y = \sum_{i=1}^N x_i y_i$,

$|x| = \sqrt{x \cdot x}$ for the Euclidean scalar product and norm, respectively. For $R > 0$, let B_R denote the open ball of radius R centered at the origin, and for $0 < r < R$ let $A_{r,R}$ denote the open annulus $B_R \setminus \bar{B}_r$. Let $\mathbb{1}_S$ denote the characteristic function of a set $S \subset \mathbb{R}^N$. We denote by $Supp(f)$ the support of the function f. As is standard, $u \wedge v$ and $u \vee v$ stand for $\inf\{u, v\}$ and $\sup\{u, v\}$, respectively. The sign $f \asymp g$ is used in the text to mean that there are two positive constants c_1 and c_2 such that $c_1 g \le f \le c_2 g$. We often use $\langle fg \rangle$ to denote $\int fg dx$, where the set of integration is clear from the context. Letters c and C are used to denote the constants whose exact value is of no importance.

As usual, for a domain G, the space $C_c^1(G)$ denotes the space of continuously differentiable functions on G of compact support. For $p \in [1, \infty]$, $\| \cdot \|_p = \| \cdot \|_{p,G}$ denote the norm in the space $L^p(G, dx)$. We write that $w \in L^p_{\text{loc}}(G)$ if $\theta w \in L^p(G)$ for any $\theta \in C_c^1(G)$.

If G is bounded, $\phi \mapsto \|\nabla\phi\|_2$ defines a Hilbert norm on $C_c^1(G)$ and the completion is denoted by $H_0^1(G)$. For an unbounded G we will consider a weighted norm $\|\phi\|_{D^1_{0,h}} := \|h^{\frac{1}{2}}\nabla\phi\|_2$ with some positive measurable weight function h, such that $h^{\pm 1} \in L^\infty_{\text{loc}}(G)$. Then the completion of $C_c^1(G)$ in this norm might cease to be a functional space. Namely, there might exist a sequence $\phi_n \in C_c^1(G)$ such that $\|h^{\frac{1}{2}}\nabla\phi_n\|_2 \to 0$ as $n \to \infty$ while $\|\phi_n\|_2 = 1$ for all $n \in \mathbb{N}$. If this is not the case then a Hardy-type inequality holds: there exists an a.s. positive weight w locally integrable over G, such that $\|h^{\frac{1}{2}}\nabla\phi\|_2 \ge \|w^{\frac{1}{2}}\phi\|_2$ for all $\phi \in C_c^1(G)$ (see [69]). We will always assume that the latter is the case and denote by $D^1_{0,h}(G)$ the completion of $C_c^1(G)$ in the norm $\| \cdot \|_{D^1_{0,h}}$. If h is a constant we omit subindex h and write simply $D^1_0(G)$. By $D^1(G)$ we denote $D^1(G) = \{u \in L^2_{\text{loc}}(G) \,:\, \nabla u \in L^2(G)\}$.

The space $D^1_{0,h}(G)$ is a Hilbert and Dirichlet space, with the dual $D_h^{-1}(G)$, see, e.g. [26]. This implies, amongst other things, that $D^1_{0,h}(G)$ is invariant under the standard truncations, e.g. $v \in D^1_{0,h}(G)$ implies that $v^+ = v \vee 0 \in D^1_{0,h}(G)$, $v^- = -(v \wedge 0) \in D^1_{0,h}(G)$. The local spaces $H^1_{\text{loc}}(G)$, $D^1_{\text{loc}}(G)$, etc. are defined in the standard way: a measurable function w belongs to $H^1_{\text{loc}}(G)$, $D^1_{\text{loc}}(G)$, if θw belongs to $H^1_0(G)$, $D^1_{0,h}(G)$, respectively, for all $\theta \in C_c^1(G)$. Similarly, $D_c^1(G)$, $H_c^1(G)$, D_c^{-1} denote the subspace of functions in $D^1_{0,h}(G)$, $H^1_0(G)$ and D_h^{-1}, respectively, with compact support in G. We denote by $\langle \cdot, \cdot \rangle$ the duality form between $D^1_{0,h}(G)$ and $D_h^{-1}(G)$, $L^p(G)$ and $L^{p'}(G)$, $D_c^1(G)$ and $D^{-1}_{\text{loc}}(G)$, $D^1_{\text{loc}}(G)$ and $D_c^{-1}(G)$ etc.

The above functional analytic setting allows us to define one of the main objects of our study, the differential operator $\mathcal{L}$ given by

$$\mathcal{L}u = -\nabla \cdot a \cdot \nabla u + b_1 \cdot \nabla u + \nabla \cdot (b_2 u) + Vu, \tag{2.1}$$

$\nabla \cdot a \cdot \nabla u := \sum_{i,j=1}^N \frac{\partial}{\partial x_i} a_{ij} \frac{\partial}{\partial x_j} u$, $a = (a_{ij}(x))_{i,j=1}^N$ is a real measurable matrix-valued function on $\mathbb{R}^N$. We assume throughout the paper that

$$\sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \ge h(x)|\xi|^2, \qquad \sum_{i,j=1}^N a_{ij}(x)\xi_i\eta_j \le Ch(x)|\xi||\eta|,$$
$$\forall \xi, \eta, x \in \mathbb{R}^N, \tag{2.2}$$

where $h^{\pm 1} \in L^\infty_{\mathrm{loc}}(G)$ is as in the definition of $D^1_{0,h}(G)$. If h is a constant, we say that the matrix a is uniformly elliptic. The lower-order terms, the vector fields b_1 and b_2 and the potential V, are supposed to be real-valued. For the sake of transparency we restrict our discussion to the case of all the coefficients of $\mathcal{L}$ being locally bounded and concentrate our attention at their behavior at infinity.

The operator $\mathcal{L}$ is defined via the bilinear form

$$\begin{aligned}\mathcal{E}(u, v) := &\int_G \nabla u \cdot a \cdot \nabla v \, dx + \int_G b_1 \cdot \nabla u v \, dx - \int_G v b_2 \cdot \nabla v \, dx \\ &+ \int_G V u^2 \, dx, u, v \in C^1_c(G). \end{aligned} \tag{2.3}$$

We always assume the conditions on b_1, b_2 and V (specified in the subsequent sections) such that $\mathcal{E}$ is a bounded coercive form on $D^1_{0,h}(G)$, that is, that the following two inequalities hold:

$$\mathcal{E}(\varphi, \varphi) \geq C \|h^{\frac{1}{2}} \nabla \varphi\|_2^2, \quad \varphi \in C^1_c(G), \tag{2.4}$$

$$|\mathcal{E}(\varphi, \psi)| \leq C \|h^{\frac{1}{2}} \nabla \varphi\|_2 \|h^{\frac{1}{2}} \nabla \psi\|_2, \quad \varphi, \psi \in C^1_c(G). \tag{2.5}$$

A standard sufficient condition for this is the *form boundedness* of the lower-order terms, that is, there exists $\varepsilon_0 \in (0, 1)$ such that

$$\begin{aligned}&\int_G \left(\frac{b_1^2 + b_2^2}{h} + |V| \right) u^2 \, dx \leq (1 - \varepsilon_0) \int_G \nabla u \cdot a \cdot \nabla u \, dx \\ &\quad \text{for all } 0 \leq u \in H^1_c(G). \end{aligned} \tag{2.6}$$

Condition (2.5) implies that $\mathcal{E}$ is a bounded bilinear form on $D^1_{0,h}(G) \times D^1_{0,h}(G)$ and a continuous bilinear form on $D^1_{\mathrm{loc}}(G) \times D^1_c(G)$. Hence the form defines a bounded linear operator $\mathcal{L} : D^1_0(G) \to D^{-1}(G)$ which is extended to a continuous linear operator $D^1_{\mathrm{loc}}(G) \to D^{-1}_{\mathrm{loc}}(G)$, by

$$\langle \mathcal{L}u, \theta \rangle := \mathcal{E}(u, \theta), \quad \theta \in C^1_c(G).$$

In particular, we say that a function u is a weak supersolution (subsolution) to the equation $\mathcal{L}v = f$ in G with $f \in D^{-1}_{\mathrm{loc}}(G)$ if $u \in D^1_{\mathrm{loc}}(G)$ and

$$\mathcal{E}(u, \phi) \geq \langle f, \phi \rangle, \ (\leq \langle f, \phi \rangle) \quad \phi \in C^1_c(G), \phi \geq 0. \tag{2.7}$$

If u is both a supersolution and a subsolution, we say that u is a solution to $\mathcal{L}v = f$ in G. Condition (2.4) implies that $\mathcal{L} : D^1_{0,h}(G) \to D^{-1}_h(G)$ is a homeomorphism. Hence the following lemma holds, which is a straightforward consequence of the Lax–Milgram theorem.

LEMMA 2.1. *Let assumptions* (2.5) *and* (2.4) *hold. Then the problem* $\mathcal{L}v = f$, $v \in D^1_{0,h}(G)$ *has a unique solution for every* $f \in D^{-1}_h(G)$.

In particular, $v = 0$ is the only $D^1_{0,h}(G)$-solution to the equation $\mathcal{L}v = 0$ in G.

COROLLARY 2.2. *The cone of positive supersolutions to the equation $\mathcal{L}v = 0$ is infinite dimensional.*

PROOF. For $n \in \mathbb{N}$, let $f_1, f_2, \dots, f_n \in C_c(G)$ be nonnegative disjointly supported functions and, for $k = 1, \dots, n$, let $v_k \in D^1_{0,h}(G)$ be the solutions to the equations $\mathcal{L}v_k = f_k$. Then v_k, $k = 1, \dots, n$, are linearly independent positive supersolutions to $\mathcal{L}v = 0$. □

Let $\Gamma \subseteq \partial G$. We say that $w \in D^1_{\mathrm{loc}}(G)$ satisfies $w = 0$ on Γ if $\theta w \in D^1_0(G)$ for all $\theta \in C^1_c(\mathbb{R}^N)$, such that $\operatorname{Supp}(\theta) \cap (\partial G \setminus \Gamma) = \emptyset$. We say that $w \geq 0$ on Γ if $w^- = 0$ on Γ in the sense described above and that $u, v \in H^1_{\mathrm{loc}}(G)$ satisfy $u \geq v$ on Γ if $(u - v) > 0$ on Γ.

LEMMA 2.3 (Kato-type inequality). *(see [2, Lemmas 2.7, 2.8, 2.10]) Let v be a subsolution to the equation $\mathcal{L}u = f$. Then $|v|$ is a subsolution to the equation $\mathcal{L}u = f \operatorname{sign} v$.*

Let v be a subsolution to the equation $\mathcal{L}u = f$ such that $v = 0$ on $\Gamma \subset \partial G$. Then $\mathcal{E}(u, \theta) \leq \langle f, \theta\rangle$ for all $\theta \in C^1_c(\mathbb{R}^N)$, $\operatorname{Supp}(\theta) \cap (\partial G \setminus \Gamma) = \emptyset$.

PROOF. The result follows by choosing $\phi = \frac{v}{v_\varepsilon}\theta$ in (2.7), with $v_\varepsilon = \sqrt{u^2 + \varepsilon^2}$. Note that $\nabla \frac{v}{v_\varepsilon} = \frac{\varepsilon^2}{v_\varepsilon^2}\nabla v$ so that $\nabla v \cdot a \cdot \nabla(\frac{v}{v_\varepsilon}\theta) \leq \nabla v_\varepsilon \cdot a \cdot \nabla\theta$ and $\frac{v}{v_\varepsilon} \to \operatorname{sign} v$ pointwise and $\nabla \frac{v}{v_\varepsilon} \to 0$ as $\varepsilon \to 0$ in $L^2_{\mathrm{loc}}(G)$. □

Maximum and comparison principles. The following theorem provides the maximum and comparison principles for equation $\mathcal{L}u = 0$ in G, in a form suitable for our framework. We give the full proof for completeness, though the arguments are mostly standard.

THEOREM 2.4. *Let $\mathcal{L}u \geq 0$ in G and let $v \in D^1_{0,h}(G)$ be such that $u^- \leq v$. Then $u \geq 0$.*

The proof is based on the following two lemmas. The first one is a weaker version of Theorem 2.4.

LEMMA 2.5 (Weak Maximum Principle). *Let $\mathcal{L}u \geq 0$, $u^- \in D^1_{0,h}(G)$. Then $u \geq 0$ in G.*

PROOF. Let $(\varphi_n) \subset C^\infty_c(G)$ be a sequence such that $\|u^- - \varphi_n\|_{D^1_{0,h}} \to 0$. For every $n \in \mathbb{N}$, set $u_n := 0 \vee \varphi_n \wedge u^-$. Since $0 \leq u_n \leq u^- \in D^1_{0,h}(G)$ and

$$\begin{aligned}\int_G |\nabla(u^- - u_n)|^2 h\, dx &= \int_{\{0 \leq \varphi_n \leq u^-\}} |\nabla(u^- - \varphi_n)|^2 h\, dx + \int_{\{\varphi_n \leq 0\}} |\nabla u^-|^2 h\, dx \\ &\leq \int_G |\nabla(u^- - \varphi_n)|^2 h\, dx + \int_{\{\varphi_n \leq 0\}} |\nabla u^-|^2 h\, dx \to 0,\end{aligned}$$

by the Lebesgue-dominated convergence, we conclude that $\|u^- - u_n\|_{D^1_{0,h}} \to 0$ (cf. [26]). Taking (u_n) as a sequence of test functions and using (2.4) and (2.5) we obtain

$$0 \le \mathcal{E}(u, u_n) = -\mathcal{E}(u^-, u_n) \to -\mathcal{E}(u^-, u^-) \quad \text{as } n \to \infty.$$

Hence $u^- = 0$. □

The next lemma provides an appropriate comparison principle.

LEMMA 2.6 (Weak Comparison Principle). *Let* $0 \le u \in H^1_{\mathrm{loc}}(G)$, $v \in D^1_{0,h}(G)$ *be such that* $\mathcal{L}(u - v) \ge 0$ *in* G. *Then* $u \ge v$ *in* G.

REMARK 2.7. Note that the assertion of Lemma 2.6 follows from Lemma 2.5 if one assumes in addition that $u \in H^1(G)$.

PROOF. Let $(G_n)_{n\in\mathbb{N}}$ be an exhaustion of G, i.e. an increasing sequence of bounded smooth domains such that $G_n \Subset G_{n+1} \Subset G$ and $\cup_{n\in\mathbb{N}} G_n = G$. Denote $f := \mathcal{L}v \in D_h^{-1}(G)$. Let $v_n \in D^1_{0,h}(G_n)$ be the unique weak solution to the linear problem $\mathcal{L}v_n = f$ in G_n. Then $\mathcal{L}(u - v_n) \ge 0$ in G_n with $u - v_n \in H^1(G_n)$, and $0 \le (u - v_n)^- \le v_n^+ \in D^1_{0,h}(G_n)$. Therefore $(u - v_n)^- \in D^1_{0,h}(G_n)$. By Lemma 2.5 we conclude that $v_n \le u$. Let $\bar{v}_n \in D^1_{0,h}(G)$ be defined by $\bar{v}_n = v_n$ on G_n and $\bar{v}_n = 0$ on $G \setminus G_n$. To complete the proof of the lemma it suffices to show that $\bar{v}_n \to v$ in $D^1_{0,h}(G)$. Indeed,

$$\begin{aligned} c\|\bar{v}_n\|^2_{D^1_{0,h}} &\le \mathcal{E}(\bar{v}_n) = \langle f, v_n\rangle \le c\|f\|_{D_h^{-1}(G)}\|\bar{v}_n\|_{D^1_{0,h}(G)} \\ &\le c\|v\|_{D^1_{0,h}(G)}\|\bar{v}_n\|_{D^1_{0,h}(G)}. \end{aligned}$$

Hence the sequence $(\bar{v}_n)$ is bounded so weakly compact in $D^1_{0,h}(G)$. Let $v_* \in D^1_{0,h}(G)$ be a limit point of the sequence and let $(\bar{v}_k)$ denote a subsequence, that converges weakly to v_*. Now let $\varphi \in H^1_c(G)$. Then $Supp(\varphi) \subset G_k$ for all $k \in \mathbb{N}$ large enough so $\mathcal{E}(\bar{v}_k, \varphi) = \langle f, \varphi\rangle$. Hence $\mathcal{E}(v_*, \varphi) = \lim_k \mathcal{E}(\bar{v}_k, \varphi) = \langle f, \varphi\rangle$. So $v, v_* \in D^1_0(G)$ and $\mathcal{L}v_* = \mathcal{L}v$. Hence $v_* = v$. Since v_* is an arbitrary limit point of the sequence $(\bar{v}_n)$, we conclude that $\bar{v}_n \to v$ weakly in $D^1_{0,h}(G)$. □

PROOF OF THEOREM 2.4. Let w satisfy $\mathcal{L}w \ge 0$ in G, and let $v \in D^1_{0,h}(G)$ be such that $w^- \le v$. Denote $u = w + v$. Then $u \ge 0$ and $\mathcal{L}(u - v) \ge 0$, hence the assertion follows from Lemma 2.6. □

Solutions between sub- and supersolution. The following is a fairly standard Caccioppoli inequality.

LEMMA 2.8 (Caccioppoli inequality). *Let* (2.2) *and* (2.6) *hold. Then for every two subdomains* $G' \Subset G'' \Subset G$ *there exists a constant* $c > 0$ *such that every subsolution* v *to the equation* $\mathcal{L}v = f$ *in G enjoys the following estimate:*

$$\|v\|_{H^1(G')} \le c(\|v\|_{L^2(G''\setminus G')} + \|f\|_{D^{-1}(G'')}), \quad G' \Subset G'' \Subset G.$$

PROOF. Let $\theta \in C_c^1(G)$ be such that $\mathbb{1}_{G''} \geq \theta \geq \mathbb{1}_{G'}$. Then $\mathcal{E}(v, v\theta^2) = \langle f, v\theta^2\rangle$. Now the assertion follows from the next identity:

$$\mathcal{E}(v, v\theta^2) = \mathcal{E}(v\theta, v\theta) + \int_G \Big[v\nabla(v\theta)\cdot(a - a^\top)\cdot\nabla\theta \\ - v^2\nabla\theta\cdot a\cdot\nabla\theta - v^2\theta(b_1 + b_2)\cdot\nabla\theta\Big]\,dx. \quad \square$$

In the next theorem we prove the existence of a solution between a supersolution and a subsolution. Results in this direction can be found in [19,18,63]. We give this result in the form most appropriate to our further applications.

THEOREM 2.9. *Let $u, v \in H^1_{\mathrm{loc}}(G)$ be a supersolution and a subsolution to the equation $\mathcal{L}w = \mathbf{f}(w)$ in G, respectively, $u \geq v$ a.e.*

Assume that for almost all $x \in G$, for a fixed x, the mapping $[v(x), u(x)] \ni w \mapsto f(x, w)$ is nonnegative and continuous and for every $G' \Subset G$ there exists $M = M_{G'} > 0$ such that

$$\frac{f(x, w_1) - f(x, w_2)}{w_1 - w_2} \geq -M, \quad v(x) \leq w_1 < w_2 \leq u(x),\ x \in G'. \tag{2.8}$$

Then there exists a solution $w \in H^1_{\mathrm{loc}}(G)$ to the equation $\mathcal{L}w = \mathbf{f}(w)$ in G, such that $v \leq w \leq u$ a.e.

PROOF. First we show that, for every $G' \Subset G$, there exists a solution w to the problem

$$\mathcal{L}w = \mathbf{f}(w) \quad \text{in } G' \quad \text{and} \quad w = v \quad \text{on } G \setminus G'.$$

Define the monotone correction $f_{G'}$ as follows:

$$f_{G'}(x, w) := \begin{cases} f(x, v(x)) + M_{G'}v(x) & w \leq v(x), x \in G' \\ f(x, w) + M_{G'}w & v(x) \leq w \leq u(x), x \in G' \\ f(x, u(x)) + M_{G'}u(x) & w \geq u(x), x \in G'. \end{cases}$$

Then $w \mapsto f_{G'}(x, w)$ is nondecreasing for a.a. $x \in G'$ and u and v are a supersolution and a subsolution to the equation $\mathcal{L}w + M_{G'}w = \mathbf{f}_{\mathbf{G'}}(w)$ in G'. Note that $\mathbf{f}_{\mathbf{G'}}(w) \leq \mathbf{f}_{\mathbf{G'}}(u) = \mathbf{f}(u) + M_{G'}u \in H^{-1}(G')$ for all measurable w.

Set $w_0 = v$ and, for $n = 1, 2, 3, \ldots$ let w_n be the solution to the following problem:

$$\begin{cases} \mathcal{L}w_n + M_{G'}w_n = \mathbf{f}_{\mathbf{G'}}(w_{n-1}) & \text{in } G', \\ w_n = v \quad \text{in } G \setminus G'. \end{cases}$$

Then $\|w_n\|_{H^1(G')} \leq c(\|v\|_{H^1(G')} + \|\mathbf{f}_{\mathbf{G'}}(u)\|_{H^{-1}(G')})$. Moreover, by the maximum principle,
$v = w_0 \leq w_1 \leq w_2 \leq \ldots w_n \leq w_{n+1} \leq u$.

Hence $w_n \to w_{G'} \in H^1(G')$ as $n \to \infty$ pointwise and weakly in $H^1(G')$. So $v \leq w_{G'} \leq u$, $w_{G'} = v$ in $G \setminus G'$ and $\mathcal{L}w_{G'} + M_{G'}w_{G'} = \mathbf{f}_{\mathbf{G'}}(w_{G'}) \Leftrightarrow \mathcal{L}w_{G'} = \mathbf{f}(w_{G'})$ in G'.

Observe that $w_{G'} \leq w_{G''}$ provided $G' \Subset G'' \Subset G$. Indeed, $w_{G''} \geq w_{G'}$ in $G \setminus G'$, and both $w_{G'}$ and $w_{G''}$ satisfy the equation $\mathcal{L}w + M_{G'}w = \mathbf{f}_{\mathbf{G}'}(w)$ in G'. Let $z = w_{G'} - w_{G''}$. Then $\mathcal{L}z + M_{G'}z = \mathbf{f}_{\mathbf{G}'}(w_{G'}) - \mathbf{f}_{\mathbf{G}'}(w_{G''})$ in G' and $z^+ \in H_0^1(G')$. Hence

$$\begin{aligned}\mathcal{E}(z^+) + M_{G'}\|z^+\|_2^2 &= \mathcal{E}(z, z^+) + M_{G'}\langle z, z^+\rangle \\ &= \langle \mathbf{f}_{\mathbf{G}'}(w_{G'}) - \mathbf{f}_{\mathbf{G}'}(w_{G''}), z^+\rangle \leq 0.\end{aligned}$$

Thus $z^+ = 0$, and hence $w_{G'} \leq w_{G''}$.

Let now $(G_n)_{n\in\mathbb{N}}$ be an exhaustion of G and let $w_n := w_{G_n}$. Since w_n is monotone it suffices to prove that it is bounded in $H^1_{\mathrm{loc}}(G)$ to conclude that the pointwise limit $\bar{w}$ solves the equation $\mathcal{L}w = \mathbf{f}(w)$. Let $G' \Subset G'' \Subset G$. Then $\mathcal{L}w_n + M_{G''}w_n = \mathbf{f}_{\mathbf{G}''}(w_n)$ in G'' for all n such that $G_n \Supset G''$. Hence, by Lemma 2.8,

$$\begin{aligned}\|w_n\|_{H^1(G')} &\leq c\left(\|w_n\|_{L^2(G''\setminus G')} + \|\mathbf{f}_{\mathbf{G}''}(w_n)\|_{H^{-1}(G'')}\right) \\ &\leq c\left(\|u\|_{L^2(G''\setminus G')} + \|\mathbf{f}_{\mathbf{G}''}(u)\|_{H^{-1}(G'')}\right).\end{aligned}$$

So the sequence (w_n) is bounded in $H^1_{\mathrm{loc}}(G)$. □

Minimal and large solutions. As we saw in the previous section discussing the model example, two harmonics in an exterior domain played the crucial role in the method. One was the fundamental solution of the Laplacian with singularity at zero, considered in the exterior of the unit ball. By comparison, we saw that every superharmonic outside the ball dominated this decaying at infinity solution. The other was a positive constant, the unique (up to a multiplicative constant) entire positive harmonic function, and we saw that in certain sense this one dominated the supersolutions.

So, we call it in the discussion below a minimal or small solution. The other one was just a constant. This type of solution will be later called large. Below we discuss these two types of solutions to the homogeneous equation.

Minimal positive solution. Let $G \subset \mathbb{R}^N$ be an unbounded domain and B_r a ball of radius $r > 0$ centered at the origin. Suppose that $G \cap B_r \neq \emptyset$. Let $\rho > r$. Consider the equation

$$-\nabla \cdot a \cdot \nabla u + b_1 \nabla u + \nabla(b_2 u) + Vu = 0 \quad \text{in } G \setminus \bar{B}_\rho. \tag{2.9}$$

We say that $v > 0$ is a *minimal positive solution* to (2.9) in $G \setminus \bar{B}_\rho$ if v is a solution to (2.9) in $G \setminus \bar{B}_\rho$ and if for any $r \in (0, \rho)$ and any positive supersolution $u > 0$ to (2.9) in $G \setminus \bar{B}_r$ there exists $c > 0$ such that

$$u \geq cv \quad \text{in } G \setminus \bar{B}_\rho.$$

This notion was introduced by Agmon [2] (see also [57–59,67,68,70]), where it is called the solution of minimal growth at infinity.

Below we construct a minimal positive solution to (2.9) in $G \setminus \bar{B}_\rho$. Let $0 \leq \psi \in C_c^\infty(G \setminus \bar{B}_r)$, $\psi = 1$ on Ω_1 for some $\Omega_1 \Subset G \cap \partial B_r$ and $Supp(\psi) \cap \partial B_r \subset G \cap \partial B_r$.

Thus $f_\psi := \nabla\cdot a\cdot\nabla\psi - b_1\cdot\nabla\psi - \nabla(b_2\psi) - V\psi \in D^{-1}(G\setminus\bar B_\rho)$. Let w_ψ be the unique solution to the problem

$$-\nabla\cdot a\cdot\nabla w + b_1\cdot\nabla w + \nabla(b_2 w) + Vw = f_\psi, \quad w\in D_0^1(G\setminus\bar B_\rho), \tag{2.10}$$

which is given by Lemma 2.1. Set $v_\psi := w_\psi + \psi$. Then v_ψ is the solution to the problem

$$-\nabla\cdot a\cdot\nabla v + b_1\cdot\nabla v + \nabla(b_2 v) + Vv = 0, \quad v-\psi\in D_0^1(G\setminus\bar B_\rho). \tag{2.11}$$

By the weak Harnack inequality $v_\psi > 0$ in $G\setminus\bar B_\rho$.

LEMMA 2.10. *v_ψ is a minimal positive solution to equation (2.9) in $G\setminus\bar B_\rho$.*

PROOF. Let $u>0$ be a positive supersolution to (2.9) in $G\setminus\bar B_r$. By the weak Harnack inequality there exists $m=m(\Omega_1)>0$ such that $u>m$ in Ω_1. Choose $c>0$ such that $c\psi<m$. Then $u-c\psi\ge 0$ in $G\setminus\bar B_\rho$, $cw_\psi\in D_0^1(G\setminus\bar B_\rho)$ and

$$\begin{aligned}(-\nabla\cdot a\cdot\nabla + b_1\cdot\nabla + \nabla b_2 + V)((u-c\psi)-cw_\psi)\\ = (-\nabla\cdot a\cdot\nabla + b_1\cdot\nabla + \nabla b_2 + V)u \ge 0 \quad \text{in } G\setminus\bar B_\rho.\end{aligned}$$

By Lemma 2.6 we conclude that $u-c\psi\ge cw_\psi$, that is $u\ge cv_\psi$ in $G\setminus\bar B_\rho$. □

In the next sections the above construction will be applied for the whole of $\mathbb{R}^N$ or a cone in place of G.

PROPOSITION 2.11 (Large solution). *Let (2.4) and (2.5) be fulfilled. Then there exists a positive solution v_l to the equation $\mathcal{L}v=0$ in $\bar B_1^c$ such that $v_l=0$ on ∂B_1.*

Moreover, every such solution v_l enjoys the following properties: $\limsup_{x\to\infty} v_l/u > 0$ *for every positive supersolution u to the equation $\mathcal{L}v=0$ in $\bar B_1^c$ and there exists a positive supersolution u_* to the equation $\mathcal{L}v=0$ in $\bar B_1^c$ such that* $\limsup_{x\to\infty} v/u_* = \infty$.

PROOF. Let $R_n\ge 1$, $R_n\to\infty$ as $n\to\infty$. Choose a reference point $o\in A_{1,R_1}$. Let $v_n\in H^1_{\mathrm{loc}}(\bar B_1^c)$ satisfy $\mathcal{L}v=0$ in A_{1,R_n}, $v=0$ on ∂B_1, $v=c_n$ on $B^c_{R_n}$, with c_n chosen so that $v_n(o)=1$. Then, for every $m\ge 1$ and $n\ge m+1$, by Lemma 2.8 $\|v_n\|_{H^1(A_{1,R_m})}\le c\|v_n\|_{L^2(A_{R_m,R_{m+1}})}$. Due to the Harnack inequality, $\sup_{x\in A_{1,R_{m+1}}} v_n(x)\le c_m v_n(o) = c_m$. So v_n is weakly compact in $H^1_{\mathrm{loc}}(\bar B_1^c)$. Let $v_l\in H^1_{\mathrm{loc}}(\bar B_1^c)$ be its limit point and v_k be a subsequence such that $v_k\to v_l$ weakly in $H^1_{\mathrm{loc}}(\bar B_1^c)$. Let $\phi\in C_c^1(\mathbb{R}^N)$ such that $\phi=0$ on ∂G. Then $Supp(\phi)\subset B_{R_k}$ for k large enough. Hence $\mathcal{E}(v_l,\phi)=\lim\mathcal{E}(v_k,\phi)=0$. So v_l is a solution to the equation $\mathcal{L}v=0$ in $\bar B_1^c$. Moreover, $v_l=0$ on ∂B_1 and $v_l>0$ in $\bar B_1^c$ by the Harnack inequality since $v_l(o)=1$.

Now we pass to the second assertion. Assume, to the contrary, that there exists a $u>0$ such that $\mathcal{L}u\ge 0$ and $v_l/u\to 0$ as $x\to\infty$. Fix $\varepsilon>0$. Then there exists $R>0$ such that $v_l\le\varepsilon u$ in B_R^c. Hence $v_l\le\varepsilon u$ on $\partial B_R\cup\partial B_1$ since $v_l=0$ on ∂G. So $v_l\le\varepsilon u$ in $A_{1,R}$ by the maximum principle (Theorem 2.4). Hence $v_l\le\varepsilon u$ in B_1^c so $v_l=0$ as $\varepsilon\to 0$, contradicting the fact that $v_l>0$.

Now we prove the last assertion.

Assume, to the contrary, that $\limsup_{x\to\infty} v_l/u < \infty$ for all $u > 0$ such that $\mathcal{L}u \geq 0$. Then v_l/u is bounded in $\bar{B}_1^c$ for every such u. Indeed, $\limsup_{x\to\infty} v_l/u < \infty$ implies that there exist c, $R > 0$ such that $v_l \leq cu$ in $\bar{B}_R^c$. Hence $v_l \leq cu$ on $\partial B_R \cup \partial B_1$ since $v_l = 0$ on ∂B_1. So $v_l \leq cu$ in $A_{1,R}$ by the maximum principle (Theorem 2.4) and hence $v_l \leq cu$ in $\bar{B}_1^c$. Now fix $u > 0$ such that $\mathcal{L}u \geq 0$ and u is not a multiple of v_l. Let $C := \sup_{\bar{B}_1^c} v_l/u$. Then $\hat{u} := u - \frac{1}{C} v_l > 0$ and $\mathcal{L}\hat{u} \geq 0$ in $\bar{B}_1^c$. Hence there exists $c > 0$ such that $v_l \leq c\hat{u}$, that is, $(\frac{1}{C} + \frac{1}{c})v_l \leq u$. This contradicts the definition of C. □

In the case of our assumptions of locally bounded coefficients (and even with mild singularities) in the same way (normalizing at 0) one can construct an *entire solution*, i.e. the solution defined on the whole of $\mathbb{R}^N$.

The above notions of small (minimal) and large solutions are consistent with a classification of solutions which naturally follows from the following version of the Phragmèn–Lindelöf principle (we follow [8], see also [48,49]). We formulate it in the case of exterior domains.

THEOREM 2.12 (Phragmèn–Lindelöf principle). *Let v be a positive subsolution to equation (2.9) in $\mathbb{R}^N \setminus \bar{B}_1^c$. Then one of the two assertions holds.*

(i) *For every positive supersolution u in $\mathbb{R}^N \setminus \bar{B}_1^c$*

$$\lim_{R\to\infty} \sup_{|x|>R} \frac{v(x)}{u(x)} > 0. \tag{2.12}$$

(ii) *For every positive supersolution u in $\mathbb{R}^N \setminus \bar{B}_1^c$*

$$\lim_{R\to\infty} \sup_{|x|>R} \frac{v(x)}{u(x)} < \infty. \tag{2.13}$$

PROOF. Suppose that (i) does not hold. Then there exists a supersolution $u^* > 0$ to (2.9) such that

$$\lim_{R\to\infty} \sup_{|x|>R} \frac{v(x)}{u^*(x)} = 0. \tag{2.14}$$

Let $u > 0$ be an arbitrary supersolution to (2.9). By the weak Harnack inequality for supersolutions and local supremum estimates for subsolutions [28] there exists a constant $c > 0$ such that $u \geq cv$ on $A_{1,2}$. For $\tau > 0$ define $v_\tau := cv - \tau u^*$. By (2.14) for any $\tau > 0$ there is $R > 0$ such that $v_\tau \leq 0$ for $|x| \geq R$. By the maximum principle the inequality holds for $2 < |x| < R$, and therefore for $x \in \mathbb{R}^N \setminus \bar{B}_2$. This implies (2.13). □

Example. We already discussed the case of the Laplacian on the exterior of the unit ball. There we have the fundamental solution $c|x|^{2-N}$ as a small solution and a constant as a large solution. Now let us look at the equation $-\Delta u - \frac{c}{|x|^2} u = 0$ on $\bar{B}_1^c$, with $c < \left(\frac{N-2}{2}\right)^2$. There are two solutions satisfying the boundary condition $u|_{|x|=1} = 1$, one of which is small and the other large. They are $v_s = |x|^{\gamma_-}$ and $v_l = |x|^{\gamma_+}$, where $\gamma_- < \gamma_+$ are the roots of the quadratic equation $\gamma^2 + \gamma(N-2) + c = 0$, and obviously $v_s v_l = |x|^{2-N}$. We will return to this example in the subsequent sections.

Positive supersolutions and the Hardy inequality. In the case of symmetric forms $\mathcal{E}$ without first-order terms (i.e. $b_1 = b_2 = 0$) there is a straight connection between the existence of positive supersolutions to the equation $\mathcal{L}u = 0$ and the positivity of the form $\mathcal{E}$ (see [2, Theorem3.1], [20, Theorem 4.2.1]).

The same relation for the general case in the presence of the drift terms b_1 and b_2 seems to be not true. However, the next theorem provides

THEOREM 2.13. *Let condition (2.6) be satisfied. Let $0 < u \in H^1_{\mathrm{loc}}(G)$ be a solution to $\mathcal{L}u \geq Wu$. Then there exists a constant $C > 0$ such that for every $\varphi \in C_0^\infty(G)$*

$$C\langle \nabla\varphi \cdot a \cdot \nabla\varphi\rangle \geq \langle W\varphi^2\rangle. \tag{2.15}$$

Moreover, if the matrix a is symmetric and $b_1 = b_2 = V = 0$ then one can choose $C = 1$.

PROOF. Let $\varphi \in C_0^\infty(G)$. As already mentioned, $u > 0$ in G and $u^{-1} \in L^\infty_{\mathrm{loc}}(G)$ (by Harnack). This allows us to choose $\frac{\varphi^2}{u}$ as a test function for the inequality $\mathcal{L}u \geq Wu$. We obtain

$$\begin{aligned} -\left\langle \varphi\frac{\nabla u}{u} \cdot a \cdot \varphi\frac{\nabla u}{u}\right\rangle + 2\left\langle \nabla\varphi \cdot a \cdot \varphi\frac{\nabla u}{u}\right\rangle + \left\langle \varphi(b_1 + b_2) \cdot \varphi\frac{\nabla u}{u}\right\rangle - 2\langle \varphi b_2 \cdot \nabla\varphi\rangle \\ + \langle V\varphi^2\rangle \geq \langle W\varphi^2\rangle. \end{aligned}$$

The rest is Cauchy–Schwarz and use of the conditions on b_1, b_2 and V. □

The above theorem shows that the existence of positive supersolution to the equation $\mathcal{L}u - Wu = 0$ implies that W satisfies a Hardy-type inequality (2.15). From this and the sharpness of the Hardy inequality (see Section B) in exterior and cone-like domains, one immediately derives the nonexistence principle which states that if the potential W decays at infinity slower than $\frac{C}{|x|^2}$ than the equation $\mathcal{L}u - Wu = 0$ has no nontrivial nonnegative supersolutions in any exterior or cone-like domain. We give precise formulations in the corresponding sections (compare [38], [70, p.156]).

3. Exterior domains. Divergence-type equations

In this section we study the existence and nonexistence of positive weak (super-) solutions to the equation

$$\mathcal{L}u = c_0|x|^{-\sigma}u^q \quad \text{in } \bar{B}_1^c,\, c_0 > 0, \tag{3.1}$$

where

$$\mathcal{L}u = -\nabla \cdot a \cdot \nabla u + b_1 \cdot \nabla u + \nabla(b_2 u) + Vu$$

$\nabla \cdot a \cdot \nabla u := \sum_{i,j=1}^N \frac{\partial}{\partial x_i} a_{ij} \frac{\partial}{\partial x_j} u$, $a = (a_{ij}(x))_{i,j=1}^N$ is a real measurable matrix-valued function on $\mathbb{R}^N$ which we assume to be locally elliptic, uniformly in binary annuli i.e. that there exists function h, $C^{-1} \le \frac{h(x)}{h(y)} \le C$ for $R \le |x|, |y| \le 2R$, $R > 0$, such that

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \ge h(x)|\xi|^2, \qquad \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\eta_j \le Ch(x)|\xi||\eta|,$$
$$\forall \xi, \eta, x \in \mathbb{R}^N. \tag{3.2}$$

The lower-order terms, the vector fields b_1 and b_2 and the potential V, are supposed to be real-valued, locally bounded and satisfying the following assumptions:

$$|b_i(x)| \le \frac{h(x)\beta_i(|x|)}{|x|}, \quad i = 1, 2, \qquad |V(x)| \le \frac{h(x)\beta_V(|x|)}{|x|^2}, \tag{3.3}$$

where β_i, $i = 1, 2$ and β_V are nonincreasing functions.

The conditions on the coefficients guarantee that weak solutions to (3.1) are continuous (even Hölder-continuous, see [28]). Positive weak solutions to the homogeneous equation $\mathcal{L}v = 0$ satisfy the following Harnack inequality

$$\sup_{A_{R,2R}} v(x) \le C \inf_{A_{R,2R}} v(x), \tag{3.4}$$

where the constant C neither depends on v nor on R. This follows by scaling from the standard Harnack inequality [28, Theorem 8.20]. Weak positive supersolutions to the equation $\mathcal{L}u = 0$ also obey the weak Harnack inequality [28, Theorem 8.18], which is again uniform on annuli.

If $N \ge 3$ and the matrix is uniformly elliptic, the classical result of Littman *et al.* [53] states that the fundamental solution $\Gamma(x, y)$ to the operator $-\nabla \cdot a \cdot \nabla$ satisfies the two-sided estimate

$$C^{-1}|x - y|^{2-N} \le \Gamma(x, y) \le C|x - y|^{2-N}.$$

Large solutions are constants in this case, and the result and the proof of Theorem 1.3 extends to this case without any change. A nonuniformly elliptic diffusion part $-\nabla \cdot a \cdot \nabla$, presence of lower-order terms and finally $N = 2$ are completely different matters due to possible changes in the behavior of small and large solutions at infinity, to which any mentioned factor can contribute.

Concerning the problem of the existence and nonexistence of positive solutions to (3.1) one can ask the following two different questions:

- What are the conditions on the coefficients of the operator $\mathcal{L}$ that ensure the validity of the assertion of Theorem 1.3?
- If the conditions on the coefficients are such that the nonexistence set is different from the one in Theorem 1.3, can one describe the changes quantitatively, or at least qualitatively?

While the first question is mainly perturbation theory by small lower-order terms (how small?) the second question is more involved.

We start our discussion with a model situation in which we are able to give a complete answer to the second question.

3.1. *Laplacian with Hardy potential*

In this subsection we study (1.1) in the case $\mathcal{L} = -\Delta - \frac{\mu}{|x|^2}$. The underlying domain G is the exterior of a ball. In the exposition we follow [48], sometimes skipping technicalities. So we study the equation

$$\left(-\Delta - \frac{\mu}{|x|^2}\right) u = c_0|x|^{-\sigma} u^q, \quad \text{in } G, c_0 > 0. \tag{3.5}$$

Below we denote $C_H := \frac{(N-2)^2}{4}$. If $\mu \le C_H$ then the quadratic equation

$$\gamma(\gamma + N - 2) + \mu = 0 \tag{3.6}$$

has real roots, denoted by $\gamma^- \le \gamma^+$.

For $\mu \le C_H$ we introduce the critical line $\sigma = \Lambda_*(q)$ on the (q, σ)-plane defined by

$$\Lambda_*(q) := \min\{\gamma^-(q-1) + 2, \gamma^+(q-1) + 2\} \quad (q \in \mathbb{R}),$$

and the nonexistence set as before

$$\mathcal{N} = \{(q, \sigma) \in \mathbb{R}^2 \setminus \{1, 2\} : \textit{equation (3.5) has no positive supersolutions}\}.$$

The main result of this subsection reads as follows.

THEOREM 3.1. *The following assertions are valid.*
(i) *Let $\mu < C_H$. Then $\mathcal{N} = \{\sigma \le \Lambda_*(q)\}$.*
(ii) *Let $\mu = C_H$. Then $\mathcal{N} = \{\sigma < \Lambda_*(q)\} \cup \{\sigma = \Lambda_*(q),\ q \ge -1\}$.*

We prove Theorem 3.1 in Section 3.1.2. In the next subsection we study the corresponding linear homogeneous equations.

REMARK 3.2. For $\sigma = 2$ and $q = 1$ equation (3.5) is linear. Due to the Hardy inequality (see Appendix B) it has positive supersolutions if and only if $c_0 + \mu \le C_H$.

3.1.1. *Linear equation*

Consider the linear equation

$$-\Delta v - \frac{\mu}{|x|^2} v = 0 \quad \text{in } G = \mathbb{R}^N \setminus \bar{B}_1. \tag{3.7}$$

The existence of positive supersolutions to (3.7) is equivalent to the positivity of the quadratic form $\mathcal{E}$ defined by

$$\mathcal{E}(u) = \int_G \nabla u \cdot \nabla u dx - \int_G \frac{\mu}{|x|^2} u^2 dx, \quad u \in H_c^1(G) \cap L^\infty(G) \tag{3.8}$$

(see [2] and in one direction Theorem 2.13). For $\mu < C_H$ the domain of the closure of the form is $D_0^1(G)$. For $\mu = C_H$ the improved Hardy inequality (see Appendix B) implies that the extended Dirichlet space is well defined (see Appendix A), and comparison principle (Lemma A.3) is applicable. The optimality of the Hardy inequality (B.1) immediately leads to the following

PROPOSITION 3.3. *Equation* (3.7) *has positive supersolution if and only if* $\mu \le C_H$.

For the case $\mu < C_H$ the small and large solutions to (3.7) satisfying $v_s|_{|x|=1} = v_l|_{|x|=1} = 1$ are $v_s = |x|^{\gamma^-}$ and $v_l = |x|^{\gamma^+}$.

For the case $\mu = C_H$ they are $v_s = |x|^{\frac{2-N}{2}}$ and $v_l = |x|^{\frac{2-N}{2}} \log|x|$. In the latter case v_s belongs to the extended Dirichlet space, which can be verified by a simple computation.

Using Theorem 2.12 we obtain the following statement, with the notation $m_R(u) = \min_{|x|=R} u(x)$.

THEOREM 3.4. *Let* $u \in H^1_{\rm loc}(G)$ *be a positive supersolution to* (3.7)*. There exist constants* $c_1, c_2 > 0$ *such that*

(i) *if* $\mu < C_H$ *then*

$$c_1 R^{\gamma^-} \le m_R(u) \le c_2 R^{\gamma^+}, \tag{3.9}$$

(ii) *if* $\mu = C_H$ *then*

$$c_1 R^{\frac{2-N}{2}} \le m_R(u) \le c_2 R^{\frac{2-N}{2}} \log(R). \tag{3.10}$$

3.1.2. *Semi-linear equation*

In this subsection we prove Theorem 3.1 considering separately superlinear and sublinear cases.

***Proof of Theorem* 3.1 (*Superlinear case* $q \ge 1$).** We consider separately the cases $\mu < C_H$ and $\mu = C_H$.

Case $\mu < C_H$: *Nonexistence.* The proof goes much in the same way as in Section 1.2.

First we prove the nonexistence of supersolutions in the subcritical case, i.e. for (q, σ) below the critical line Λ^*. Let $\sigma < \gamma^-(q-1) + 2$. Let $u > 0$ be a supersolution to (3.5). Then u is a supersolution to (3.7). By Theorem 3.4 there exists $c > 0$ such that

$$m_R(u) \ge c R^{\gamma^-} \quad (R > 2).$$

Linearizing (3.5) and using the bound above, we conclude that u is a supersolution to

$$-\Delta u - \frac{\mu}{|x|^2} u - \frac{V(x)}{|x|^2} u = 0 \quad \text{in } \mathbb{R}^N \setminus \bar{B}_2, \tag{3.11}$$

where $V(x) := c_0 u^{q-1} |x|^{2-\sigma}$ satisfies

$$V(x) \ge c^{q-1} |x|^{\gamma^-(q-1)+(2-\sigma)} \quad \text{in } \mathbb{R}^N \setminus \bar{B}_2.$$

Then the assertion follows by Proposition 3.3.

Now let $\sigma = \gamma^-(q-1) + 2$. Arguing as above we conclude that u is a supersolution to (3.11) in $\mathbb{R}^N \setminus \bar{B}_2$ with $V \ge \delta$ for some $\delta > 0$ (δ can be chosen small enough so that $\mu + \delta < C_H$). Again applying Theorem 3.4 yields

$$m_R(u) \ge c R^{\gamma_\delta^-} \quad (R > 3)$$

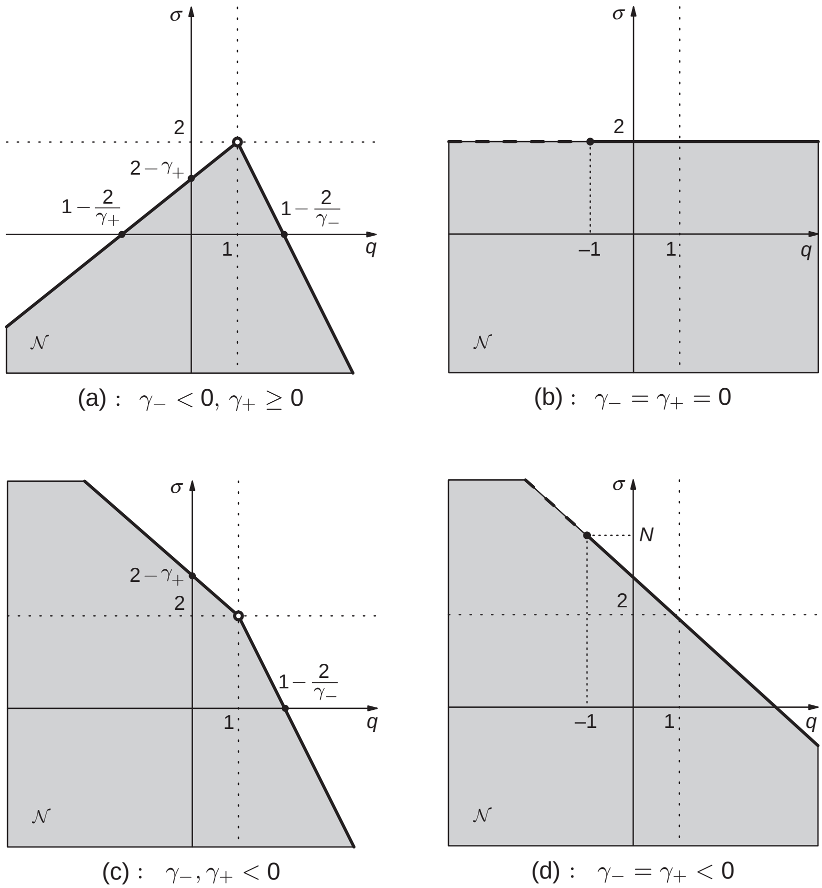

Fig. 3. The nonexistence set $\mathcal{N}$ of equation (3.5) for typical values of γ^- and γ^+.

with γ_δ^- being the smaller root of the equation $\gamma^2 + \gamma(N-2) + (\mu + \delta) = 0$, so that $\gamma_\delta^- > \gamma^-$. Using the last improved bound in the linearization of (3.5) we obtain that u is a supersolution to

$$-\Delta u - \frac{\mu}{|x|^2}u - \frac{\tilde{V}(x)}{|x|^2}u = 0 \quad \text{in } \mathbb{R}^N \setminus \bar{B}_3,$$

with $\tilde{V}(x) := c_0 u^{q-1}|x|^{2-\sigma} \geq c^{q-1}|x|^{\gamma_\delta^-(q-1)+(2-\sigma)}$ and $\gamma_\delta^-(q-1) + (2-\sigma) > 0$. The assertion now follows from Proposition 3.3.

Case $\mu < C_H$: Existence. Let $q \geq 1$ and $\sigma > \gamma^-(q-1) + 2$. Choose $\gamma \in (\gamma^-, \gamma^+)$ such that $\gamma \leq \frac{\sigma-2}{q-1}$. Then one can verify directly that the functions

$$w := \tau|x|^\gamma$$

are supersolutions to (3.5) in $\mathbb{R}^N \setminus \bar{B}_1$ for sufficiently small $\tau > 0$.

Case $\mu = C_H$: Nonexistence. Let $u > 0$ be a supersolution to (3.5). Then u is a supersolution to (3.7). By Theorem 3.4 there exists $c > 0$ such that

$$m_u(R) \geq cR^{\frac{2-N}{2}} \quad (R > 2).$$

Using this bound in the linearization in the same way as above, we obtain that u is a supersolution to $-\Delta u - \frac{\mu_1}{|x|^2} = 0$ with $\mu_1 > C_H$. Then the assertion now follows from Proposition 3.3.

Case $\mu = C_H$: Existence. Let $q \geq 1$ and $\sigma > \gamma^-(q-1)+2$. Choose $\beta \in (0,1)$. Then one verifies directly that the functions

$$w := \tau |x|^{\frac{2-N}{2}} \log^\beta |x|$$

are supersolution to (3.5) in $\mathbb{R}^N \setminus \bar{B}_1$ for sufficiently small $\tau > 0$.

Proof of Theorem 3.1 (Sublinear case $q < 1$). As before, we consider separately the cases $\mu < C_H$ and $\mu = C_H$. We start with two lemmas. The first one is already familiar in the case of Laplacian (Lemma 1.4), and the proof is the same.

LEMMA 3.5. *Let $q < 1$. Let $u > 0$ be a supersolution to (3.5) in $G = \mathbb{R}^N \setminus \bar{B}_1$. Then there exists $c > 0$ such that*

$$m_R(u) \geq cR^{\frac{2-\sigma}{1-q}}, \quad R > 2. \tag{3.12}$$

LEMMA 3.6. *Let $q < 1$, $\mu \leq C_H$ and $\sigma \in \mathbb{R}$. Suppose that (3.5) has a positive supersolution in $G = \mathbb{R}^N \setminus \bar{B}_1$. Then there exists a positive solution to (3.5) in G.*

PROOF. Let $u > 0$ be a supersolution to (3.5) in G. Then, as was shown before, $u \geq cv$ in G, where $v(x) = c|x|^{\gamma^-}$ ($\gamma^- = \frac{2-N}{2}$ if $\mu = C_H$). Obviously, $v_\psi > 0$ is a subsolution to (3.5) in $\mathbb{R}^N \setminus \bar{B}_2$. Thus we can proceed via the standard sub and supersolutions techniques to prove existence of a solution to (3.5) in $\mathbb{R}^N \setminus \bar{B}_2$, located between cv and u (cf. [40, Proposition 1.1(iii)] and Section 2). Finally, after a suitable scaling we obtain a solution to (3.5) in G. □

Now we are ready to prove Theorem 3.1 in the sublinear case.

Case $\mu < C_H$: Nonexistence. We distinguish between the subcritical and critical cases. When (q, σ) is below the critical line, the proof of the nonexistence is straightforward.

So, first let $q < 1$ and $\sigma < \gamma^+(q-1)+2$. Let $u > 0$ be a supersolution to (3.5). Then u is a supersolution to the linear equation

$$-\Delta u - \frac{\mu}{|x|^2} u = 0 \quad \text{in } G. \tag{3.13}$$

By Theorem 3.4 we conclude that

$$m_R(u) \leq cR^{\gamma^+}, \quad R > 2. \tag{3.14}$$

This contradicts (3.12).

Next we consider the case when $\sigma = \gamma^+(q-1)+2$ is on the critical line, and hence (3.14) is comparable with (3.12). Let $u > 0$ be a supersolution to (3.5). Due to Lemma 3.6 there is a solution $w > 0$ to (3.5) such that $w \le u$. Linearizing (3.5) and using the upper bound (3.12) we conclude that $w > 0$ is a solution to

$$-\Delta w - \frac{\mu}{|x|^2} w - \frac{V(x)}{|x|^2} w = 0 \quad \text{in } G, \tag{3.15}$$

where $V(x) := c^{q-1}|x|^{2-\sigma} w^{q-1}$ satisfies $V(x) \le c_1$ in $\mathbb{R}^N \setminus \bar{B_2}$. Therefore by the strong Harnack inequality

$$M_R(w) := \max_{|x|=R} w(x) \le C_s m_R(w), \qquad R > 2,$$

with C_s independent from R. Hence combining this with (3.14) we obtain

$$M_R(w) \le c R^{\gamma^+}, \qquad R > 2.$$

This implies that $V(x) \ge \delta$ for $|x| > 2$, for some $\delta > 0$. Choosing δ such that $\mu+\delta < C_H$ conclude that w is a supersolution to the linear equation

$$-\Delta w - \frac{\mu+\delta}{|x|^2} w = 0 \quad \text{in } \mathbb{R}^N \setminus \bar{B_2}. \tag{3.16}$$

Applying Theorem 3.4 to (3.16) we infer that

$$m_R(w) \le c R^{\gamma_\delta^+} \quad (R > 3)$$

with γ_δ^+ being the larger root of the equation $\gamma^2 + \gamma(N-2) + (\mu+\delta) = 0$, so that $\gamma_\delta^+ < \gamma^+$. Now we reach a contradiction with (3.12).

Case $\mu < C_H$: Existence. Let $\sigma > \gamma^+(q-1)+2$. Choose $\gamma \in (\gamma^-, \gamma^+)$ such that $\gamma \ge \frac{\sigma-2}{q-1}$. Direct computation shows that

$$w := \tau |x|^\gamma$$

are supersolutions to (3.5) in G for a sufficiently large $\tau > 0$.

Case $\mu = C_H$: Nonexistence. Recall that in this case $\gamma^+ = \frac{2-N}{2}$. For (q, σ) below critical line ($\sigma < \gamma^+(q-1)+2$) the proof is the same as for $\mu < C_H$, so we omit it.

Now let $q \in [-1, 1)$ and $\sigma = \gamma^+(q-1)+2$. Let $u > 0$ be a supersolution to (3.5) in G. As above, we can find a solution $w > 0$ to (3.5) in G such that $w \le u$. Arguing as above by Theorem 3.4 we obtain

$$cR^{\gamma^+} \le m_R(w) \le cR^{\gamma^+} \log(R).$$

The lower bound shows applicability of the strong Harnack inequality to equation (3.15) which is obtained by linearization as the potential $V(x) := c^{q-1}|x|^{2-\sigma} w^{q-1}$ satisfies $V(x) \le c_1$ in $\mathbb{R}^N \setminus \bar{B_2}$. Therefore we infer that

$$M_R(w) \le C R^{\gamma^+} \log(R).$$

This implies the lower bound on the potential in (3.15): $V(x) \ge C(\log|x|)^{q-1}$. For $q > -1$ this contradicts the improved Hardy inequality (see Section B).

The remaining case $q = -1$ is the most delicate. w becomes a supersolution to the equation

$$-\Delta w - \frac{C_H}{|x|^2} w - \tilde{V} w = 0, \quad |x| > 2,$$

with $\tilde{V}(x) = c_0|x|^{-N} w^{-2} \geq c|x|^{-N}(\log|x|)^{-2}$. This is of course a limiting potential for the improved Hardy inequality, and one expects that the upper estimate on w can be improved as it was done in the previous cases. And indeed constructing an explicit subsolution to the equation

$$-\Delta w - \frac{C_H}{|x|^2} w - \frac{\varepsilon}{|x|^N \log^2|x|} w = 0, \quad |x| > 2,$$

in the form $\tau|x|^{\gamma^+}(\log|x|)^\beta$ with $\beta^+ < \beta < 1$, β^+ being the larger root of the equation $\beta(1-\beta) = \varepsilon$, one obtains the required improvement (again using Theorem 2.12 and the Harnack inequality). This leads again to a contradiction to the improved Hardy inequality.

Case $\mu = C_H$: *Existence.* One can construct a supersolution explicitly in the form $u(x) = \tau|x|^{\frac{2-N}{2}}(\log|x|)^\beta$ choosing appropriately $\tau > 0$ and $\beta \in (0, 1)$. □

3.2. *Uniformly elliptic case*

In this section we study positive supersolutions to equation (3.1) with the coefficients a, b_1, b_2, V satisfying (3.2) and (3.3) with $h = const$. We impose the following additional quantitative assumption on a, b_1, b_2, V:

there exists $\varepsilon_0 \in (0, 1)$ such that

$$\beta_1^2 + \beta_2^2 + \beta_V \leq (1 - \varepsilon_0)\frac{(N-2)^2}{4}. \tag{3.17}$$

The latter assumption guarantees the validity of (2.6) and hence (2.4) and (2.5).

As in the previous sections, we introduce the critical line $\sigma = \Lambda(q)$ on the (q, σ)-plane defined by

$$\Lambda(q) := \min\{(2-N)(q-1) + 2, 2\} \quad (q \in \mathbb{R}),$$

and the nonexistence set

$$\mathcal{N} = \{(q, \sigma) \in \mathbb{R}^2 \setminus \{1, 2\} : \textit{equation (3.1) has no positive supersolutions}\}.$$

The main result of this subsection reads as follows.

THEOREM 3.7. *Let $N \geq 3$ and conditions* (3.2), (3.3), (3.17) *hold, with $h = const$ and $\beta_1(r)$, $\beta_2(r)$, $\beta_V(r) \to 0$ as $r \to \infty$. Then*

$$\{(q, \sigma) : \sigma < \Lambda(q)\} \subset \mathcal{N} \subset \{(q, \sigma) : \sigma \leq \Lambda(q)\}.$$

If, in addition, $\beta_1(r) + \beta_V(r)$ is a Dini function at infinity, then

$$\mathcal{N} \supset \{(q, \sigma) : \sigma = \Lambda(q), q \geq 1\}.$$

If instead, $\beta_2(r)+\beta_V(r)$ is a Dini function at infinity, then

$$\mathcal{N}\supset\{(q,\sigma):\sigma=\Lambda(q),\, q\le 1\}.$$

So $\mathcal{N}=\{(q,\sigma):\sigma\le\Lambda(q)\}$ provided $\beta_1(r)+\beta_2(r)+\beta_V(r)$ is a Dini function at infinity.

REMARK 3.8.
(1) The diagram of the existence and nonexistence zones on the plane (q,σ) is the same as in Figure 2 with the exception of the critical line.
(2) For $(q,\sigma)=(1,2)$ equation (3.1) becomes linear. So the existence of a positive supersolution depends on the constant c_0.
(3) The example below shows that the conditions on the lower-order terms are optimal in a sense. Namely, for every $q\in\mathbb{R}$ one can produce a potential and drifts such that the functions β_V,β_1,β_2 are not Dini at infinity and positive supersolutions exist with value (q,σ) on the critical line.

EXAMPLE 3.9. *The function $u(x)=c|x|^{2-N}(\log|x|)^{-\alpha}$ is a supersolution to the equation*

$$-\Delta u+\frac{k}{|x|^2\log|x|}=c_0|x|^{-\sigma}u^{\frac{N-\sigma}{N-2}},\qquad |x|>1,\sigma<2,$$

with $k>\frac{(N-2)^2}{2-\sigma}$ and $\frac{N-2}{2-\sigma}<\alpha<\frac{k}{N-2}$.

The function $u(x)=c(\log|x|)^{\alpha}$ is a supersolution to the equation

$$-\Delta u+\frac{k}{|x|^2\log|x|}=c_0|x|^{-2}u^{q},\qquad |x|>1,q<1,$$

with $k>\frac{N-2}{1-q}$ and $\frac{1}{1-q}<\alpha<\frac{k}{N-2}$.

REMARK 3.10. In Theorem 3.7 it was not assumed that the matrix a is symmetric. If one assumes that a is symmetric then the Kelvin transform is applicable (see, e.g. [34]), and Theorem 3.7 can be applied to equations in punctured balls. For instance, in this way one obtains the following result which is a natural extension of [15, Theorem 0.2] to the case of divergence-type equations with measurable coefficients. For the case of the Laplacian, the result in [15, Theorem 0.2] is stronger, as it asserts the nonexistence of distributional solutions.

THEOREM 3.11. *Let $N\ge 3$, $q>1$. Let a be symmetric and uniformly elliptic. Let $v\in H^1_{\mathrm{loc}}(B_1\setminus\{0\})$, $v\ge 0$, be a solution to $-\nabla\cdot a\cdot\nabla v\ge c_0|x|^{-2}v^q$ in $B_1\setminus\{0\}$. Then $v\equiv 0$.*

The rest of the section is organized as follows. In the next subsection we provide the estimates for small and large solutions to the linear equation $\mathcal{L}u=0$. In the subsequent subsection we apply the obtained results to prove Theorem 3.7.

3.2.1. *Linear equations*

Here we study positive solutions to the linear homogeneous equation

$$\mathcal{L}u=0. \tag{3.18}$$

Throughout this section we assume that $N\ge 3$ and h from (3.2) is a positive constant. The main results of this subsection are the following four theorems.

The first theorem provides the estimates for large solutions, existence of which was proved in Section 2.

THEOREM 3.12. *Let conditions (3.2), (3.3), (3.17) be fulfilled and $\beta_2(r), \beta_V(r) \to 0$ as $r \to \infty$. Then for every positive solution $v_l \in H^1_{\rm loc}(\mathbb{R}^N)$ to the equation $\mathcal{L}v = 0$ in $\mathbb{R}^N$ and for every $\varepsilon > 0$ there exists $c \geq 1$ independent of x such that*

$$\frac{1}{c}|x|^{-\varepsilon} \leq v_l(x) \leq c|x|^{\varepsilon}, \quad x \in \bar{B}_1^c.$$

If, in addition, $\beta_2 + \beta_V$ is a Dini function at infinity, then there exists $c \geq 1$ independent of x such that

$$\frac{1}{c} \leq v_l(x) \leq c, \quad x \in \mathbb{R}^N.$$

The next result gives the estimates of small solutions at infinity.

THEOREM 3.13. *Let conditions (3.2), (3.3), (3.17) be fulfilled and $\beta_1(r), \beta_V(r) \to 0$ as $r \to \infty$. Then for a small positive solution $v_s \in D^1_0(\mathbb{R}^N)$ to the equation $\mathcal{L}v = 0$ in $\bar{B}_1^c$, and for every $\varepsilon > 0$ there exists $c \geq 1$ independent of x, such that*

$$\frac{1}{c}|x|^{2-N-\varepsilon} \leq v_s(x) \leq c|x|^{2-N+\varepsilon}, \quad x \in \bar{B}_1^c.$$

If, in addition, $\beta_1 + \beta_V$ is a Dini function at infinity, then there exists $c \geq 1$ independent of x such that

$$\frac{1}{c}|x|^{2-N} \leq v_s(x) \leq c|x|^{2-N}, \quad x \in \bar{B}_1^c.$$

The following theorem provides a polynomial correction to the estimates when $\mathcal{L}$ is perturbed by a negative Hardy-type potential $\frac{\mu}{|x|^2}$. This result is vital for our proof of nonexistence for the semi-linear equations in the critical case $\sigma = \Lambda(q)$.

THEOREM 3.14. *Let conditions (3.2), (3.3), (3.17) be fulfilled and $\beta_1(r),\ \beta_2(r),\ \beta_V(r) \to 0$ as $r \to \infty$. Let $v_l \in H^1_{\rm loc}(\mathbb{R}^N)$ and $v_s \in D^1_0(\mathbb{R}^N)$ be a large and a small positive solutions to the equation $\mathcal{L}v = \frac{\mu}{|x|^2}v$ on $\bar{B}_1^c$, respectively, with $0 < \mu < \varepsilon_0 h\left(\frac{N-2}{2}\right)^2$ and ε_0 is from (3.17). Then there exist $\gamma > 0$ and $C,\ c > 0$ such that*

$$v_l(x) \leq C|x|^{-\gamma} \quad \text{and} \quad v_s(x) \geq c|x|^{2-N+\gamma}, \quad x \in \bar{B}_1^c.$$

One of the key results leading to the assertions of Theorems 3.13 and 3.14 is the next theorem establishing a relationship between small solutions of the homogeneous equation $\mathcal{L}v = 0$ and large solutions of the formally adjoint equation $\mathcal{L}^*v = 0$, where

$$\mathcal{L}^*u = -\nabla \cdot a^\top \cdot \nabla u - \nabla(b_1 u) - b_2 \nabla u + Vu.$$

The result is interesting and important in its own right.

THEOREM 3.15. *Let assumptions* (3.2), (3.3) *and* (3.17) *be fulfilled. Let* $v_l \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$ *be a positive solution to* $\mathcal{L}^* v = 0$ *in* $\mathbb{R}^N$. *Let* $v_s \in D^1_0(\mathbb{R}^N)$ *be defined by*

$$\begin{cases} \mathcal{L} v_s = 0 & \text{in } \bar{B}_1^c, \\ v_s = v_l, & \text{in } \bar{B}_1. \end{cases} \tag{3.19}$$

Then there exists $C \geq 1$ *such that*

$$C^{-1}|x|^{2-N} \leq v_l(x) v_s(x) \leq C|x|^{2-N}, \quad x \in \bar{B_1}^c.$$

Before passing to the proof of the above theorems we make a number of relevant comments.

(1) Note that the above theorems are proved without the assumption that the matrix is symmetric. For a symmetric matrix a and without first-order terms the result of Theorem 3.12 holds for a more general class of potentials as was shown in [29,30]. Recently, global Gaussian bounds on the fundamental solutions of the parabolic equation with potential were obtained in [92,52]. Integrating these over the time variable one obtains two-sided estimates on the fundamental solution to the elliptic equation:

$$\Gamma(x, y) = \int_0^\infty p(t, x, y) dt,$$

where Γ is the fundamental solution to the elliptic equation and p is the fundamental solution to the parabolic equation (with time-independent coefficients).

In order to formulate the result we need the notion of a Green-bounded potential. Let $\Gamma_a(x, y)$ be the positive minimal Green function to

$$-\nabla \cdot a \cdot \nabla v = 0 \quad \text{in } \mathbb{R}^N.$$

We say that a potential $V \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ is Green bounded and write $V \in GB$ if

$$\|V\|_{GB,a} := \sup_{x \in \mathbb{R}^N} \int_{\mathbb{R}^N} \Gamma_a(x, y)|V(y)| dy < \infty,$$

which is equivalent up to a constant factor to the condition $\sup_{x\in\mathbb{R}^N} \int_{\mathbb{R}^N} |x - y|^{2-N}|V(y)|dy < \infty$, but we will use below the numerical value of $\|V\|_{GB,a}$. One can see, e.g. by the Stein interpolation theorem (see, e.g. [82,84]), that if $V \in GB$ then V is form bounded in the sense of (2.6). If $V \in L^\infty_{\mathrm{loc}}$, $|V(x)| \leq \frac{\beta_V(x)}{|x|^2}$ with β_V Dini at infinity then $V \in GB$ with respect to $-\Delta$, which can be easily verified.

The following theorem is a direct consequence of the result in [92,52].

THEOREM 3.16. *Let* $V \in GB$ *and* $\|V^-\|_{GB,a} < 1$. *Let* $u \in D^1_0(\mathbb{R}^N)$, $u > 0$, *be the solution to the equation*

$$-\nabla \cdot a \cdot \nabla u + Vu = 0 \quad \text{in } \mathbb{R}^N \setminus \bar{B}_1. \tag{3.20}$$

Then there is a constant C such that

$$C^{-1}|x|^{2-N} \leq u(x) \leq C|x|^{2-N}. \tag{3.21}$$

And of course, combining the above theorem with Theorem 3.15 we obtain the corresponding result for large solutions in the whole space (see also [29,30,38,58,59,66, 68] for the case of symmetric matrices).

THEOREM 3.17. *Let $V \in GB$ and $\|V^-\|_{GB,a} < 1$. Then there exists a solution $v > 0$ to the equation* (3.22)

$$-\nabla \cdot a \cdot \nabla u + Vu = 0 \quad \text{in } \mathbb{R}^N, \tag{3.22}$$

such that $0 < c < v < c^{-1}$ in $\mathbb{R}^N$.

(2) Concerning estimates for large and small solutions to equations with the first-order terms the situation in the literature is quite different (as far as the authors can judge). For the Laplacian perturbed by a drift term $b \cdot \nabla$ the estimates for the fundamental solution to the elliptic equation are valid under the corresponding Green-bound-type conditions for the drift term [91,42]. Local in-time estimates for fundamental solutions to the parabolic equations $u_t = \nabla \cdot a \cdot \nabla u + b \cdot \nabla u$ with general uniformly elliptic a and b satisfying some integral conditions of Kato-type are obtained in [76]. However, the authors are not aware of any results on the estimates at infinity of the fundamental solutions to the elliptic equations for general uniformly elliptic non-smooth matrix a and non-smooth drifts b_1 and b_2. So Theorem 3.13 seems to be a new result in this direction. The authors believe though that its improvement in the spirit of Theorem 3.16 should be true.

(3) As we already saw in the previous subsection, the potential decaying exactly as $\frac{\mu}{|x|^2}$ affects the estimates of both large and small solutions. In case of the Laplacian this change in the behavior at infinity is expressed precisely via the constant μ. Theorem 3.14 provides the "correction" in the general case of uniformly elliptic matrices in presence of all lower-order terms and the negative inverse square potential.

(4) Theorems 3.12–3.15 in this form seem to be new and are being published here for the first time. These results are of interest in their own rights as they provide estimates at infinity for the fundamental solution of the full divergence-type second-order elliptic equation with close to optimal conditions on the coefficients and for large solutions. The assumption of local boundedness of the lower-order terms was made only in order to simplify the exposition. The results should remain true for equations with mildly singular lower-order terms of Kato-class type (see, e.g. [43] for the corresponding local results).

***Proof of Theorems* 3.12–3.15.** Below we use the notation $m_R(v_l) = \min_{|x|=R} v_l(x)$, $M_R(v_l) = \max_{|x|=R} v_l(x)$, and respectively for v_s.

The proof of Theorem 3.15 requires the following rough estimate:

LEMMA 3.18. *Let conditions* (3.2), (3.3), (3.17) *be fulfilled. Let $v_s \in D_0^1(\mathbb{R}^N)$ solve the equation $\mathcal{L}v = 0$ in $\bar{B}_1^c$. Then $v_s(x)|x|^{\frac{N-2}{2}} \to 0$ as $x \to \infty$.*

PROOF. Since $v_s \in D_0^1(\mathbb{R}^N)$, the Hardy inequality implies that

$$\|v_s\|^2_{D_0^1(\mathbb{R}^N)} \geq c \int_{\bar{B}_1^c} \frac{v_s^2}{|x|^2} dx = c \sum_{k=0}^{\infty} \int_{A_{2^k,2^{k+1}}} \frac{v_s^2}{|x|^2} dx$$
$$\geq c \sum_{k=0}^{\infty} \inf_{2^k \leq |x| \leq 2^{k+1}} v_s(x)(2^k)^{N-2}.$$

Hence $(2^k)^{N-2} \inf_{2^k \leq |x| \leq 2^{k+1}} v_s(x) \to 0$ as $k \to \infty$. The assertion follows by the Harnack inequality. □

Let us introduce two cut-off functions.

$$\xi = \begin{cases} 1 & \text{if } |x| < R, \\ 2 - \frac{|x|}{R} & \text{if } R < |x| < 2R, \\ 0 & \text{if } |x| > 2R. \end{cases} \tag{3.23}$$

$$\eta = \begin{cases} 0 & \text{if } |x| < \frac{R}{2}, \\ \frac{2}{R}|x| - 1 & \text{if } \frac{R}{2} < |x| < R, \\ 1 & \text{if } |x| > R. \end{cases} \tag{3.24}$$

Note that ξ is suitable to cut-off a large solution v_l and to use subsequently $\xi^2 v_l$ as a test function.

The next lemma is a fairly standard estimate of the Caccioppoli type.

LEMMA 3.19. *Let conditions of Theorem 3.15 hold and let ξ and η be defined by (3.23), (3.24). Then there is a constant $C > 0$ independent of R such that*

$$\|\nabla(\xi v_l)\|_2^2 \leq C\|v_l \nabla \xi\|_2^2,$$
$$\|\nabla(\eta v_s)\|_2^2 \leq C\|v_s \nabla \eta\|_2^2.$$

PROOF. The proof is fairly standard. First, note that

$$\langle \xi^2 v_l, \mathcal{L}^* v_l \rangle = 0.$$

From this integrating by parts, we obtain

$$\mathcal{E}(\xi v_l, \xi v_l) = \langle v_l \nabla \xi \cdot (a - a^\top) \cdot \nabla(\xi v_l) \rangle - \langle \xi v_l^2 (b_1 - b_2) \cdot \nabla \xi \rangle.$$

The rest is the Cauchy–Schwarz inequality and use (3.3). The second estimate is proved similarly. □

COROLLARY 3.20. *There exist $\gamma > 0$ and $c > 0$ such that $v_l(x) \geq c|x|^{\frac{2-N}{2}+\gamma}$ for $x \in \bar{B}_c^1$.*

PROOF. Set $m_r = m_r(v_l)$. Lemma 3.19 and the Hardy inequality imply

$$\int_{B_R} \frac{v_l^2}{|x|^2} dx \leq \frac{c}{R^2} \int_{A_{R,2R}} v_l^2 dx.$$

Hence by the Harnack inequality there is a constant $C_0 > 0$ independent of R such that

$$\int_0^R m_r^2 r^{N-3} dr \leq C_0 m_R^2 R^{N-2}.$$

This already implies the assertion with $\gamma = \frac{1}{C_0}$: just integrate the differential inequality $y(R) \leq C_0 R y'(R)$ with $y(R) = \int_1^R m_r r^{N-3} dr$. □

PROPOSITION 3.21. *Let the assumptions of Theorem 3.15 be fulfilled. Then there exists $C > 0$ such that, for $R > 1$,*

$$m_R(v_l) m_R(v_s) \leq C R^{2-N}.$$

PROOF. For $R > 0$, set $m_R = m_R\left(\frac{v_s}{v_l}\right) = \min_{|x|=R} \frac{v_s}{v_l}$. Note that $\frac{v_s}{v_l} \to 0$ as $x \to \infty$ by Lemma 3.18 and Corollary 3.20. Let

$$\xi = \begin{cases} v_l & \text{if } |x| < R, \\ \frac{v_s}{m_R} \wedge v_l, & \text{if } |x| > R. \end{cases}$$

Denote $\Omega_R := \{|x| > R : v_s < m_R v_l\}$. Note that Ω_R is open and its complement is compact since v_l and v_s are continuous and $\frac{v_s}{v_l} \to 0$ as $x \to \infty$. Then $v_s - \xi \in D_0^1(\bar{B}_1^c)$. Hence we have

$$\langle \mathcal{L} v_s, v_s - \xi \rangle = 0 \iff \mathcal{E}(v_s, v_s - \xi) = 0.$$

Therefore

$$\mathcal{E}(v_s, v_s) = \mathcal{E}(v_s, \xi) = m_R \mathcal{E}\left(\frac{v_s}{m_R} - \xi, \xi\right) + m_R \mathcal{E}(\xi, \xi).$$

Note that $\frac{v_s}{m_R} - \xi \in H_0^1(\bar{\Omega}_R^c)$ and $\xi = v_l$ on $\bar{\Omega}_R^c$. Hence

$$\mathcal{E}\left(\frac{v_s}{m_R} - \xi, \xi\right) = \mathcal{E}\left(\frac{v_s}{m_R} - \xi, v_l\right) = 0.$$

Therefore we obtain the equality

$$\mathcal{E}(v_s, v_s) = m_R \mathcal{E}(\xi, \xi). \tag{3.25}$$

On the other hand,

$$\mathcal{E}(\xi, \xi) \geq C \|\nabla \xi\|_2^2 = C m_R^2(v_l) \left\| \nabla \left(\frac{\xi}{m_R(v_l)} \right) \right\|_2^2.$$

Since $\frac{\xi}{m_R(v_l)} \geq 1$ on B_R, by the definition of capacity we have

$$\left\| \nabla \left(\frac{\xi}{m_R(v_l)} \right) \right\|_2^2 \geq \operatorname{cap}(B_R) = C R^{N-2}.$$

By (3.25) we then obtain

$$\mathcal{E}(v_s, v_s) \geq C m_R m_R^2(v_l) R^{2-N}.$$

Now the asserted upper bound follows from the Harnack inequality. □

The above proposition provides an upper bound on the product $m_R(v_s)m_R(v_l)$. Next we obtain a lower bound on this product.

PROPOSITION 3.22. *Let assumptions of Theorem 3.15 be fulfilled. Then there exists $C > 0$ such that, for $R > 1$,*

$$M_R(v_l)M_R(v_s) \geq CR^{2-N}.$$

PROOF. Let ξ and η be defined by (3.23), (3.24). Then $v_s - \xi v_l$ is a test function for the equation $\mathcal{L}v_s = 0$ and $(1-\eta)v_s$ is a test function for the equation $\mathcal{L}^* v_l = 0$. Hence we have

$$\mathcal{E}(v_s, v_s - \xi v_l) = 0.$$

Therefore we obtain

$$\mathcal{E}(v_s, v_s) = \mathcal{E}(\eta v_s, \xi v_l) + \mathcal{E}((1-\eta)v_s, \xi v_l).$$

Since $\xi = 1$ on the support of $1-\eta$, using the equation for v_l we conclude that

$$\mathcal{E}((1-\eta)v_s, \xi v_l) = \mathcal{E}((1-\eta)v_s, v_l) = 0.$$

By Lemma 3.19 and the Harnack inequality

$$\mathcal{E}(v_s, v_s) = \mathcal{E}(\eta v_s, \xi v_l) \leq C\|\nabla(\eta v_s)\|_2 \|\nabla(\xi v_l)\|_2 \leq CM_R(v_s)M_R(v_l)R^{N-2}.$$

Finally, by the definition of capacity

$$\mathcal{E}(v_s, v_s) \geq C\|\nabla v_s\|_2^2 \geq m_1^2(v_l)\mathrm{cap}(B_1). \quad \square$$

The next two propositions form a proof of Theorem 3.12.

PROPOSITION 3.23. *Let* (3.2), (3.3), (3.17) *be fulfilled, and $\beta_2(r)r^\nu$, $\beta_V(r)r^\nu$ be increasing for some $\nu \in (0, \frac{N-2}{2})$. Let $v_l \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$ be a positive solution to $\mathcal{L}v = 0$ in $\mathbb{R}^N$. Then there exists a constant $c > 0$ such that, for every $R > 2$ one has*

$$m_{R/2}(v_l) \geq m_R(v_l)\big(1 - c[\beta_2(R) + \beta_V(R)]\big) \quad \textit{and}$$
$$M_{R/2}(v_l) \leq M_R(v_l)\big(1 + c[\beta_2(R) + \beta_V(R)]\big).$$

PROOF. We prove the first assertion. The second one is proved similarly. For shortness we write m_R in place of $m_R(v_l)$. Let $w = m_R - v_l$. Then $m_{R/2} \geq m_R - \sup_{A_{\frac{R}{2},R}} w^+$. For w we have

$$\mathcal{L}w = m_R(V + \mathrm{div} b_2) \quad \text{in } B_R, \quad \text{and} \quad w^+ \in H^1_0(B_R). \tag{3.26}$$

Then by [28, Theorem 8.25] we obtain

$$\sup_{A_{\frac{R}{2},R}} w^+ \le C\left\{\left[R^{-N}\int_{A_{\frac{R}{4},R}}(w^+)^2dx\right]^{1/2} + m_R \sup_{A_{\frac{R}{4},R}}\left(R^2|V(x)| + R|b_2(x)|\right)\right\}. \tag{3.27}$$

Testing (3.26) by w^+ and using Cauchy–Schwarz and the Hardy inequality we arrive at

$$\begin{aligned}\mathcal{E}(w, w^+) &= m_R\left(\int_{B_R} Vw^+dx - \int_{B_R} b_2\nabla w^+dx\right)\\ &\le Cm_R\|\nabla w^+\|_2\left(\int_{B_R}(V^2|x|^2 + |b_2|^2)dx\right)^{1/2}.\end{aligned}$$

Therefore by (2.4)

$$\|\nabla w^+\|_2^2 \le Cm_R^2\left(\int_{B_R}(V^2|x|^2 + |b_2|^2)dx\right).$$

By the Hardy inequality we have

$$\|\nabla w^+\|_2^2 \ge C\int_{B_R}\left(\frac{w^+}{|x|}\right)^2dx \ge CR^{-2}\int_{A_{\frac{R}{4},R}}(w^+)^2dx.$$

Combining this with (3.27) we have

$$\begin{aligned}\sup_{A_{\frac{R}{2},R}} w^+ \le Cm_R&\left\{\left[R^{2-N}\int_{B_R}(V^2|x|^2 + |b_2|^2)dx\right]^{1/2}\right.\\ &\left.+ \sup_{A_{\frac{R}{4},R}}\left(R^2|V(x)| + R|b_2(x)|\right)\right\}.\end{aligned}$$

Using (3.3) we rewrite

$$\begin{aligned}\sup_{A_{\frac{R}{2},R}} w^+ \le Cm_R&\left\{\left[R^{2-N}\int_0^R \frac{\beta_V^2(r) + \beta_2^2(r)}{r}r^{N-2}dr\right]^{1/2}\right.\\ &\left.+ \sup_{A_{\frac{R}{4},R}}\left(R^2|V(x)| + R|b_2(x)|\right)\right\}.\end{aligned}$$

From the monotonicity of the functions $\beta_V r^\nu$ and $\beta_2 r^\nu$ we have two consequences

$$R^{2-N}\int_0^R \frac{\beta_V^2(r)+\beta_2^2(r)}{r} r^{N-2}dr \le \frac{1}{N-2-2\nu}(\beta_V^2(R)+\beta_2^2(R))$$
$$\text{and} \quad \beta_V(R/2) \le 2^\nu \beta_V(R), \quad \beta_2(R/2) \le 2^\nu \beta_2(R).$$

This implies that there is a constant C such that

$$\sup_{A_{\frac{R}{2},R}} w^+ \le C m_R[\beta_V(R)+\beta_2(R)]. \quad \square$$

COROLLARY 3.24. *In the assumptions of Proposition 3.23 for a positive large solution v_l to the equation $\mathcal{L}u = 0$ there are positive constants C_1, C_2, C_3, C_4 such that*

$$m_R(v_l) \le C_1 e^{C_2\int_0^R[\beta_V(r)+\beta_2(r)]\frac{dr}{r}},$$

$$M_R(v_l) \ge C_3 e^{-C_4\int_0^R[\beta_V(r)+\beta_2(r)]\frac{dr}{r}}.$$

PROOF. Iterating the inequality from Proposition 3.23 from $|x| = R$ to $|x| = 1$ we obtain

$$\frac{m_1(v_l)}{m_R(v_l)} \ge \prod_{k=1}^{[\log_2 R]+1}\Big[1 - C(\beta_V+\beta_2)(R2^{(1-k)})\Big] \ge e^{-C\sum_{k=1}^{[\log_2 R]+1}(\beta_V+\beta_2)(R2^{(1-k)})}$$
$$\ge e^{-C\int_1^R(\beta_V+\beta_2)(r)\frac{dr}{r}}.$$

The second estimate is proved in the same way. $\square$

REMARK 3.25. The assumption that the functions $\beta_V r^\nu$ and $\beta_2 r^\nu$ are increasing is purely technical. One can easily see that the function

$$\tilde\beta(r) := \nu r^{-\nu}\int_0^r \beta(s)s^\nu \frac{ds}{s}$$

satisfies the properties

(i) $\tilde\beta(r) \ge \beta(r)$, $\tilde\beta$ is decreasing and $\tilde\beta(r)$ tends to zero as $r \to \infty$ and so does $\beta(r)$;
(ii) $\tilde\beta(r)r^\nu$ is increasing;
(iii) $\int_1^R \tilde\beta(r)\frac{dr}{r} \le \int_0^R \beta(r)(r^\nu \wedge 1)\frac{dr}{r}$.

So if a function β is a Dini function at infinity, that is

$$\int^\infty \beta(r)\frac{dr}{r} < \infty, \tag{3.28}$$

then $\tilde\beta$ is also Dini at infinity.

PROOF OF THEOREM 3.12. The assertion follows directly from Corollary 3.24. $\square$

PROOF OF THEOREM 3.13. The assertion follows by combining the result of Theorem 3.12 to the adjoint equation $\mathcal{L}^*v = 0$ with Theorem 3.15. $\square$

The next proposition forms the main part of the proof of Theorem 3.14.

PROPOSITION 3.26. *Let conditions* (3.2), (3.3) *and* (3.17) *be fulfilled. Let* $v_l^* \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$ *be a positive solution to* $\mathcal{L}^* u = 0$ *in* $\mathbb{R}^N$ *and let* $v \in D^1_0(\mathbb{R}^N)$ *be the solution to the problem*

$$\begin{cases} \mathcal{L}v = \frac{\mu}{|x|^2} v & \text{on } \bar{B}_1^c, \\ v = v_l^* & \text{on } \bar{B}_1, \end{cases} \tag{3.29}$$

with $0 < \mu < \varepsilon_0 h\left(\frac{N-2}{2}\right)^2$, *and* ε_0 *is from* (3.17). *Then there exist* $\gamma > 0$ *and* $C > 0$ *such that*

$$m_R(v) m_R(v_l^*) \geq C R^{2-N+\gamma}, \qquad R > 1.$$

PROOF. For $R > 1$, let ξ, η be as in (3.23), (3.24). Due to this choice equation (3.29) can be tested by $v - \xi v_l^*$. Then we have

$$\mathcal{E}(v, v - \xi v_l^*) = \mu \int_{|x|>1} v(v - \xi v_l^*)|x|^{-2} dx.$$

Therefore

$$\mathcal{E}(v, v) - \mu \int_{|x|>1} v^2 |x|^{-2} dx = \mathcal{E}(v, \xi v_l^*) - \mu \int_{|x|>1} \xi v v_l^* |x|^{-2} dx.$$

In the same way as in the proof of Proposition 3.22 we obtain

$$\mathcal{E}(v, \xi v_l^*) = \mathcal{E}(\eta v, \xi v_l^*) \leq C R^{N-2} M_R(v) M_R(v_l^*).$$

On the other hand,

$$\mathcal{E}(v, v) - \mu \int_{|x|>1} v^2 |x|^{-2} dx > 0.$$

Hence we conclude that

$$\int_{|x|>1} v v_l^* \xi |x|^{-2} dx \leq C R^{N-2} M_R(v) M_R(v_l^*).$$

From this by the Harnack inequality it follows that

$$\int_1^R m_r(v) m_r(v_l^*) r^{N-3} dr \leq C_0 m_R(v) m_R(v_l^*) R^{N-2}.$$

This implies the assertion with $\gamma = \frac{1}{C_0}$, see the end of the proof of Corollary 3.20. □

PROOF OF THEOREM 3.14. The assertion follows from the Proposition 3.26 combined with (3.22) and Theorem 3.15. □

REMARK 3.27. Theorem 3.14 improves the estimate given in [38, Prop. 3.7]. In the proof we combined ideas from the proof of Proposition 3.22 and [50].

3.2.2. *Semi-linear equations*
In this section we prove Theorem 3.7. Again we give separate proofs in super- and sublinear cases.

Superlinear case—Nonexistence. We start with a nonexistence lemma which is a direct consequence of Theorem 2.13.

LEMMA 3.28 (Nonexistence Lemma). *Let conditions* (3.2), (3.3) *and* (3.17) *be satisfied. Let* $0 \le W \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N \setminus \bar{B}_1)$ *be such that* $W(x)|x|^2 \to \infty$ *as* $|x| \to \infty$. *Then the equation* $\mathcal{L}u - Wu = 0$ *has no nontrivial nonnegative supersolutions in* $\mathbb{R}^N \setminus \bar{B}_R$ *for any* $R \ge 1$.

Now we are ready to prove Theorem 3.7 for the case $q \ge 1$. The strategy is the same as in Section 1.2.1.

First, we prove nonexistence for (q, σ) below the critical line. Let $\sigma < 2+(2-N)(q-1)$. Choose $\varepsilon > 0$ and $\delta > 0$ such that $2 - \sigma + (2 - N)(q - 1) = \varepsilon(q - 1) + \delta$. Let u be a positive supersolution to (3.1) and v_s be a small solution to $\mathcal{L}v = 0$. Then $u \ge cv_s$ since u is a supersolution to the equation $\mathcal{L}v = 0$. Hence $u(x) \ge c|x|^{2-N-\varepsilon}$ for $x \in \bar{B}_1^c$, by Theorem 3.13. So the potential $W(x) := c_0|x|^{-\sigma} u^{q-1}(x)$ satisfies the estimate $W(x) \ge c|x|^{-2+\delta}$ for $x \in \bar{B}_1^c$ and u is a supersolution to the equation $\mathcal{L}u - Wu = 0$. This contradicts Lemma 3.28.

Now let $\sigma = 2 + (2 - N)(q - 1)$ and $\beta_1 + \beta_V$ be a Dini function at infinity. Let u be a positive supersolution to (3.1) and v_s be a small solution to $\mathcal{L}v = 0$. Then $v_s \ge c|x|^{2-N}$, by Theorem 3.13. So $u \ge cv_s \ge c|x|^{2-N}$ and $W(x) := c_0|x|^{-\sigma} u^{q-1}(x) \ge c|x|^{-2}$ for $x \in \bar{B}_1^c$, by the preceding argument. Hence u is a supersolution to the equation $\mathcal{L}v = \mu|x|^{-2}v$ with $\mu > 0$ small enough. Theorem 3.14 implies that a small solution $\bar{v}_s$ to the latter equation enjoys the estimate $\bar{v}_s(x) \ge c|x|^{2-N+\gamma}$ for $x \in \bar{B}_1^c$, with some $\gamma > 0$. Hence $u \ge c|x|^{2-N+\gamma}$ and $W(x) := c_0|x|^{-\sigma} u^{q-1}(x) \ge c|x|^{-2+(q-1)\gamma}$ for $x \in \bar{B}_1^c$. Since u is a supersolution to the equation $\mathcal{L}u - Wu = 0$, this contradicts Lemma 3.28.

Superlinear case—Existence. Let $\sigma > 2 + (2 - N)(q - 1)$. Fix $\varepsilon, \delta > 0$ such that $\sigma = 2 + (2 - N)(q - 1) + (q - 1)\varepsilon + \delta$. Choose μ such that $0 < \mu < \varepsilon_0 h \left(\frac{N-2}{2}\right)^2$ with h as in (3.2) and ε_0 as in (3.17). Let u be a small solution to the equation $\mathcal{L}u = \mu|x|^{-2-\delta}u$ in $\bar{B}_1^c$. Then $u \le c|x|^{2-N+\varepsilon}$ for $x \in B_1^c$, by Theorem 3.13. For a given $c_0 > 0$, choose $\tau > 0$ such that $(\tau c)^{q-1}c_0 \le \mu$. Then τu is a supersolution to the equation (3.1) since

$$c_0|x|^{-\sigma}(\tau u)^{q-1} \le (\tau c)^{q-1} c_0 |x|^{-\sigma+(2-N)(q-1)+\varepsilon(q-1)} \le \mu|x|^{-2-\delta}, \quad |x| > 1.$$

On the other hand, there exists a small solution v_s to the equation $\mathcal{L}v = 0$ in $\bar{B}_1^c$ such that $v_s \le \tau u$ in $\bar{B}_1^c$. So v_s is a subsolution to the equation (3.1). Hence, by Theorem 2.9, there exists a solution v to the latter equation.

Sublinear case—Nonexistence. We start with the following lemma which is a complete analogue to Lemma 1.4.

LEMMA 3.29. *Let $u > 0$ be a supersolution to (3.1) in G. Then there exists a constant $C > 0$ such that*

$$u(x) \geq C|x|^{\frac{2-\sigma}{1-q}}, \quad |x| > 2.$$

First, suppose that $\sigma < 2$. Set $\varepsilon < \frac{2-\sigma}{1-q}$. Let u be a positive supersolution to (3.1). Then $u(x) \geq c|x|^{\frac{2-\sigma}{1-q}}$ for $x \in \bar{B}_1^c$, by Lemma 3.29.

By Theorem 3.12, an entire solution v_l to the equation $\mathcal{L}v = 0$ satisfies the estimate $v_l(x) \leq c|x|^\varepsilon$ for $x \in \bar{B}_1^c$. Since $\frac{v_l(x)}{u(x)} \to 0$ as $x \to \infty$, we reach a contradiction to Theorem 2.12.

Next, let $\sigma = 2$ and $\beta_2 + \beta_V$ be a Dini function at infinity and let u be a positive supersolution to (3.1). Then $u \geq c$, by Lemma 3.29, and $\frac{1}{c} \leq v_l \leq c$, by Theorem 3.12. Since v_l is a subsolution to (3.1), there exists a solution to (3.1), by Theorem 2.9. This solution will be also denoted by u. Since $u \geq c$, the potential $c_0|x|^{-2}u^{q-1}$ satisfies (3.3). So u enjoys the Harnack inequality and hence is bounded between two positive constants. Therefore u is a supersolution to the equation $\mathcal{L}v = \mu|x|^{-2}v$ in $\bar{B}_1^c$ for $\mu > 0$ small enough. However, the large solution $\bar{v}_l$ to the latter equation satisfies the upper bound $\bar{v}_l \leq c|x|^{-\gamma}$ in $\bar{B}_1^c$ with some $\gamma > 0$, by Theorem 3.14. So $\bar{v}_l/u \to 0$ as $x \to \infty$ contradicting Theorem 2.12.

Sublinear case—Existence. Let $\sigma > 2$. Fix $\varepsilon, \delta > 0$ such that $\sigma = 2 + (1-q)\varepsilon + \delta$. Choose μ such that $0 < \mu < \varepsilon h\left(\frac{N-2}{2}\right)^2$ with h as in (3.2) and ε_0 as in (3.17). Let u be a large solution to the equation $\mathcal{L}u = \mu|x|^{-2-\delta}u$ in $\bar{B}_1^c$. Then $u \geq c|x|^{-\varepsilon}$ for $x \in B_1^c$, by Theorem 3.12. For a given $c_0 > 0$, choose $\tau > 0$ such that $(\tau c)^{q-1}c_0 \leq \mu$. Then τu is a supersolution to the equation $\mathcal{L}v = c_0|x|^{-\sigma}v^q$ since

$$c_0|x|^{-\sigma}(\tau u)^{q-1} \leq (\tau c)^{q-1}c_0|x|^{-\sigma+\varepsilon(1-q)} \leq \mu|x|^{-2-\delta}, \quad |x| > 1.$$

On the other hand, there exists a small solution v_s to the equation $\mathcal{L}v = 0$ in $\bar{B}_1^c$ such that $v_s \leq \tau u$ in $\bar{B}_1^c$. So v_s is a subsolution to the equation $\mathcal{L}v = c_0|x|^{-\sigma}v^q$. Hence, by Theorem 2.9, there exists a solution v to the latter equation.

3.3. *Nonuniformly elliptic case*

In this subsection we are concerned with positive supersolutions to the equation

$$-\nabla \cdot a \cdot \nabla u + Vu = c_0|x|^{-\sigma}u^q \quad \text{in } \mathbb{R}^N \setminus \bar{B}_1, \tag{3.30}$$

with $h(x) = |x|^a$ in (3.2).

As in the previous sections, we introduce the critical line $\Lambda_\alpha(q)$ on the (q, σ)-plane

$$\Lambda_\alpha(q) := \min\{(2-N-\alpha)(q-1) + 2 - \alpha, 2 - \alpha\} \quad (q \in \mathbb{R}),$$

and the nonexistence set

$$\mathcal{N}_\alpha = \{(q,\sigma) \in \mathbb{R}^2 \setminus \{1,2\} : \textit{equation } (3.30) \textit{ has no positive supersolutions}\}.$$

The first main result of this subsection reads as follows.

THEOREM 3.30. *Let* $\alpha > 2 - N$ *and* $|V(x)| \le \frac{\beta_V(|x|)}{|x|^{2-\alpha}}$ *with* $\beta_V(r) \downarrow 0$ *as* $r \to \infty$. *Then*

$$\{(q,\sigma) : \sigma < \Lambda_\alpha(q)\} \subset \mathcal{N}_\alpha \subset \{(q,\sigma) : \sigma \le \Lambda_\alpha(q)\}.$$

If in addition $\beta_V(r)$ *is a Dini function at infinity, then*

$$\mathcal{N}_\alpha = \{(q,\sigma) : \sigma \le \Lambda_\alpha(q)\}.$$

The next theorem concerns the case when a large solution to the corresponding linear equation grows polynomially and constants become small solutions.

THEOREM 3.31. *Let a be symmetric. Let* $\alpha < 2 - N$ *and* $|V(x)| \le \frac{\beta_V(|x|)}{|x|^{2-\alpha}}$ *with* $\beta_V(r) \downarrow 0$ *as* $r \to \infty$. *Then*

$$\{(q,\sigma) : \sigma < \Lambda_\alpha(q)\} \subset \mathcal{N}_\alpha \subset \{(q,\sigma) : \sigma \le \Lambda_\alpha(q)\}.$$

If in addition $\beta_V(r)$ *is a Dini function at infinity, then*

$$\mathcal{N}_\alpha = \{(q,\sigma) : \sigma \le \Lambda_\alpha(q)\}.$$

Note that in the special case $\alpha = 2 - N$ the critical line becomes $\sigma = N$. The last theorem of this subsection deals with this special case.

THEOREM 3.32. *Let* $\alpha = 2 - N$. *Then*

$$\mathcal{N}_\alpha \supseteq \{(q,\sigma) : \sigma < N\} \cup \{(q,\sigma) : \sigma = N, q \ge -1\}.$$

REMARK 3.33. For the case of the Laplacian ($a = Id$) we proved in Section 3.1 that $\mathcal{N}_\alpha = \{(q,\sigma) : \sigma < N\} \cup \{(q,\sigma) : \sigma = N, q \ge -1\}$. We expect this to be true for the general case as well.

3.3.1. *Linear equations*

In this auxiliary subsection we study only the case when h in (3.2) satisfies $h(x) = |x|^{2-N}$, so a is uniformly elliptic only if $N = 2$, and without lower-order terms in $\mathcal{L}$.

PROPOSITION 3.34. *For* $R > 1$, *let* v_R *be the solution of the boundary value problem*

$$\begin{cases} \nabla \cdot a \cdot \nabla v = 0 & \text{in } A_{1,R}, \\ v = 1 \quad \text{in } B_1, & v = 0 \quad \text{in } \bar{B}_R^c. \end{cases}$$

Then $v_R \uparrow 1$ *as* $R \to \infty$.

PROOF. Let $\phi_R \in D^1_{0,h}(B_R)$ be defined as follows:

$$\phi_R(x) = \begin{cases} 1, & |x| < R^{\frac{1}{e}}; \\ 1 + \ln\ln R^{\frac{1}{e}} - \ln\ln|x|, & R^{\frac{1}{e}} < r < R. \end{cases}$$

Note that

$$\|h^{\frac{1}{2}}\nabla\phi\|_2^2 = c\int_{R^{\frac{1}{e}}}^{R} \frac{dr}{r\ln^2 r} \to 0 \quad \text{as } R \to \infty.$$

Since $v_R - \phi_R \in D^1_{0,h}(A_{1,R})$, we conclude that

$$\langle \nabla v_R \cdot a \cdot \nabla(v_R - \phi_R)\rangle = 0 \Rightarrow \|v_R\|_{D^1_{0,h}(\mathbb{R}^N)} \le c\|\phi_R\|_{D^1_{0,h}(B_R)} \to 0 \quad \text{as } R \to \infty.$$

Finally, observe that $v_R \le v_{R'}$ provided $R \le R'$, by the maximum principle. Hence the assertion follows. □

We start with the two-sided estimate for large solutions v_l. For shortness we drop the subindex l in the next proposition.

PROPOSITION 3.35. *Let $R^* > 2$. Let $v \in H^1(A_{1,R^*})$, $v > 0$ be the solution to the problem*

$$\begin{cases} -\nabla \cdot a \cdot \nabla v = 0 & \text{in } A_{1,R^*}, \\ v|_{|x|=1} = 0, \quad v|_{|x|=R^*} = \kappa, \end{cases}$$

where $\kappa > 1$ is such that $m_2(v) =: \inf_{|x|=2} v(x) = 1$. Then there exists a constant $C > 0$ such that for $x \in A_{1,R}$ one has

$$C^{-1}\log|x| \le v(x) \le C\log|x|.$$

First, we prove the identity

$$\int_{\{v<m\}} \nabla v \cdot a \cdot \nabla v dx = m\int_{\{v<1\}} \nabla v \cdot a \cdot \nabla v dx, \quad 0 < m < \kappa. \tag{3.31}$$

For this set $\xi = \left(m \vee \frac{1}{m}\right) v \wedge 1 \wedge m$. Then $(\xi - v)^+ \in H^1_0(\{v < m \vee 1\})$ since $\xi = m \vee 1$ on $\{v \ge m \vee 1\}$. Thus using $(\xi - v)^+$ as a test function, we obtain

$$\int_{\{v<m\vee 1\}} \nabla v \cdot a \cdot \nabla v dx = \int_{\{v<m\vee 1\}} \nabla v \cdot a \cdot \nabla \xi dx.$$

Note that $\nabla\xi = \left(m \vee \frac{1}{m}\right) \mathbb{1}_{\{v<m\wedge 1\}}\nabla v$. Hence we have

$$\int_{\{v<m\vee 1\}} \nabla v \cdot a \cdot \nabla v dx = \left(m \vee \frac{1}{m}\right)\int_{\{v<m\wedge 1\}} \nabla v \cdot a \cdot \nabla v dx,$$

which is (3.31).

Now let $R \in (1, R^*)$ and $m = m_R(v)$. Then by the maximum principle $v \geq m_R$ on A_{R,R^*}. Using the definition and monotonicity of the relative h-capacity we obtain

$$\int_{\{v<m_R\}} \nabla v \cdot a \cdot \nabla v dx \geq c m_R^2 \mathrm{cap}_h \left(\{v \geq m_R\}, \bar{B_1}^c\right)$$
$$\geq c m_R^2 \mathrm{cap}_h \left(\bar{B_R}^c, \bar{B_1}^c\right) = c m_R^2 \mathrm{cap}_h \left(B_1, B_R\right), \tag{3.32}$$

where the weighted relative capacity is defined by

$$\mathrm{cap}_h(B_1, B_2)$$
$$= \inf\left\{\int h(x)|\nabla\varphi|^2 dx \; : \; \varphi \in C_0^\infty(E_2), \varphi = 1 \text{ on } E_1\right\}, \quad E_1 \Subset E_2.$$

(We refer the reader to [31] for estimates of weighted capacities.) In particular,

$$\int_{\{v<1\}} \nabla v \cdot a \cdot \nabla v dx \geq c\, \mathrm{cap}_h(B_1, B_2).$$

Then from (3.31) and (3.32) it follows that

$$m_R \leq \frac{\int_{\{v<1\}} \nabla v \cdot a \cdot \nabla v dx}{c\, \mathrm{cap}_h \left(B_1, B_R\right)}. \tag{3.33}$$

Next, set $m = M_R(v)$. Let $\eta \in C^1(B_{R^*})$ be such that $\eta = 1$ on $\{v > M_R\}$ and $\eta = 0$ on B_1. Testing the equation by $(M_R\eta - v)^+$ we obtain

$$\int_{\{v<M_R\}} \nabla v \cdot a \cdot \nabla v dx = M_R \int_{\{v<M_R\}} \nabla v \cdot a \cdot \nabla \eta dx.$$

By Cauchy–Schwarz

$$\int_{\{v<M_R\}} \nabla v \cdot a \cdot \nabla v dx \leq c M_R^2 \int_{\{v<M_R\}} \nabla \eta \cdot a \cdot \nabla \eta dx.$$

Optimizing the RHS over η we arrive at

$$\int_{\{v<M_R\}} \nabla v \cdot a \cdot \nabla v dx \leq c M_R^2 \mathrm{cap}_h \left(\{v \geq M_R\}, \bar{B_1}^c\right)$$
$$= c M_R^2 \mathrm{cap}_h \left(B_1, \{v < M_R\}\right).$$

By the maximum principle $B_R \subset \{v < M_R\}$. Therefore

$$\int_{\{v<M_R\}} \nabla v \cdot a \cdot \nabla v dx \leq c M_R^2 \mathrm{cap}_h(B_1, B_R).$$

In particular,

$$\int_{\{v<1-m_2\}} \nabla v \cdot a \cdot \nabla v dx \leq \int_{\{v<M_2\}} \nabla v \cdot a \cdot \nabla v dx \leq c M_2 \mathrm{cap}_h(B_1, B_2).$$

Thus we have

$$M_R \geq c \frac{\int_{\{v<1\}} \nabla v \cdot a \cdot \nabla v dx}{\operatorname{cap}_h(B_1, B_R)} \geq c \frac{\operatorname{cap}_h(B_1, B_2)}{\operatorname{cap}_h(B_1, B_R)} = c \log R,$$

and (3.32) yields

$$m_R \leq C M_2 \frac{\operatorname{cap}_h(B_1, B_2)}{\operatorname{cap}_h(B_1, B_R)} = C \log R. \quad \square$$

We need a further improvement of the upper bound of the large solution for the case of the critical negative potential. In comparison with the case of a uniformly elliptic matrix in dimension three or more, we are only able to show a qualitative fact.

PROPOSITION 3.36. *Let* $v_0, v \in H^1_{\mathrm{loc}}(\mathbb{R}^n \setminus B_1)$, $v_0, v > 0$ *satisfy* $-\nabla \cdot a \cdot \nabla v_0 = 0$ *and* $-\nabla \cdot a \cdot \nabla v_0 = \frac{\mu}{|x|^N \log^2 |x|} v$ *such that* $v_0|_{|x|=1} = v|_{|x|=1} = 0$. *(*$\mu > 0$ *is sufficiently small so that such* v *exists.) Then*

$$\lim_{x \to \infty} \frac{v(x)}{v_0(x)} = 0.$$

PROOF. Set $m_R := \min_{|x|=R} \frac{v(x)}{v_0(x)}$. By the maximum principle $\frac{v}{m_R} \geq v_0$ on $A_{1,R}$. Hence m_r is decreasing.

Without loss we assume that $\sup v > 1$, otherwise the assertion follows immediately from Proposition 3.35. Let $m > 1$, $\xi = m(v \wedge 1)$. Then $Supp(\xi - v)^+ = \{v < m\}$ and $\nabla \xi = m \mathbb{1}_{\{v<1\}} \nabla v$. Testing the equation by $(\xi - v)^+$ we have

$$\int_{\{v<m\}} \nabla v \cdot a \cdot \nabla(\xi - v) dx = \mu \int_{\{v<m\}} v(\xi - v)|x|^{-N} \log^{-2} |x| dx.$$

Then by the Hardy inequality for the degenerate case, we have

$$\mu \int_{\{v<m\}} v\xi |x|^{-N} \log^{-2} |x| dx \leq m\mu \int_{\{v<1\}} \nabla v \cdot a \cdot \nabla v dx,$$

so that

$$\mu \int_{\{1<v<m\}} v|x|^{-N} \log^{-2} |x| dx \leq \int_{\{v<1\}} \nabla v \cdot a \cdot \nabla v dx.$$

As $\frac{v}{\log |x|}\big|_{|x|=r} \geq c m_r (\frac{v}{v_0})$, we conclude that

$$\int_2^\infty m_r r^{-1} dr < \infty,$$

which together with monotonicity of m_r implies the assertion. $\square$

3.3.2. *Semi-linear equations*

PROOF OF THEOREM 3.30. The change of the independent variables

$$x = y|y|^{\frac{N-2}{N+\alpha-2}-1} \tag{3.34}$$

transforms (3.30) into

$$-\nabla\cdot\tilde{a}\cdot\nabla u + \tilde{V}u = c_0|y|^{-\tilde{\sigma}}u^q$$

with the uniformly elliptic matrix $\{\tilde{a}_{ij}\}_{i,j=1}^N$ given by

$$\tilde{a}_{ij}(y) = \sum_{k,l=1}^N \frac{a_{ij}(x(y))}{h(x(y))}\left(\delta_{ik} + \frac{\alpha}{N-2}\frac{y_i y_k}{|y|^2}\right)\left(\delta_{jl} + \frac{\alpha}{N-2}\frac{y_j y_l}{|y|^2}\right), \tag{3.35}$$

the potential $\tilde{V}(y) = V(x(y))|y|^{-\frac{\alpha N}{N+\alpha-2}}$ and $\tilde{\sigma} = \sigma + \frac{\alpha(N-\sigma)}{N+\alpha-2}$. Now the assertion follows from Theorem 3.7. □

PROOF OF THEOREM 3.31. Note that the change of the independent variables (3.34) maps the exterior of the unit ball into punctured interior. So we combine (3.34) with the Kelvin transform (hence the restriction of symmetry on a)

$$y = \frac{z}{|z|^2} \quad \text{and} \quad \hat{u}(z) = \frac{u(x(y))}{\omega(y)},$$

where ω is the fundamental solution to $-\nabla\cdot\tilde{a}\cdot\nabla v = 0$ with the pole at zero ($\tilde{a}$ is defined by (3.35)). Then equation (3.30) transforms into

$$-\nabla\cdot\hat{a}\cdot\nabla u + \hat{V}u = c_0|z|^{-\hat{\sigma}}u^q$$

with the uniformly elliptic matrix $\{\hat{a}_{ij}\}_{i,j=1}^N$ given by

$$\hat{a}_{ij}(z) = \sum_{k,l=1}^N \frac{a_{ij}(x(z))}{h(x(z))}\left(\delta_{ik} - \frac{2(N-2)+\alpha}{N-2}\frac{z_i z_k}{|z|^2}\right)\left(\delta_{jl} - \frac{2(N-2)+\alpha}{N-2}\frac{z_j z_l}{|x|^2}\right),$$

the potential $\hat{V}(z) = V(x(z))|z|^{\frac{\alpha N}{N+\alpha-2}-4}$ and $\hat{\sigma} = -\sigma - \frac{\alpha(N-\sigma)}{N+\alpha-2} - (q-1)(N-2)+4$. Now the assertion follows from Theorem 3.7. □

For the proof of Theorem 3.32 we need the two following lemmas. The nonexistence lemma playing the role of Lemma 3.28 in the case $\alpha = 2-N$ looks as follows.

LEMMA 3.37 (Nonexistence lemma). *Let* (3.2) *be satisfied with* $h(x) = |x|^{2-N}$ *and let* $0 \le W \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N\setminus\bar{B}_1)$ *be such that* $W(x)|x|^N\ln^2|x| \to \infty$ *as* $|x|\to\infty$. *Then the equation* $-\nabla\cdot a\cdot\nabla v - Wv = 0$ *has no nontrivial nonnegative supersolutions in* $\mathbb{R}^N\setminus\bar{B}_R$ *for any* $R>1$.

PROOF. Direct application of Theorem 2.13 with appropriate choice of φ. □

The next Keller–Ossermann-type estimate follows by the same argument as in Lemma 1.4.

LEMMA 3.38. *Let condition* (3.2) *hold with* $h(x) = |x|^{2-N}$ *and* $q < 1$. *Let* $u > 0$ *be a supersolution to the equation* $-\nabla\cdot a\cdot\nabla v = c_0|x|^{-\sigma}v^q$ *in* $\bar{B}_1^c$. *Then there exists* $c > 0$ *such that* $u \geq c|x|^{\frac{N-\sigma}{1-q}}$.

PROOF OF THEOREM 3.32. Let $u > 0$ satisfy $-\nabla\cdot a\cdot\nabla u \geq c_0|x|^{-\sigma}u^q$ in $\bar{B}_1^c$. Then u is a supersolution to the equation $-\nabla\cdot a\cdot\nabla v = 0$ in $\bar{B}_1^c$ and hence $u \geq c$ on $\bar{B}_1^c$ since constants are small solutions to the equation. Hence, for $q \geq 1$ and $\sigma \leq N$, $W(x) = c_0|x|^{-\sigma}u^{q-1} \geq c|x|^{-N}$ in $\bar{B}_1^c$ and so the existence of a positive supersolution u to the equation $-\nabla\cdot a\cdot\nabla v - Wv = 0$ in $\bar{B}_1^c$ is ruled out by Lemma 3.37.

Let now $q < 1$ and $\sigma < N$. Then u is a supersolution to the equation $-\nabla\cdot a\cdot\nabla v = 0$ in $\bar{B}_1^c$ enjoying the estimate of Lemma 3.38. However, the large solution v_l to the equation satisfies $v_l(x) \asymp \ln|x|$ so $v_l/u \to 0$ as $x \to \infty$, contradicting Theorem 2.12.

To consider the limit case $\sigma = N$, we apply Theorem 2.9 to conclude that we may consider u as a solution to the equation $-\nabla\cdot a\cdot\nabla v = c_0|x|^{-N}v^q$ in $\bar{B}_1^c$, without loss of generality. Since $u \geq c$, the potential $W(x) = c_0|x|^{-N}u^{q-1} \leq c|x|^{-N}$ in $\bar{B}_1^c$. Hence u enjoys the Harnack inequality in $A_{R,2R}$, $R > 1$, with the constant independent of R. Since $\limsup_{x\to\infty} v_l/u > 0$, we conclude that $\min_{|x|=R} u \leq c\ln R$ so $u(x) \leq c\ln|x|$ in $\bar{B}_1^c$. Hence $W(x) = c_0|x|^{-N}u^{q-1} \geq c|x|^{-N}\ln^{q-1}|x|$ in $\bar{B}_1^c$. Since u is a solution to the equation $-\nabla\cdot a\cdot\nabla u = Wu$, we arrive at a contradiction to Lemma 3.37 for $q > -1$.

For the case $\sigma = N$ and $q = -1$, the function u is a supersolution to the equation $-\nabla\cdot a\cdot\nabla v = \mu|x|^{-N}(\ln|x|)^{-2}v$ with $\mu > 0$ small enough. By Proposition 3.36, a large solution $\bar{v}_l$ to the latter equation has the following behavior at infinity: $\frac{\bar{v}_l(x)}{\ln|x|} \to 0$ as $x \to \infty$. Since $\limsup_{x\to\infty} \bar{v}_l/u > 0$ and u enjoys the Harnack inequality, we conclude that $\frac{u(x)}{\ln|x|} \to 0$ as $x \to \infty$. Hence

$$W(x)|x|^N\ln^2|x| = c_0|x|^{-N}u^{-2}|x|^N\ln^2|x| = c_0\frac{\ln^2|x|}{u^2(x)} \to \infty \quad \text{as } x \to \infty.$$

So the existence of u contradicts Lemma 3.37. □

3.4. *Discussion*

(1) The results of Section 3.2 generalize the respective results in [38], including first-order terms. In the last subsection we confined ourselves to the case of equations without lower first-order terms, and in the degenerate case $h(x) = |x|^{2-N}$ even without a potential. We expect that for all the nonuniformly elliptic cases with the polynomial behavior of h it should be possible to carry out the same programme with presence of all lower-order terms. This issue is open at the moment. It would be certainly interesting to investigate the existence of positive supersolutions for general locally bounded h.

(2) It is an interesting open problem to describe the existence and nonexistence of positive supersolutions to the equation $\mathcal{L}u = c_0u^q$ with $\mathcal{L} = -\nabla\cdot a\cdot\nabla u + \frac{\mu}{|x|^2}u$ and a being a uniformly elliptic matrix. Based on the results of Sections 3.1 one can make certain predictions on the existence and rough position of critical exponents (there may be two of them: $q^* > 1$ and $q_* < 1$). However, it will not be possible to determine their exact value

(see the corresponding results on cone-like domains in the next section). The situation will be different if one assumes that the matrix a tends to the identity matrix at infinity. Then the values of the critical exponents and even critical lines should be the same as for the Laplacian perturbed by a Hardy potential (see Section 3.1). Another matter is the question of the nonexistence of positive supersolutions on the critical line. Simple examples show that the issue will depend on the the speed with which the matrix a approaches its limit at infinity.

4. Cone-like domains

In this section we study positive supersolutions to linear and semi-linear equations in cone-like domains in $\mathbb{R}^N$, $N \geq 3$. The exposition is based on [40,41,48].

Let $S^{N-1} = \{x \in \mathbb{R}^N : |x| = 1\}$ and $\Omega \subseteq S^{N-1}$ be a subdomain of S^{N-1}. Throughout this section, for $0 \leq \rho < R \leq +\infty$, we denote

$$\mathcal{C}_\Omega^{(\rho,R)} := \{(r,\omega) \in \mathbb{R}^N : \omega \in \Omega,\ r \in (\rho, R)\}, \qquad \mathcal{C}_\Omega^\rho := \mathcal{C}_\Omega^{(\rho,+\infty)}.$$

Accordingly, $\mathcal{C}_\Omega = \mathcal{C}_\Omega^0$ and $\mathcal{C}_{S^{N-1}} = \mathbb{R}^N \setminus \{0\}$.

We would like to emphasize that we do not make any assumptions on the smoothness of the domain Ω. Our results below show that the existence and nonexistence of positive supersolutions in cone-like domains do not depend on the smoothness of the boundary of the cross-section Ω of the cone. Most of the proofs in this section can be simplified if one assumes that the boundary of Ω is Lipschitz.

4.1. *Linear equations*

Here we discuss some facts concerning the linear equation

$$-\nabla \cdot a \cdot \nabla u - Vu = 0 \quad \text{in } \mathcal{C}_\Omega, \tag{4.1}$$

where $0 \leq V \in L^\infty_{\text{loc}}(\mathcal{C}_\Omega^\rho)$ is a form-bounded potential. The matrix a is assumed to be real, **symmetric** and uniformly elliptic.

We start with establish rough bounds on positive supersolutions to the equation

$$-\nabla \cdot a \cdot \nabla v = 0 \quad \text{in } \mathcal{C}_\Omega. \tag{4.2}$$

Given a function $0 < u \in H^1_{\text{loc}}(\mathcal{C}_\Omega^{R/2,R})$ and a subdomain $\Omega' \subseteq \Omega$, denote

$$m_R(u,\Omega') = \inf_{\mathcal{C}_{\Omega'}^{(R/2,R)}} u, \qquad M_R(u,\Omega') = \sup_{\mathcal{C}_{\Omega'}^{(R/2,R)}} u.$$

LEMMA 4.1. *Let $\Omega \subseteq S^{N-1}$ be a domain and $\Omega' \Subset \Omega$. Then there exist $\alpha \leq 2-N$ and $\beta > 0$ such that for any positive supersolution u to equation (4.2) in $\mathcal{C}_\Omega^\rho$ and any $\rho > 0$ there are two constants $c, C > 0$ such that for any $R > 2\rho$*

$$cR^\alpha \leq m_R(u,\Omega') \leq CR^\beta. \tag{4.3}$$

PROOF. Recall that by the weak Harnack inequality

$$C_W \inf_{C_{\Omega'}^{(r,R)}} u \geq \fint_{C_{\Omega'}^{(r,R)}} u dx,$$

where a standard notation $\fint_G u dx = \frac{1}{|G|}\int_G u dx$ is used with $|G|$ being the Lebesgue measure of G. The constant $C_W > 1$ depends on Ω' but not on r and R. Let $r \geq 2\rho$. Denote $m_r = \inf_{\mathcal{C}_{\Omega'}^{(ra,rb)}} u$ (in this proof only).

Set $a = 1/2$, $b = 2$. Then $m_R(u, \Omega') \geq m_r$ if $r \leq R \leq 2r$. Then

$$\begin{aligned} m_r &\leq \fint_{\mathcal{C}_{\Omega'}^{(ra,rb)}} u dx \leq \left(\frac{2b-a}{b-a}\right)^N \fint_{\mathcal{C}_{\Omega'}^{(ra,2rb)}} u dx \\ &\leq C_W \left(\frac{2b-a}{b-a}\right)^N \inf_{\mathcal{C}_{\Omega'}^{(ra,2rb)}} u \qquad (4.4) \\ &\leq C_W \left(\frac{2b-a}{b-a}\right)^N \inf_{\mathcal{C}_{\Omega'}^{(2ra,2rb)}} u = C_W \left(\frac{2b-a}{b-a}\right)^N \\ &\qquad m_{2r} = C_1 m_{2r}, \quad C_1 > 1. \qquad (4.5) \end{aligned}$$

Let $r_n = 2^n\rho$ and $n \in \mathbb{N}$. Iterating (4.5) we obtain $m_{r_n} \geq C_1^{1-n} m_{2\rho}$. Choosing n such that $R > 2ar_n$ we obtain the lower bound of (4.3) with $\alpha = -\log_2 C_1$ and $c = (4a\rho)^{-\alpha} m_{2\rho}$. Taking into account (3.21) we conclude that $\alpha \leq 2 - N$.

To prove the upper bound of (4.3) set $a = 1/2$, $b = 3/2$. Then $m_R(u, \Omega') \leq m_r$ if $r \leq R \leq 2r$. Then arguing in the same way one obtains the upper bound of (4.3) with $\beta = \log_2 C_1$ and $C = (2b\rho)^{-\beta} m_{2\rho}$. The details are left to the reader. □

Remark 4.2. (1) A similar argument was used before by Pinchover [67, Lemma 6.5].

(2) Due to the strong Harnack inequality (4.3) provides two-sided rough polynomial bounds at infinity for the *solutions* to (4.2).

Minimal solutions. The construction of the minimal solution provided in Lemma 2.10 is appropriate for (4.1) in the cone-like domain $\mathcal{C}_\Omega$ with the following changes. Fix a smooth proper subdomain $\Omega' \Subset \Omega$ and a function $0 \leq \psi \in C_c^\infty(\Omega')$, $\psi \neq 0$. Choose a function $\theta(r) \in C^{0,1}[1, +\infty)$ such that $0 \leq \theta(r) \leq 1$, $\theta(1) = 1$ and $\theta(r) = 0$ for $r \geq 2$. Take $\theta(r)\psi(\omega)$ in place of ψ in (2.10). Then v_ψ is a minimal solution to (4.1).

Our next step is to obtain a polynomial upper bound on the minimal positive solutions to the equation

$$-\nabla \cdot a \cdot \nabla v - Vv = 0 \quad \text{in } \mathcal{C}_\Omega,$$

with a Green-bounded potential V. In application in Section 4.2.3 we will use a special Green-bounded potential which will be specified there. We will use the important property of Green-bounded potentials, formulated in Theorem 3.17. Using this theorem we first prove the required upper bound in the case of the "half-space" cone $\mathcal{C}_+ = \{x_N > 0\}$ with the cross-section $S_+ = \{|x| = 1, x_N > 0\}$. For a given uniformly elliptic matrix a and a potential V defined on $\mathcal{C}_+$ we denote by $\bar{a}$ and $\bar{V}$ the extensions of a and V from $\mathcal{C}_+$ to $\mathbb{R}^N$ by reflection, so that $\bar{a}(\cdot, -x_N) = \bar{a}(\cdot, x_N)$ and $\bar{V}(\cdot, -x_N) = \bar{V}(\cdot, x_N)$. Thus the matrix $\bar{a}$ is uniformly elliptic on $\mathbb{R}^N$ with the same ellipticity constant as a.

The next theorem provides a polynomial improvement for the upper bound on the minimal solutions in "half-space" cone compared to the respective bound in the exterior domains. It was first proved in [40], and we follow the proof given there.

THEOREM 4.3. *Let* $0 \le V \in L^1_{\mathrm{loc}}(\mathcal{C}_+)$ *be a potential such that* $\|\bar{V}\|_{GB,\bar{a}} < 1$. *Let* $v_\psi > 0$ *be a minimal positive solution in* $\mathcal{C}^1_+$ *to the equation*

$$-\nabla \cdot a \cdot \nabla v - V v = 0 \quad \text{in } \mathcal{C}_+,$$

as constructed in (2.10). Then there exists $\gamma \in (0,1)$ *such that*

$$0 < v_\psi \le c|x|^{2-N-\gamma} \quad \text{in } \mathcal{C}^1_+. \tag{4.6}$$

PROOF. Let $\bar{v}$ denote the extension of v_ψ from $\mathcal{C}^1_+$ to B^c_1 by reflection, that is $\bar{v}(\cdot, x_N) = -v_\psi(\cdot, -x_N)$. Thus $\bar{v}(x)$ is a solution to the equation

$$-\nabla \cdot \bar{a} \cdot \nabla \bar{v} - \bar{V}\bar{v} = 0 \quad \text{in } B^c_1.$$

Let w be a solution to (3.22) given by Lemma 3.17. One can check by direct computation (see [38, Lemma 3.4]), that $v_1 := \bar{v}/w$ is a solution to the equation

$$-\nabla \cdot (w^2\bar{a}) \cdot \nabla v_1 = 0 \quad \text{in } B^c_1, \tag{4.7}$$

where the matrix $w^2\bar{a}$ is clearly uniformly elliptic. Let $\Gamma(x) := \Gamma_{w^2\bar{a}}(x, 0)$ be the positive minimal Green function to the equation $-\nabla \cdot (w^2\bar{a}) \cdot \nabla u = 0$ in $\mathbb{R}^N$. The classical estimate of the fundamental solution [53] says that there are two constants $c_1, c_2 > 0$ such that

$$c_1|x|^{2-N} \le \Gamma(x) \le c_2|x|^{2-N} \quad \text{in } B^c_1. \tag{4.8}$$

Applying Lemma 2.6 to v_1 and Γ on C^1_+ and by the construction of v_1 we conclude that

$$|v_1(x)| \le c_3\Gamma(x) \quad \text{on } B^c_1. \tag{4.9}$$

Applying the Kelvin transformation $y = y(x) = x/|x|^2$ and $x = x(y) = y/|y|^2$ to (4.7) we see that the function $\tilde{v}_1(y) = v_1(x(y))/\Gamma(x(y))$, $\tilde{v} \in L^\infty(B_1)$, solves the equation

$$-\nabla \cdot \tilde{a} \cdot \nabla \tilde{v}_1 = 0 \quad \text{in } B_1,$$

where the matrix $\tilde{a}(y)$ is uniformly elliptic on B_1. It follows that $\tilde{v}_1 \in H^1_{\mathrm{loc}}(B_1)$ (see, e.g., [79]). Then, by the De Giorgi–Nash regularity result [28], $\tilde{v}_1 \in C^{0,\gamma}(B_1)$ for some $\gamma \in (0,1)$. Notice that

$$\tilde{v}_1(y) = 0 \quad \text{in } \{y \in B_1,\ y_N = 0\}$$

by the construction. Therefore $\tilde{v}_1(0) = 0$, hence

$$|\tilde{v}_1(y)| \le c|y|^\gamma \quad \text{in } B_1.$$

We conclude that

$$|\bar{v}| \le c_3|\bar{v}_1(x)| \le c_4|x|^{2-N-\gamma} \quad \text{in } B_1^c,$$

as required. □

The next theorem shows that a polynomial improvement of the upper bound for minimal solutions compared to the upper bound in the whole of $\mathbb{R}^N$ is a general phenomenon in cones.

THEOREM 4.4. *Let $\Omega \subset S^{N-1}$ be a domain such that $S^{N-1} \setminus \Omega$ has a nonempty interior. Let $W > 0$ be Green bounded with respect to a. Then there exists $\epsilon > 0$ and $\beta = \beta(\epsilon) < 2 - N$ such that any minimal positive solution v_ψ in $\mathcal{C}_\Omega^1$ to the equation*

$$-\nabla \cdot a \cdot \nabla v - \epsilon W v = 0 \quad \text{in } \mathcal{C}_\Omega$$

has the polynomial upper bound

$$v_\psi \le c|x|^\beta \quad \text{in } \mathcal{C}_\Omega^1. \tag{4.10}$$

REMARK 4.5. By the maximum principle the assertion of Theorem 4.4 is true for $\varepsilon = 0$.

PROOF. If $\mathcal{C}_\Omega \subseteq \mathcal{C}_+$ then (4.10) follows from (4.6) by Lemma 2.6. We shall consider the case $\mathcal{C}_\Omega$ is not a subset of $\mathcal{C}_+$.

Without loss of generality we can assume that $(0, \dots, 0, -1) \notin \Omega$. Set $\hat{x} = (x_1, \dots, x_{N-1})$ and $\sigma = \inf\{|\hat{x}| : x \in \Omega, x_N < 0\}$. Let $D_\sigma = \{x \in S^{N-1} : |\hat{x}| \le \sigma, x_N < 0\}$ and $\hat{D}_\sigma := S^{N-1} \setminus D_\sigma$. Then $\mathcal{C}_\Omega \subseteq \mathcal{C}_{\hat{D}_\sigma}$. Extend the matrix a by the identity matrix from $\mathcal{C}_\Omega$ to $\mathcal{C}_{\hat{D}_\sigma}$. Let w_ψ be a minimal positive solution in $\mathcal{C}_{\hat{D}_\sigma}^1$ to the equation

$$-\nabla \cdot a \cdot \nabla w - \epsilon W w = 0.$$

To complete the proof we need only to show that w_ψ satisfies (4.10) in $\mathcal{C}_{\hat{D}_\sigma}^1$. Then the same bound on minimal positive solutions in $\mathcal{C}_\Omega^1$ follows from Lemma 2.6.

Consider the transformation

$$y = y(x) = \left(x_1, \dots, x_{N-1}, x_N + k|\hat{x}|\right),$$

where $k = \sqrt{\sigma^{-2} - 1}$. Then $y : \mathcal{C}_{\hat{D}_\sigma} \to \mathcal{C}_+$ is a bijection, the Jacobi matrix of $y(x)$ is nondegenerate and has the determinant equal to 1 everywhere. Moreover, $|x| \le |y(x)| \le \varkappa|x|$ for all $x \in \mathcal{C}_{\hat{D}_\sigma}$, where $\varkappa = \sqrt{2 + k^2}$. Therefore $\hat{w}(y) := w_\psi(x(y))$ solves the equation

$$-\nabla \cdot \hat{a} \cdot \nabla \hat{w} - \hat{W}_\epsilon \hat{w} = 0 \quad \text{in } \mathcal{C}_+^\kappa,$$

with the uniformly elliptic matrix $\hat{a}(y) := a(x(y))$ and $\hat{W}_\epsilon(y) := \epsilon W(x(y))$. One can easily check by direct computation that $\bar{\hat{W}}_\epsilon \in GB$. Fix $\epsilon > 0$ such that $\|\bar{\hat{W}}_\epsilon\|_{GB,\bar{\hat{a}}} < 1$. Then by Theorem 4.3 we conclude that $\hat{w}(y)$ satisfies (4.6). Therefore $w_\psi(x)$ obeys (4.10) with $\beta := 2 - N - \gamma$ as required. □

From the above results we conclude that small solutions to the homogeneous equation $-\nabla \cdot a \cdot \nabla v - Vv = 0$ (with Green-bounded potential) "decay" faster in the case of cone-like domains than in the case of exterior domains. Although we do not have a result like Theorem 3.15 for the case of cone-like domains (this question is open at the moment), we still observe the same phenomenon. We will see below that the large solutions "grow" faster in the case of cone-like domains than in the case of exterior domains.

Large solutions and particularly their behavior at infinity is the key tool to study positive supersolutions to the semi-linear equations in the sublinear case (compare with Section 3.2.2). In order to study large supersolutions (compare with Section 3) we need a version of Phragmèn–Lindelöf comparison principle. First, we construct a sequence of appropriate comparison functions to the equation

$$(-\nabla \cdot a \cdot \nabla - V)v = 0 \quad \text{in } \mathcal{C}_\Omega^\rho. \tag{4.11}$$

Fix a subdomain $\Omega' \subseteq \Omega$ and a function $0 \le \psi \in C_c^\infty(\Omega')$, $\psi \not\equiv 0$, as above. Let $\tilde{\theta} \in C^\infty[1/2, 1]$ be such that $\tilde{\theta}(1) = 1$, $0 \le \tilde{\theta} \le 1$ and $\tilde{\theta}(1/2) = 0$. Assume $R \ge 4\rho$ and set $\theta_R(r) := \tilde{\theta}(r/R)$ $(r \in [R/2, R])$. Thus $\theta_R \psi \in D^1(\mathcal{C}_{\Omega'}^{(R/2,R)})$. By $w_{\psi,R}$ we denote the unique solution to the problem

$$(-\nabla \cdot a \cdot \nabla - V)w = 0, \qquad w - \theta_R \psi \in D_0^1(\mathcal{C}_\Omega^{(\rho,R)}).$$

The constructed $w_{\psi,R}$ depends only on the choice of ψ and R, as one can see from the maximum principle.

REMARK 4.6. Note that $w_{\psi,R}$ is positive. Indeed, $(w_{\psi,R})^- \le (w_{\psi,R} - \theta_R\psi)^- \in D_0^1(\mathcal{C}_\Omega^{(\rho,R)})$. Thus $w_{\psi,R} > 0$ in $\mathcal{C}_\Omega^{(\rho,R)}$, by the weak maximum principle and weak Harnack's inequality.

REMARK 4.7. Assume that $V \in GB(\mathbb{R}^N)$ is a Green-bounded potential, see Theorem 3.17. Then the functions $w_{\psi,R}$ are uniformly bounded in $R \ge 4\rho$. Indeed, let $w_0 > 0$ be a quasi-constant solution to equation $(-\nabla \cdot a \cdot \nabla - V)w = 0$ in $\mathbb{R}^N$, that is $\epsilon \le w_0 \le \epsilon^{-1}$ for some $\epsilon > 0$, see Theorem 3.17. Without loss of generality we may assume that $w_0 \ge \max_\Omega \psi$. Then

$$\begin{aligned} &(-\nabla \cdot a \cdot \nabla - V)((w_0 - \theta_R\psi) - (w_{\psi,R} - \theta_R\psi)) \\ &\quad = (-\nabla \cdot a \cdot \nabla - V)(w_0 - w_{\psi,R}) = 0 \quad \text{in } \mathcal{C}_\Omega^{(\rho,k)}. \end{aligned}$$

Thus the comparison principle (see Lemma 2.6) implies that $w_0 \ge w_{\psi,R}$ in $\mathcal{C}_\Omega^{(\rho,R)}$.

Fix a compact $K_0 \subset \mathcal{C}_\Omega^{(\rho,2\rho)}$. Set

$$v_{\psi,R} := \frac{w_{\psi,R}}{\inf_{K_0} w_{\psi,R}}.$$

Then $\inf_{K_0} v_{\psi,R} = 1$ and $v_{\psi,R}$ is a solution to the equation

$$(-\nabla \cdot a \cdot \nabla - V)v = 0 \quad \text{in } \mathcal{C}_\Omega^{(\rho,R)}. \tag{4.12}$$

We call such constructed family $(v_{\psi,R})_{R \ge 4\rho}$ a family of *comparison functions* to equation (4.11). Note that construction of each $v_{\psi,R}$ depends only on the choice of K_0, ψ and R.

LEMMA 4.8 (Phragmèn–Lindelöf type comparison principle). *Let $\Omega' \subseteq \Omega$ be a domain, $\psi \in H_0^1(\Omega')$, $K_0 \subset \mathcal{C}_\Omega^{(\rho,2\rho)}$ a compact, and let $(v_{\psi,R})_{R\geq 4\rho}$ be a family of comparison functions to (4.11), as constructed above. Let $0 < u \in H^1_{\mathrm{loc}}(\mathcal{C}_\Omega^\rho)$ be a supersolution to (4.11). Then there exists $\mu = \mu(\Omega', \psi, K_0, u) > 0$ such that for any $R \geq 4\rho$ one has*

$$m_R(u, \Omega') \leq \mu M_R(v_{\psi,R}, \Omega'). \tag{4.13}$$

PROOF. Set $\nu_R = \inf_{K_0} w_{\psi,R}$. Suppose for any $\mu > 0$ there exists $R \geq 4\rho$ such that

$$u > \mu v_{\psi,R} = \frac{\mu}{\nu_R} w_{\psi,R} \quad \text{in } \mathcal{C}_{\Omega'}^{(R/2,R)}. \tag{4.14}$$

Let $\psi_R > 0$ be the unique solution to the problem

$$(-\nabla \cdot a \cdot \nabla - V)w = 0, \qquad w - \theta_R \psi \in D_0^1(\mathcal{C}_{\Omega'}^{R/2,R}). \tag{4.15}$$

Clearly

$$(-\nabla \cdot a \cdot \nabla - V)(w_{\psi,R} - \psi_R) = 0 \quad \text{in } \mathcal{C}_{\Omega'}^{(R/2,R)}.$$

Observe that $w_{\psi,R} > 0$ in $\mathcal{C}_\Omega^{(\rho,R)} \setminus \mathcal{C}_{\Omega'}^{(R/2,R)}$. Thus $(w_{\psi,R} - \psi_R)^- \in D_0^1(\mathcal{C}_{\Omega'}^{R/2,R})$. Hence, by Lemma 2.5,

$$w_{\psi,R} \geq \psi_R \quad \text{in } \mathcal{C}_{\Omega'}^{(R/2,R)}.$$

Therefore

$$\begin{aligned}(-\nabla \cdot a \cdot \nabla - V)(u - \mu v_{\psi,R}) &= (-\nabla \cdot a \cdot \nabla - V) \\ &\quad \times \left(\left(u - \frac{\mu}{\nu_R}\bar{\psi}_R\right) - \frac{\mu}{\nu_R}(w_{\psi,R} - \bar{\psi}_R)\right) \\ &\geq 0 \quad \text{in } \mathcal{C}_\Omega^{(\rho,R)}.\end{aligned}$$

Thus Lemma 2.6 implies that

$$u \geq \mu v_{\psi,R} \quad \text{in } \mathcal{C}_\Omega^{(\rho,R)}.$$

Since $\mu > 0$ is arbitrary, we conclude that $\inf_{K_0} u = +\infty$, a contradiction. □

REMARK 4.9. Condition (4.13) implies that

$$\limsup_{R\to\infty} m_R\left(\frac{u}{v_{\psi,R}}, \Omega'\right) < \infty. \tag{4.16}$$

COROLLARY 4.10 (Strong Phragmèn–Lindelöf type comparison principle). *Assume that*

$$V(x) \leq c|x|^{-2} \quad \textit{in } \mathcal{C}_\Omega^\rho. \tag{4.17}$$

Let $\Omega' \Subset \Omega$ be a domain, $\psi \in H_0^1(\Omega')$, $K_0 \subset \mathcal{C}_\Omega^{(\rho,2\rho)}$ a compact, and let $(v_{\psi,R})_{R\geq 4\rho}$ be the family of comparison functions to (4.11), as constructed above. Let $0 < u \in H^1_{\mathrm{loc}}(\mathcal{C}_\Omega^\rho)$ be a solution to (4.11). Then there exists $\hat{\mu} = \hat{\mu}(\Omega', \psi, K_0, u) > 0$ such that for any $R \geq 4\rho$ one has

$$M_R(u, \Omega') \leq \hat{\mu} M_R(v_{\psi,R}, \Omega'). \tag{4.18}$$

PROOF. Assumption (4.17) implies that the solution u satisfies the strong Harnack inequality

$$M_R(u,\Omega') \le c_H m_R(u,\Omega'),$$

where the constant $c_H = c_H(\Omega') > 0$ does not depend on R (this can be seen by scaling). Thus we deduce from (4.13) that

$$M_R(u,\Omega') \le c_H m_R(u,\Omega') < c_H \mu M_R(v_{\psi,R},\Omega'),$$

so $\hat{\mu} = c_H\mu > 0$ does not depend on R. □

Now we construct a large solution to the homogeneous equation $-\nabla\cdot a\cdot\nabla v - Vv = 0$ on a proper cone-like domain, i.e. a cone-like domain with a cross-section Ω such that $S^{N-1}\setminus\Omega$ contains an open set. Let $U\subset S^{N-1}$ be a "spherical" subdomain such that $\Omega \Subset U$. Consider a class of equations

$$-\nabla\cdot a\cdot\nabla v - Vv = 0 \quad \text{in } \mathcal{C}_U, \tag{4.19}$$

where $0 \le V \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$. Let $\bar{a}$ be a symmetric measurable and uniformly elliptic extension of the matrix a to $\mathbb{R}^N$ and $\bar{V} \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ be an extension of V. We assume $\bar{V}$ to be Green bounded (GB) with respect to $\bar{a}$.

One can verify by a direct computation that the potential

$$\bar{V}_\epsilon(x) = \frac{\varepsilon}{|x|^2\log^2|x|}\wedge 1,$$

that extends to $\mathbb{R}^N$ is Green bounded (GB) with respect to $\bar{a}$, for small enough $\varepsilon > 0$.

We construct a family of comparison functions to equation (4.19). Fix smooth subdomains $U'' \Subset U' \Subset U$ such that $\Omega' \Subset U''$, and a function $0 \lneq \psi \in C_0^\infty(U')$ such that $0 \le \psi \le 1$ and $\psi \equiv 1$ on U''. Let $\tilde{\theta} \in C^\infty[1/2,1]$ be such that $\tilde{\theta}(1) = 1$, $0 \le \tilde{\theta} \le 1$ and $\tilde{\theta}(1/2) = 0$. Assume that $R \ge 1$ and set $\theta_R(r) := \tilde{\theta}(r/R)$ $(r \in [R/2,R])$. Thus $\theta_R\psi \in D^1(\mathcal{C}_{U'}^{(R/2,R)})$. By $w_{\psi,R}$ we denote the unique solution to the problem

$$-\nabla\cdot a\cdot\nabla w - Vw = 0, \qquad w - \theta_R\psi \in D_0^1(\mathcal{C}_U^{(0,R)}).$$

REMARK 4.11. (i) Such constructed $w_{\psi,R}$ depends only on R and ψ, but does not depend on the choice of $\tilde{\theta}$.
(ii) Observe that $w_{\psi,R}$ is positive. Indeed, $(w_{\psi,R})^- \le (w_{\psi,R} - \theta_R\psi)^- \in D_0^1(\mathcal{C}_U^{(0,R)})$. Thus $w_{\psi,R} > 0$ in $\mathcal{C}_U^{(0,R)}$, by Lemma 2.5 and the weak Harnack inequality.

Fix a compact $K_0 \subset \mathcal{C}_U^{(0,1/2)}$. Set

$$v_{\psi,R} := \frac{w_{\psi,R}}{\inf_{K_0} w_{\psi,R}}.$$

Then $\inf_{K_0} v_{\psi,R} = 1$ and $(v_{\psi,R})_{R\ge 1}$ is a family of comparison functions to the equation

$$(-\nabla\cdot a\cdot\nabla - V)v = 0 \quad \text{in } \mathcal{C}_U^{(0,R)}. \tag{4.20}$$

LEMMA 4.12. *There exists $M_\infty > 0$ such that $\|w_{\psi,R}\|_{L^\infty} \le M_\infty$ for every $R > 1$.*

PROOF. Let $w_0 > 0$ be a quasi-constant solution to (3.22) that satisfies $0 < \epsilon < w_0 < \epsilon^{-1}$ in $\mathbb{R}^N$. Without loss of generality we may assume that $w_0 \ge \max_U \psi = 1$. Then

$$\begin{aligned}&(-\nabla\cdot a\cdot\nabla - V)((w_0-\theta_R\psi)-(w_{\psi,R}-\theta_R\psi))\\&\quad=(-\nabla\cdot a\cdot\nabla - V)(w_0-w_{\psi,R})\\&\quad=0\quad\text{in } \mathcal{C}_U^{(0,R)}.\end{aligned}$$

Thus Lemma 2.6 implies that $w_{\psi,R} \le w_0$ in $\mathcal{C}_U^{(0,R)}$, uniformly in $R \ge 1$. □

PROPOSITION 4.13. *There exist $\gamma > 0$ and $C > 0$ such that for $R \ge 1$ one has*

$$m_R(v_{\psi,R},\Omega') \ge CR^\gamma.$$

PROOF. Let w_0 be a quasi-constant solution to (3.22) given by Theorem 3.17. One can check by direct computation (see, e.g. [38, Lemma 3.4]), that $w_R := w_{\psi,R}/w_0$ is a solution to the equation

$$-\nabla\cdot A\cdot\nabla w = 0\quad\text{in } \mathcal{C}_U^{(0,R)},\tag{4.21}$$

where $A := w_0^2 a$. Clearly the matrix A is uniformly elliptic with the ellipticity constant $\nu(A)$.

Applying the scaling $y = x/R$ to (4.21) we see that the function $\hat{w}_R(y) = w_R(Ry)$ solves the equation

$$-\nabla\cdot \hat{A}_R\cdot\nabla\hat{v}_R = 0\quad\text{in } \mathcal{C}_U^{(0,1)},$$

where the matrix $\hat{A}_R(y) = A(Ry)$ is uniformly elliptic with the same ellipticity constant $\nu = \nu(A)$ as A.

Observe that $\partial\mathcal{C}_U^{(0,1)}$ satisfies the exterior cone condition. In particular, every boundary point of $\partial\mathcal{C}_U^{(0,1)}$ is regular. Thus, by the boundary regularity result [28, Theorem 8.27] applied at $x = 0$ we conclude that there exist $\gamma > 0$ and $C_0 > 0$ such that

$$\operatorname{osc}_{\mathcal{C}_U^{(0,1/R)}}\hat{w}_R(y) \le CR^{-\gamma}\sup_{\mathcal{C}_U^{(0,1/2)}}\hat{w}_R(y) \le C_0M_\infty R^{-\gamma}.$$

The constants $\gamma > 0$ and $C_0 > 0$ depend only on the ellipticity constant $\nu(a)$ and do not depend on R.

By the same regularity result [28, Theorem 8.27] applied at Ω (considered as a portion of the boundary of $\mathcal{C}_U^{(0,1)}$) we conclude that for some $\delta \in (0, 1/2)$ there exist $C_1 > 0$ and $\gamma_1 > 0$ such that

$$\operatorname{osc}_{\mathcal{C}_\Omega^{(1-\delta,1)}}\hat{w}_R(y) \le C_1\delta^{\gamma_1}\sup_{\mathcal{C}_\Omega^{(0,1/2)}}\hat{w}_R(y) \le C_1M_\infty\delta^{\gamma_1}.$$

Here $\gamma_1 > 0$ and $C_1 > 0$ depend only on $\nu(a)$ and do not depend on R. Hence the strong Harnack inequality implies that there exists a constant $M_1 > 0$ such that

$$\inf_{\mathcal{C}_\Omega^{(1/2,1)}} \hat{w}_R(y) \geq M_1.$$

Applying the inverse scaling $x = Ry$, we conclude that $m_{v_{\psi,R}}(R, \Omega') \geq CR^\gamma$ with some $C > 0$. □

For completeness we give the proof of the existence of a large solution.

PROPOSITION 4.14. *There exists a large solution to equation* (4.19) *in* $\mathcal{C}_\Omega$.

PROOF. By the Harnack inequality for any compact $K \subset \mathcal{C}_\Omega^{(0,R)}$ such that $K_0 \subset K$ one has

$$\sup_K v_{\psi,R} \leq c \inf_K v_{\psi,R} \leq c \inf_{K_0} v_{\psi,R} = c,$$

where $c = c(K) > 0$. Let $R_n \to \infty$. Hence the sequence (v_{ψ,R_n}) is locally bounded in $\mathcal{C}_\Omega^\rho$. By the standard Caccioppoli and diagonalization arguments (see, e.g. [40, Proposition 1.1]) one can construct a function $v_\psi \in H^1_{\mathrm{loc}}(\mathcal{C}_\Omega)$ that is a solution (4.19) in $\mathcal{C}_\Omega$ and satisfies $v_\psi \geq v_{\psi,R_n}$ for each $n \in \mathbb{N}$. Therefore v_ψ is a large solution to (4.19) in $\mathcal{C}_\Omega$. □

Further we derive sharp two-sided estimates for small and large supersolutions for the particular case of the Laplacian on the cone perturbed by the Hardy-type potential $V(x) = \frac{V_1(\omega)}{|x|^2}$ with $V_1 \in L^\infty(\Omega)$.

Estimates of positive supersolutions to $-\Delta v - \frac{V(\omega)}{|x|^2} v = 0$. Consider the linear equation

$$-\Delta v - \frac{V(\omega)}{|x|^2} v = 0 \quad \text{in } \mathcal{C}_\Omega^\rho, \tag{4.22}$$

where $\Omega \subseteq S^{N-1}$ $(N \geq 2)$ is a domain, $V \in L^\infty(\Omega)$ and $\rho \geq 1$. Recall that in the polar coordinates (r, ω) the operator $-\Delta - \frac{V(\omega)}{|x|^2}$ has the form

$$-\frac{\partial^2}{\partial r^2} - \frac{N-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\{-\Delta_\omega - V(\omega)\},$$

where Δ_ω denotes the Dirichlet Laplace–Beltrami operator on Ω. In what follows $(\lambda_{V,k})_{k\in\mathbb{N}}$ denotes the sequence of eigenvalues of the operator $-\Delta_\omega - V$ in $L^2(\Omega)$,

$$\lambda_{V,1} < \lambda_{V,2} \leq \cdots \leq \lambda_{V,k} \leq \ldots.$$

(If $V = 0$, the strict inequality $\lambda_{1,V} < \lambda_{2,V}$ is standard, for $V \neq 0$ see, e.g. [74], Sec. XIII.12.) By $(\phi_{V,k})_{k\in\mathbb{N}}$ we denote the corresponding orthonormal basis of eigenfunctions in $L^2(\Omega)$, with the positive principal eigenfunction $\phi_{V,1} > 0$. If $V = 0$ we write λ_k and ϕ_k.

Let $\psi \in L^2(\Omega)$. Then

$$\psi = \sum_{k=1}^{\infty} \psi_k \phi_{V,k}, \quad \text{where } \psi_k = \int_{\Omega} \psi \phi_{V,k} \, d\omega, \tag{4.23}$$

and the series converges in $L^2(\Omega)$ with $\|\psi\|_{L^2}^2 = \sum_{k=1}^{\infty} \psi_k^2$. If, in addition, $\psi \in H_0^1(\Omega)$ then (4.23) converges in $H_0^1(\Omega)$ with $\|\nabla\psi\|_{L^2}^2 \asymp \sum_{k=1}^{\infty} \lambda_{V,k}\psi_k^2$. If $\psi \in C_c^{\infty}(\Omega)$, then it is not difficult to show that the series (4.23) converges also in $L^{\infty}(\Omega)$.

The existence of positive solutions to (4.22) is equivalent to the positivity of the quadratic form

$$\mathcal{E}_V(v, v) := \int_{\mathcal{C}_{\Omega}^{\rho}} \left(|\nabla v|^2 - \frac{V(\omega)}{|x|^2} v^2 \right) dx \qquad (v \in H_c^1(\mathcal{C}_{\Omega}^{\rho}) \cap L_c^{\infty}(\mathcal{C}_{\Omega}^{\rho})),$$

that corresponds to (4.22) [2].

As in Proposition 3.3 the optimality of the Hardy inequality on the cone-like domains (B.4) implies the following nonexistence result.

PROPOSITION 4.15. *Equation* (4.22) *has a positive supersolution if and only if* $C_H + \lambda_{V,1} \geq 0$.

If $C_H + \lambda_1 \geq 0$ then the roots of the quadratic equation

$$\alpha(\alpha + N - 2) = \lambda_{V,k} \tag{4.24}$$

are real, for each $k \in \mathbb{N}$. Denote these roots by $\alpha_{V,k}^- \leq \alpha_{V,k}^+$. If $C_H + \lambda_{V,1} = 0$ and $k = 1$ then (4.24) has the unique root, denoted by $\alpha_* := \alpha_{V,1}^{\pm} = \frac{2-N}{2}$.

For a positive function $u \in H_{\text{loc}}^1(\mathcal{C}_{\Omega}^1)$ and a subdomain $\Omega' \subseteq \Omega$, denote

$$m_R(u, \Omega') := \inf_{\mathcal{C}_{\Omega'}^{(R/2,R)}} u, \qquad M_R(u, \Omega') := \sup_{\mathcal{C}_{\Omega'}^{(R/2,R)}} u.$$

The next theorem provides two-sided bound for supersolutions to (4.22).

THEOREM 4.16. *Let* $u \in H_{\text{loc}}^1(\mathcal{C}_{\Omega}^{\rho})$ *be a positive supersolution to* (4.22). *Then for every proper subdomain* $\Omega' \Subset \Omega$ *and* $R \geq 2\rho$ *there exist constants* $c_1, c_2 > 0$ *such that*

(i) *if* $C_H + \lambda_{V,1} > 0$ *then*

$$c_1 R^{\alpha_{V,1}^-} \leq m_R(u, \Omega') \leq c_2 R^{\alpha_{V,1}^+}, \tag{4.25}$$

(ii) *if* $C_H + \lambda_{V,1} = 0$ *then*

$$c_1 R^{\alpha_*} \leq m_R(u, \Omega') \leq c_2 R^{\alpha_*} \log(R). \tag{4.26}$$

Remark 4.17. (1) The above estimates are sharp, as one sees comparing with the explicit it solutions $r^{\alpha_{V,1}^{\pm}}\phi_{V,1}$ in the case (i) and $r^{\alpha_*}\phi_{V,1}$ and $r^{\alpha_*}\log(r)\phi_{V,1}$ in the case (ii).
(2) If $v(\cdot)$ is a supersolution to (4.22) in $\mathcal{C}_{\Omega}^{\rho}$, then, for $\tau > 0$, $v(\tau\,\cdot)$ is a supersolution to (4.22) in $\mathcal{C}_{\Omega}^{\tau\rho}$, so we can always choose $\rho > 0$ as convenient.

Proof of Theorem* 4.16 *(i). The lower bound in (4.25) follows from the next two lemmas.

LEMMA 4.18. *Let* $\psi \in C_c^\infty(\Omega)$. *Then*

$$v_\psi(x) = \sum_{k=1}^\infty \psi_k r^{\alpha^-_{V,k}} \phi_{V,k}(\omega), \quad \text{where } \psi_k = \int_\Omega \psi(\omega)\phi_{V,k}(\omega)\, d\omega, \tag{4.27}$$

is a minimal positive solution to equation (4.22) *in* $\mathcal{C}^1_\Omega$.

PROOF. Set $v_k(x) := r^{\alpha^-_{V,k}}\phi_{V,k}(\omega)$. Then a direct computation gives that

$$-\Delta v_k - \frac{V(\omega)}{|x|^2} v_k = 0 \quad \text{in } \mathcal{C}^1_\Omega.$$

Recall that $\nabla = \mathbf{n}\frac{\partial}{\partial r} + \frac{1}{r}\nabla_\omega$, where $\mathbf{n} = \frac{x}{|x|} \in \mathbb{R}^N$. Since

$$\int_\Omega |\nabla_\omega \phi_{V,k}|^2 d\omega - \int_\Omega V(\omega)|\phi_{V,k}|^2 d\omega = \lambda^-_{V,k},$$

we obtain

$$\begin{aligned}
\varepsilon_0 \|\nabla v_k\|^2_{L^2} &\le \int_{\mathcal{C}^\rho_\Omega} \left(|\nabla v_k|^2 - \frac{V(\omega)}{|x|^2}|v_k|^2 \right) dx \\
&= \int_1^\infty \int_\Omega \left(\left| \frac{\partial}{\partial r} r^{\alpha^-_{V,k}} \phi_{V,k}(\omega) \right|^2 + \frac{|r^{\alpha^-_{V,k}} \nabla_\omega \phi_{V,k}(\omega)|^2}{r^2} - \frac{V(\omega)|r^{\alpha^-_{V,k}}\phi_{V,k}|^2}{r^2} \right) \\
&\quad \times r^{N-1}\, d\omega dr \\
&= \int_1^\infty r^{2\alpha^-_{V,k}+N-3}((\alpha^-_{V,k})^2 + \tilde\lambda_k)\, dr = \frac{(\alpha^-_{V,k})^2 + \tilde\lambda_k}{2 - N - 2\alpha^-_{V,k}} = -\tilde\alpha_k,
\end{aligned}$$

where $\varepsilon_0 > 0$ is the constant in the inequality $\int V\varphi^2 dx \le (1-\varepsilon_0)\int |\nabla\varphi|^2 dx$. Now it is straightforward that $v_k - \phi_{V,k}\theta_1 \in D^1_0(\mathcal{C}^1_\Omega)$, so v_k is the solution to the problem

$$-\Delta v - \frac{V(\omega)}{|x|^2} v = 0, \qquad v - \phi_{V,k}\theta_1 \in D^1_0(\mathcal{C}^1_\Omega).$$

Hence we have

$$\begin{aligned}
\varepsilon_0 \|\nabla v_\psi\|_2^2 &\le \int_{\mathcal{C}^1_\Omega} \left(|\nabla v_\psi|^2 - \frac{V(\omega)}{|x|^2}|v_\psi|^2 \right) dx = \sum_{k=1}^\infty \psi_k^2(-\alpha^-_{V,k}) \\
&\le \frac{N-2}{2}\|\psi\|_2^2 + \|\psi\|_2 \left(\int_\Omega \left(|\nabla_\omega \psi|^2 - V(\omega)\psi^2 + \left(\frac{N-2}{2}\right)^2 \psi^2 \right) d\omega \right)^{\frac{1}{2}}.
\end{aligned}$$

Hence $v_\psi - \psi\theta_1 \in D^1_0(\mathcal{C}^1_\Omega)$, so v_ψ solves the problem

$$-\Delta v - \frac{V(\omega)}{|x|^2} v = 0, \qquad v - \psi\theta_1 \in D^1_0(\mathcal{C}^1_\Omega).$$

By the uniqueness (maximum principle) we conclude that v_ψ defined by (4.27) coincides with the minimal solution v_ψ as constructed in (2.10), (4.15). □

LEMMA 4.19. *Let $v_\psi > 0$ be a minimal solution (4.27) to equation (4.22) in $\mathcal{C}^1_\Omega$. Then for any $\Omega' \Subset \Omega$ and $\rho > 1$ there exists $c = c(\Omega', \rho) > 0$ such that*

$$v_\psi(x) \geq c r^{\alpha^-_{V,1}} \quad \text{in } \mathcal{C}^\rho_{\Omega'}. \tag{4.28}$$

PROOF. By (4.27) one can represent v_ψ as $v_\psi(x) = \psi_1 r^{\alpha^-_{V,1}} \phi_{V,1}(\omega) + w(x)$, where

$$w(x) = \sum_{k=2}^{\infty} \psi_k r^{\alpha^-_{V,k}} \tilde{\phi}_{V,k}(\omega).$$

Notice that $w(x)$ satisfies

$$-\Delta w - \frac{V(\omega)}{|x|^2} w = 0 \quad \text{in } \mathcal{C}^1_\Omega.$$

Thus by the standard elliptic estimate (see, e.g. [28, Theorem 8.17]) for any $\Omega' \Subset \Omega$ and $\rho > \frac{4}{3}$ one has

$$\sup_{\mathcal{C}^{(\rho,2\rho)}_{\Omega'}} |w|^2 \leq c\rho^{-N} \int_{\mathcal{C}^{(\frac{3\rho}{4},\frac{9\rho}{8})}_\Omega} |w|^2 \, dx,$$

where the constant $c > 0$ does not depend on ρ. Therefore

$$\begin{aligned} \sup_{\mathcal{C}^{(\frac{3\rho}{4},\frac{9\rho}{8})}_{\Omega'}} |w|^2 &\leq c\rho^{-N} \int_{\frac{3\rho}{4}}^{\frac{9\rho}{8}} r^{N-1} \int_\Omega |w|^2 \, d\omega \, dr \\ &= c\rho^{-N} \int_{\frac{3\rho}{4}}^{\frac{9\rho}{8}} r^{N-1} \sum_{k=2}^{\infty} \psi_k^2 r^{2\alpha^-_{V,k}} \, dr \\ &\leq c \int_{\frac{3\rho}{4}}^{\frac{9\rho}{8}} r^{2\alpha^-_{V,2}-1} \, dr \|\psi - \psi_1\phi_{V,1}\|_2^2 = c_1 \rho^{2\alpha^-_{V,2}}. \end{aligned} \tag{4.29}$$

So we conclude that

$$v_\psi(x) \geq \psi_1 r^{\tilde{\alpha}_1} \phi_1(\omega) - c r^{\tilde{\alpha}_2} \quad \text{in } \mathcal{C}^\rho_{\Omega'}.$$

Since $\alpha^-_{V,2} < \alpha^-_{V,1} < 0$ this implies (4.28). □

For the upper bound we will use the Phragmèn–Lindelöf type comparison principle. For that we need to construct the family of comparison functions.

Fix a subdomain $\Omega' \subseteq \Omega$ and a function $0 \leq \psi \in C_c^\infty(\Omega')$, $\psi \not\equiv 0$. Let $R \geq 4$. For $(r, \omega) \in \mathcal{C}^{(1,R)}_\Omega$ and $k \in \mathbb{N}$ define

$$w_{k,R}(r,\omega) := \eta_{k,R}(r)\phi_{V,k}(\omega), \quad \text{where } \eta_{k,R}(r) := \left\{ \frac{r^{\alpha^+_{V,k}} - r^{\alpha^-_{V,k}}}{R^{\alpha^+_{V,k}} - R^{\alpha^-_{V,k}}} \right\}. \tag{4.30}$$

Let $\tilde{\theta} : [0,1] \to \mathbb{R}$ be a smooth function such that $0 \leq \tilde{\theta} \leq 1$, $\tilde{\theta}(1) = 1$ and $\tilde{\theta}(\xi) = 0$ for $\xi \in [0, 1/2]$. For $r \in [R/2, R]$ set $\theta_R(r) := \tilde{\theta}(r/R)$.

A direct computation shows that $w_{k,R}$ is a solution to the problem

$$\left(-\Delta - \frac{V(\omega)}{|x|^2}\right) v = 0, \qquad v - \theta_R \phi_{V,k} \in H_0^1(\mathcal{C}_\Omega^{(1,R)}). \tag{4.31}$$

Let

$$w_{\psi,R} := \sum_{k=1}^{\infty} \psi_k w_{k,R},$$

where ψ_k are the Fourier coefficients of ψ as in (4.23). Thus $w_{\psi,R} \in H^1_{\mathrm{loc}}(\mathcal{C}_\Omega^{1,R})$ is a solution to (4.31) and $w_{\psi,R}(R,\omega) = \psi(\omega)$.

Fix a compact $K_0 \subset \mathcal{C}_\Omega^{(2,4)}$. Define the family of the comparison functions $v_{\psi,R}$ by

$$v_{\psi,R} := \frac{w_{\psi,R}}{\inf_{K_0} w_{\psi,R}}.$$

Then $\inf_{K_0} v_{\psi,R} = 1$.

The lemma is a reformulation of Lemma 4.8 adopted to the above construction.

LEMMA 4.20 (Phragmen–Lindelöf type comparison principle). *Let $0 < u \in H^1_{\mathrm{loc}}(\mathcal{C}^1_\Omega)$ be a supersolution to (4.22) in $\mathcal{C}^1_\Omega$. Then*

$$m_R(u, \Omega') \le c M_R(v_{\psi,R}, \Omega'), \quad R \ge 8.$$

Now the upper bound in (4.25) follows from the next lemma.

LEMMA 4.21. *$M_{v_{\psi,R}}(R, \Omega') \asymp R^{\alpha^+_{V,1}}$ as $R \to \infty$.*

PROOF. Choosing $u := r^{\alpha^+_{V,1}} \phi_1$ as a (super) solution in Lemma 4.20 we conclude that

$$M_R(v_{\psi,R}, \Omega') \ge c R^{\alpha^+_{V,1}}, \quad R \gg 1.$$

Now we estimate $M_R(v_{\psi,R}, \Omega')$ from above.

First, observe that Lemma 2.6 and arguments, similar to those in Lemmas 4.18, 4.19 imply the upper bound

$$v_{\psi,R}(r,\omega) \le c \eta_{1,R}(r) \phi_{V,1}(\omega) \quad \text{in } \mathcal{C}_\Omega^{(1,R)},$$

where $c > 0$ is chosen so that $\psi \le c\phi_{V,1}$ in Ω. Clearly, if $\alpha^+_{V,1} \ge 0$ then $\eta_{1,R}(r) \le 1$. However, if $\alpha^+_{V,1} < 0$ then $\eta_{1,R}(r)$ attains its maximum at $r_* \in (1, R)$ with $\eta_{1,R}(r_*) \to \infty$ as $R \to \infty$. Nevertheless, one can readily verify that

$$\max_{r \in [R/2, R]} \eta_{k,R}(r) \le \max\{1, 2^{-\alpha^+_{V,1}}\}, \quad R \gg 1.$$

Therefore

$$M_{v_{\psi,R}}(\Omega, R) \le c_1, \quad R \gg 1.$$

To estimate $\inf_{K_0} v_{\psi,R}$ from below, note that

$$v_{\psi,R} = \psi_1 v_{1,R} + \tilde{v}_{\psi,R}, \quad \text{where } \tilde{v}_{\psi,R} = \sum_{k=2}^{\infty} \psi_k v_{k,R}.$$

Then similarly to (4.29) we obtain

$$\sup_{K_0} |\tilde{v}_{\psi,R}| \le \max_{r\in(2,3)} \eta_k(r) \sum_{k=2}^{\infty} |\psi_k||\phi_{V,k}(\omega)| \le \frac{c_1}{R^{\alpha^+_{V,2}} - R^{\alpha^-_{V,2}}}.$$

We conclude that

$$\inf_{K_0} v_{\psi,R} \ge \inf_{K_0} \psi_1 v_{1,R}(r) - \sup_{K_0} |\tilde{v}_{\psi,R}| \ge \frac{c_2}{R^{\alpha^+_1}} - \frac{c_3}{R^{\alpha^+_2}}.$$

This completes the proof since $\alpha^+_{V,2} > \alpha^+_{V,1}$. □

Proof of Theorem* 4.16 *(ii). Let $\rho \ge 1$. Then Hardy's Inequality (B.4) implies that the form $\mathcal{E}_V$ satisfies the λ-property (A.4) with $\lambda(x) = \frac{1/4}{|x|^2 \log^2 |x|}$. Hence the extended Dirichlet space $\mathcal{D}(\mathcal{E}_V, \mathcal{C}^\rho_\Omega)$ is well defined (see Appendix A), and in particular, the Comparison Principle (Lemma A.3) is valid. The space $\tilde{D}^1_0(\mathcal{C}^2_\Omega)$ is larger then $D^1_0(\mathcal{C}^2_\Omega)$ (cf. [25,88]). In order to see this, for $\beta \in [0, 1]$ consider

$$w_\beta(r, \omega) := r^{\alpha_*} \log^\beta(r)\phi_{V,1}(\omega). \tag{4.32}$$

Clearly, $w_\beta \in C^\infty_{\mathrm{loc}}(\mathcal{C}^\rho_\Omega)$ but $\nabla w_\beta \notin L^2(\mathcal{C}^\rho_\Omega)$. Let $\theta(r) \in C^{0,1}[\rho, +\infty)$ be such that $0 \le \theta(r) \le 1$, $\theta(\rho) = 1$ and $\theta(r) = 0$ for $r \ge \rho + 1$. For the proof of Theorem 4.16 (ii) we need the following lemma.

LEMMA 4.22. *$w_\beta - \theta\phi_{V,1} \in \tilde{D}^1_0(\mathcal{C}^\rho_\Omega)$ for each $\beta \in [0, 1/2)$.*

PROOF. Define the cut-off function $\theta_R(r) \in C^{0,1}_c(\mathcal{C}^1_\Omega)$ by

$$\theta_R(r) := \begin{cases} 1, & 1 \le r \le R, \\ \frac{\log(R^2/r)}{\log R}, & R \le r \le R^2, \\ 0, & r \ge R^2. \end{cases}$$

Let $w_R := \theta_R(w_\beta - \theta\phi_{V,1})$. A direct computation shows that

$$\mathcal{E}_V(w_R, w_R) = \int_\rho^\infty \left|\nabla \left(\log^\beta(r)\theta_R(r)\right)\right|^2 r\,dr \le c_1 + c_2 \log^{2\beta-1}(R) \le c.$$

Hence $\mathcal{E}_V(w_{R_n}, w_{R_n})$ is a Cauchy sequence, for an appropriate choice of $R_n \to \infty$. Since $(w_{R_n}) \subset C^{0,1}_{\mathrm{loc}}(\mathcal{C}^\rho_\Omega)$ converges pointwise to the function v_β, the assertion follows. □

Now we are ready to sketch the proof of (4.26). We only mention main steps as the proofs are completely similar to the proofs for the lower bound of (4.25) (see [48] for details).

As before, fix a proper smooth subdomain $\Omega' \Subset \Omega$ and a function $0 \le \psi \in C_c^\infty(\Omega')$, $\psi \ne 0$. For $(r, \omega) \in \mathcal{C}_\Omega^1$ set

$$v_*(r, \omega) := c_* r^{\alpha_*} \phi_{V,1}(\omega),$$

where $c_* > 0$ chosen so that $v_*(1, \omega) = \phi_{V,1}(\omega)$. Clearly $v_* \in H^1_{\mathrm{loc}}(\mathcal{C}^1_\Omega)$ is a solution to (4.22) in $\mathcal{C}^1_\Omega$. Define v_ψ by

$$v_\psi := \psi_1 v_* + \sum_{k=2}^{\infty} \psi_k v_k, \tag{4.33}$$

where ψ_k are the Fourier coefficients of ψ as in (4.23) and $v_k = r^{\alpha^-_{V,k}} \phi_{V,k}$, $k \ge 2$. Thus $v_\psi(1, \omega) = \psi(\omega)$. The remaining part is completely similar to the proof of the lower bound of (4.25). We leave the details to the reader.

As before, for the upper bound we use the Phragmèn–Lindelöf type comparison principle, we need to construct the family of comparison functions.

Fix a subdomain $\Omega' \subseteq \Omega$ and a function $0 \le \psi \in C_c^\infty(\Omega')$, $\psi \ne 0$. Let $R \ge 4$. For $(r, \omega) \in \mathcal{C}_\Omega^{(1,R)}$ and $k \in \mathbb{N}$ define

$$w_{*,R}(r, \omega) := \eta_{*,R}(r) \phi_{V,1}(\omega), \quad \text{where } \eta_{*,R}(r) := \frac{\log(r)}{\log(R)} \left(\frac{r}{R}\right)^{\alpha_*}. \tag{4.34}$$

Let $\theta_R : [0, 1] \to \mathbb{R}$ be as in the proof of the upper bound of (4.25). Let

$$w_{\psi,R} := \psi_1\, w_{*,R} + \sum_{k=2}^{\infty} \psi_k\, w_{k,R},$$

where ψ_k are the Fourier coefficients of ψ as in (4.23). A direct computation shows that $w_{*,R}$ and $w_{k,R}$ $(k \ge 2)$, defined by (4.30) are solutions to the problems

$$\left(-\Delta - \frac{V(\omega)}{|x|^2}\right) w = 0, \qquad w - \theta_R \phi_{V,k} \in H_0^1(\mathcal{C}_\Omega^{(1,R)}). \tag{4.35}$$

The remaining part is completely similar to the proof of the upper bound of (4.25). We leave the details to the reader (see [48] for details). □

4.2. *Semi-linear equations*

In this section we turn to the problem of the existence of positive supersolutions to the semi-linear equation

$$\mathcal{L}u = c_0 |x|^{-\sigma} u^q, \quad c_0 > 0, \tag{4.36}$$

in cone-like (unbounded) domains. We start with a model case $\mathcal{L} = -\Delta$, then proceed to a more general case $\mathcal{L} = -\nabla \cdot |x|^A \cdot \nabla - B|x|^{A-2}$ with $A, B \in \mathbb{R}$, which allows for precise quantitative results for (4.36) and exhibits interesting phenomena. Finally, we study the case of general divergence-type operators $\mathcal{L} = -\nabla \cdot a \cdot \nabla$ with symmetric uniformly elliptic matrix a, where mostly only qualitative results for (4.36) (with $\sigma = 0$) are available.

The value of the critical exponent for the equation $-\Delta u = u^p$ in $\mathcal{C}_\Omega$ with $\Omega \subseteq S^{N-1}$ satisfying mild regularity assumptions was first established by Bandle and Levine [7] (see also [6]). They reduce the problem to an ODE by averaging over Ω. The nonexistence of positive solutions without any smoothness assumptions on Ω has been proved by Berestycki *et al.* [9] by means of a proper choice of a test function.

4.2.1. *Laplacian*

In this subsection we study the existence and nonexistence of positive solutions and supersolutions to the equation

$$-\Delta u = \frac{c_0}{|x|^\sigma} u^q \quad \text{in } \mathcal{C}_\Omega^\rho, \quad c_0 > 0. \tag{4.37}$$

Let $\lambda_1 = \lambda_1(\Omega) \geq 0$ denote the principal eigenvalue of the Dirichlet Laplace–Beltrami operator $-\Delta_\omega$ on Ω. Let $\alpha_1^+ \geq 0 > \alpha_1^-$ denote the roots of the quadratic equation

$$\alpha(\alpha + N - 2) = \lambda_1.$$

The mentioned result for (4.37) in [7,9] reads as follows.

THEOREM 4.23. *Let $q > 1$. Let $\Omega \subseteq S^{N-1}$ be a domain. Then the equation $-\Delta u = u^q$ has no positive supersolutions in $\mathcal{C}_\Omega^1$ if and only if $q \leq 1 - \frac{2}{\alpha_1^-}$.*

In order to formulate the main result on the existence of positive supersolutions to (4.37), as before, we introduce the critical line and the nonexistence set on the plane (q, σ). The critical line $\sigma = \Lambda(q)$ on the (q, σ)-plane is defined by

$$\Lambda(q) := \min\{\alpha_1^-(q-1) + 2, \alpha_1^+(q-1) + 2\} \quad (q \in \mathbb{R}),$$

and the nonexistence set

$$\mathcal{N} = \{(q, \sigma) \in \mathbb{R}^2 \setminus \{1, 2\} : \textit{equation (4.37) has no positive supersolutions}\}.$$

The main result of this subsection can be written now as follows.

THEOREM 4.24. $\mathcal{N} = \{\sigma \leq \Lambda(q)\}$.

REMARK 4.25. If $\sigma = 2$ and $q = 1$ equation (4.37) becomes a linear equation with the potential $c_0|x|^{-2}$, which has a positive supersolution if and only if $c_0 \leq \frac{(N-2)^2}{4} + \lambda_1$.

Applying the Kelvin transformation $y = y(x) = \frac{x}{|x|^2}$ we see that if u is a positive solution to (4.37) in $\mathcal{C}_\Omega^1$ then $\hat{u}(y) = |y|^{2-N} u(x(y))$ is a positive solution to

$$-\Delta \hat{u} = \frac{c_0}{|y|^s} \hat{u}^q \quad \text{in } \widehat{\mathcal{C}}_\Omega^1, \tag{4.38}$$

where $s = (N+2) - q(N-2) - \sigma$ and $\widehat{\mathcal{C}}_\Omega^1 := \{(r, \omega) \in \mathbb{R}^N : \omega \in \Omega,\ 0 < r < 1\}$.

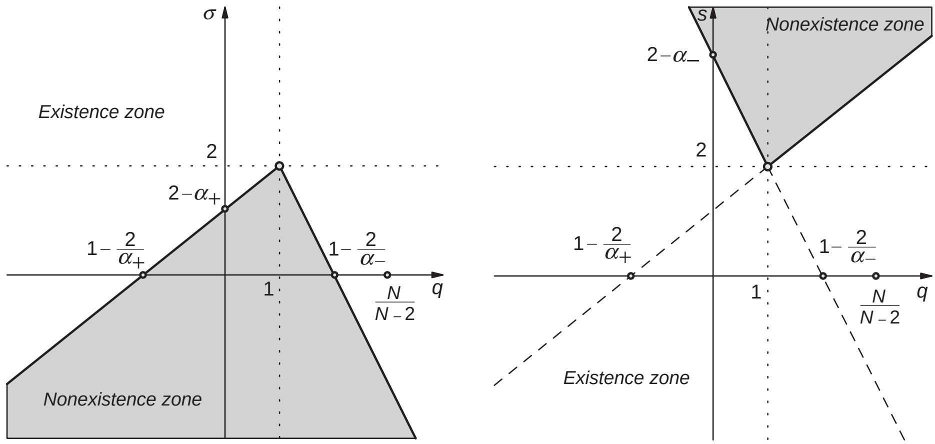

Fig. 4. Existence and nonexistence zones for equations (4.37) (left) and (4.38) (right).

We define the critical line for (4.38) $s = \widehat{\Lambda}(q)$ on the (q, σ)-plane by

$$\widehat{\Lambda}(q) := \max\{\alpha_1^-(q-1)+2, \alpha_1^+(q-1)+2\} \quad (q \in \mathbb{R}),$$

and the nonexistence set

$$\widehat{\mathcal{N}} = \{(q,\sigma) \in \mathbb{R}^2 \setminus \{1,2\} : \textit{equation (4.38) has no positive supersolutions}\}.$$

Theorem 4.24 via the Kelvin transform yields the next result.

THEOREM 4.26. $\widehat{\mathcal{N}} = \{s \geq \widehat{\Lambda}(q)\}$.

REMARK 4.27. The diagram of the existence and nonexistence zones for equations 4.37 and 4.38 on the plane (q, σ) is shown on Figure 4. Observe that in view of the scaling invariance of the Laplacian the critical line does not depend on $\rho > 0$.

The rest of this subsection contains the proof of Theorem 4.24. In the proof of Theorem 4.24 we distinguish between the superlinear case $q \geq 1$ and sublinear case $q < 1$.

Proof of Theorem 4.24 (Superlinear case). We distinguish the subcritical and critical cases.

Nonexistence. Subcritical case $\sigma < \alpha_1^-(q-1)+2$. Let $w > 0$ be a supersolution to (4.37) in $\mathcal{C}^1_\Omega$. Then w is a supersolution to the linear equation

$$-\Delta w = 0 \quad \text{in } \mathcal{C}^1_\Omega. \tag{4.39}$$

Choose a proper subdomain $\Omega' \Subset \Omega$. Then, by Theorem 4.16, there exists $c > 0$ such that

$$m_w(R, \Omega') \geq c R^{\alpha_1^-} \quad (R \geq 2).$$

Linearizing (4.37) and using the bound above, we conclude that $w > 0$ is a supersolution to

$$-\Delta w - \frac{V(x)}{|x|^2} w = 0 \quad \text{in } \mathcal{C}^2_{\Omega'}, \tag{4.40}$$

where $V(x) := c_0 w^{q-1}|x|^{2-\sigma}$ satisfies

$$V(x) \geq c^{q-1}|x|^{\alpha^-(q-1)+(2-\sigma)} \quad \text{in } \mathcal{C}^2_{\Omega'}.$$

Then the assertion follows by Corollary 4.15 or by Lemma 3.28.

Critical case $\sigma = \alpha_1^-(q-1)+2$. Let $w > 0$ be a supersolution to (4.37) in $\mathcal{C}^1_\Omega$. Then arguing as in the previous case we conclude that w is a supersolution to (4.40) with $V(x) := c_0 w^{q-1}|x|^{2-\sigma} \geq \delta$ in $\mathcal{C}^2_{\Omega'}$, for some $\delta > 0$. Thus w is a supersolution to the linear equation

$$-\Delta w - \frac{W(\omega)}{|x|^2} w = 0 \quad \text{in } \mathcal{C}^2_{\Omega}, \tag{4.41}$$

where $W(\omega) := \varepsilon\chi_{\Omega'}$, with a fixed $\varepsilon \in (0, \delta]$. Choosing $\varepsilon < C_H + \lambda_1$ and applying Theorem 4.16 to (4.41) we conclude that

$$m_R(u, \Omega') \geq cR^{\tilde{\alpha}}, \qquad R \geq 4,$$

with $\alpha_1^- < \tilde{\alpha} < \alpha_*$. Therefore $w > 0$ is a supersolution to

$$-\Delta w - \frac{\tilde{V}(x)}{|x|^2} w = 0 \quad \text{in } \mathcal{C}^4_{\Omega'},$$

where $\tilde{V}(x) := c_0 w^{q-1}|x|^{2-\sigma} \geq c^{q-1}|x|^{\tilde{\alpha}(q-1)+(2-\sigma)}$ in $\mathcal{C}^4_{\Omega'}$, with $\tilde{\alpha}(q-1)+(2-\sigma) > 0$. Then the assertion follows from Corollary 4.15 or from Lemma 3.28.

Existence. Let $q > 1$ and $\sigma > \alpha_1^-(q-1)+2$. Choose $\alpha \in (\alpha_1^-, \alpha_1^+)$ such that $\alpha \leq \frac{\sigma-2}{q-1}$. Then one can verify directly that the functions

$$w := \tau r^\alpha \phi_1(\omega)$$

are supersolutions to (4.37) in $\mathcal{C}^1_\Omega$ for sufficiently small $\tau > 0$.

Proof of Theorem* 4.24 *(Sublinear case). We start with the following a priori estimate which is a complete analogue of Lemma 1.4.

LEMMA 4.28. *Let $q < 1$. Let $w > 0$ be a supersolution to (4.37) in $\mathcal{C}^1_\Omega$. Then for each proper subdomain $\Omega' \Subset \Omega$ there exists $C > 0$ such that*

$$m_R(w, \Omega') \geq C\, R^{\frac{2-\sigma}{1-q}}, \qquad R > 2. \tag{4.42}$$

PROOF. Let $w > 0$ be a supersolution to (4.37). Then $-\Delta w \geq 0$ in $\mathcal{C}^1_\Omega$ and the weak Harnack inequality for negative exponents (see, e.g. [28, Theorem 8.18]) is applicable to w. Testing the inequality $-\Delta w \geq c_0|x|^{-\sigma}w^q$ by $\frac{w^2}{\theta}$ we obtain

$$\int_{\mathcal{C}^1_\Omega} |\nabla\varphi|^2 dx \geq c_0 \int_{\mathcal{C}^1_\Omega} |x|^{-\sigma} w^{q-1}\varphi^2 dx, \quad \forall \varphi \in H^1_c(\mathcal{C}^1_\Omega) \cap H^\infty_c(\mathcal{C}^1_\Omega). \tag{4.43}$$

Fix a proper subdomain $\Omega' \Subset \Omega$. Choose $\psi \in C^\infty_c(\Omega)$ such that $\psi = 1$ on Ω'. Choose $\theta_R(r) \in C^{0,1}_c(1,+\infty)$ such that $0 \leq \theta_R \leq 1$, $\theta_R = 1$ for $r \in [R/2, R]$, $Supp(\theta_R) = [R/4, 2R]$ and $|\nabla\theta_R| < c/R$. Then

$$\int_{\mathcal{C}^1_\Omega} |\nabla(\theta_R\psi)|^2 dx \leq cR^{N-2}. \tag{4.44}$$

On the other hand,

$$\int_{\mathcal{C}^1_{\Omega'}} |x|^{-\sigma} w^{q-1}(\theta_R\psi)^2\, dx \geq R^{-\sigma} \int_{\mathcal{C}^{(R/2,R)}_{\Omega'}} w^{q-1} dx. \tag{4.45}$$

Combining (4.43), (4.44) and (4.45) we derive

$$cR^{\sigma-2} \geq R^{-N} \int_{\mathcal{C}^{(R/2,R)}_{\Omega'}} w^{q-1} dx.$$

Using the weak Harnack inequality we obtain

$$cR^{\frac{\sigma-2}{1-q}} \geq \left(R^{-N} \int_{\mathcal{C}^{(R/2,R)}_{\Omega'}} w^{-(1-q)}\, dx\right)^{\frac{1}{1-q}} \geq c_1 \left(\inf_{\mathcal{C}^{(5R/8,7R/8)}_{\Omega'}} w\right)^{-1}.$$

The assertion follows. □

Now we are ready to prove the nonexistence part of Theorem 4.24.

Nonexistence. *Subcritical case* $\sigma < \alpha_1^-(q-1)+2$.
Let $w > 0$ be a supersolution to (4.37). Then by Theorem 4.16 we conclude that

$$m_R(w, \Omega') \leq cR^{\alpha_1^+}, \quad R > 2. \tag{4.46}$$

This contradicts (4.42).
Critical case $\sigma = \alpha_1^-(q-1)+2$. Let $w > 0$ be a supersolution to (4.37). In the same way as in Lemma 3.6 there exists a solution to (4.37) (see Theorem 2.9). Without risk of

confusion we still denote it by w. Choose a proper subdomain $\Omega' \Subset \Omega$. Linearizing (4.37) and using the upper bound (4.42) we conclude that $w > 0$ is a solution to

$$-\Delta w - \frac{V(x)}{|x|^2} w = 0 \quad \text{in } \mathcal{C}^1_{\Omega}, \tag{4.47}$$

where $V(x) := c_0|x|^{2-\sigma} w^{q-1}$ satisfies $V(x) \le c_1$ in $\mathcal{C}^{\rho_1}_{\Omega'}$, with a fixed $\rho_1 > 2$. This implies, in particular, that w satisfies the strong Harnack inequality with r-independent constants. More precisely, for a given subdomain $\Omega'' \Subset \Omega'$ one has

$$M_R(w, \Omega'') \le C_s\, m_R(w, \Omega''), \qquad R > 2\rho, \tag{4.48}$$

where $C_s = C_s(\Omega'') > 0$ does not depend on R. Using (4.48) and the upper bound (4.46) we conclude that

$$M_R(w, \Omega'') \le c_2 R^{\alpha_1^+}, \qquad R > 2\rho. \tag{4.49}$$

This implies that $V(x) \ge \delta$ in $\mathcal{C}^{\rho_2}_{\Omega''}$, for some $\delta > 0$ and $\rho_2 > 2\rho_1$. Hence $w > 0$ is a supersolution to the linear equation

$$-\Delta w - \frac{W_\varepsilon(\omega)}{|x|^2} w = 0 \quad \text{in } \mathcal{C}^{\rho_2}_{\Omega}, \tag{4.50}$$

where $W_\varepsilon(\omega) := \varepsilon \chi_{\Omega''}$, with a fixed $\varepsilon \in (0, \delta]$. Choosing $\varepsilon > 0$ small enough and applying Theorem 4.16 to (4.50) we conclude that

$$m_R(w, \Omega') \le c_2 R^{\tilde{\alpha}}, \tag{4.51}$$

with $\tilde{\alpha} < \alpha_1^+$. Now (4.51) contradicts the upper bound (4.42).

Existence. For $\sigma > 2$ positive supersolutions to (4.37) exist in the exterior of a ball (see Section 3.1), and therefore in a cone-like domain. Let $\sigma \le 2$, $q < 1 - \frac{2-\sigma}{\alpha_+}$.

Assume that $0 \le q < 1$. Choose $\alpha \in (\alpha_1^-, \alpha_1^+)$ such that $\alpha \ge \frac{\sigma-2}{q-1}$. Then there exists a unique bounded positive solution to the problem

$$-\Delta_\omega \phi - \alpha(\alpha + N - 2)\phi = 1, \qquad \phi \in H^1_0(\Omega).$$

Further, a direct computation verifies that the functions

$$w := \tau r^\alpha \phi(\omega)$$

are supersolutions to (4.37) in $\mathcal{C}^1_\Omega$ for a sufficiently large $\tau > 0$.

Now assume that $q < 0$. Choose α as above, so there exists a unique bounded positive solution of the problem

$$-\Delta_\omega \bar{\phi} - \alpha(\alpha + N - 2)(\bar{\phi} + 1) = 1, \qquad \bar{\phi} \in H^1_0(\Omega).$$

Then

$$w := \tau r^\alpha \bar{\phi}(\omega)$$

are supersolutions to (4.37) in $\mathcal{C}^1_\Omega$ for sufficiently large $\tau > 0$. □

4.2.2. *Laplacian with Hardy potential*
In this subsection we discuss the existence and nonexistence of positive supersolutions to the singular semi-linear elliptic equation with critical potential

$$-\nabla\cdot(|x|^A\nabla u) - B|x|^{A-2}u = C|x|^{A-\sigma}u^q \quad \text{in } \mathcal{C}^\rho_\Omega. \tag{4.52}$$

Here $A, B \in \mathbb{R}$, $C > 0$ and $(q,\sigma) \in \mathbb{R}^2$.

The results of this subsection are taken from [48]. Here we only formulate main results and outline the ideas used in the proofs, which are an appropriate combination of ideas described in Sections 3.1 and 4.2.1.

Below we denote $\tilde{C}_H := \frac{(2-N-A)^2}{4}$, while $\lambda_1 = \lambda_1(\Omega) \geq 0$ denotes the principal Dirichlet eigenvalue of the Laplace–Beltrami operator $-\Delta_\omega$ on Ω.

First, we formulate the result in the special linear case.

THEOREM 4.29. *Let $(q,\sigma) = (1,2)$. Then equation (4.52) has no positive supersolutions if and only if $B + C > \tilde{C}_H + \lambda_1$.*

If $B \leq \tilde{C}_H + \lambda_1$ then the quadratic equation

$$\gamma(\gamma + N - 2 + A) = \lambda_1 - B \tag{4.53}$$

has real roots, denoted by $\gamma^- \leq \gamma^+$. Note that if $B = \tilde{C}_H + \lambda_1$, then $\gamma^\pm = (2-N-A)/2$.

For $B \leq \tilde{C}_H + \lambda_1$ we introduce the critical line $\Lambda_*(q, A, B, \Omega)$ on the (q,σ)-plane

$$\Lambda_*(q, A, B, \Omega) := \min\{\gamma^-(q-1) + 2, \gamma^+(q-1) + 2\} \quad (q \in \mathbb{R}),$$

and the nonexistence set

$$\mathcal{N} = \{(q,\sigma) \in \mathbb{R}^2 \setminus \{1,2\} : \textit{equation } (4.52) \textit{ has no positive supersolutions}\}.$$

(The notation $\gamma, \gamma^+, \gamma^-$, the critical line and the nonexistence set are restricted to this subsection.)

The main result of this subsection reads as follows.

THEOREM 4.30. *The following assertions are valid.*

(i) *Let $B < \tilde{C}_H + \lambda_1$. Then $\mathcal{N} = \{\sigma \leq \Lambda_*(q)\}$.*

(ii) *Let $B = \tilde{C}_H + \lambda_1$. Then*
$$\{\sigma < \Lambda_*(p)\} \cup \{\sigma = \Lambda_*(q),\ q \geq -1\} \subseteq \mathcal{N} \subseteq \{\sigma \leq \Lambda_*(q)\}.$$
If $\Omega = S^{N-1}$ then $\mathcal{N} = \{\sigma < \Lambda_*(q)\} \cup \{\sigma = \Lambda_*(q),\ q \geq -1\}$.

The next lemma, the proof of which is straightforward, shows that a simple transformation allows one to reduce equation (4.52) to the uniformly elliptic case $A = 0$.

LEMMA 4.31. *The function u is a (super-) solution to equation (4.52) if and only if $w(x) = |x|^{\frac{A}{2}}u(x)$ is a (super-) solution to the equation*

$$-\Delta w - \frac{\mu}{|x|^2}w = \frac{C}{|x|^s}w^q \quad \textit{in } \mathcal{C}^\rho_\Omega, \tag{4.54}$$

where $\mu = B - \frac{A}{2}(\frac{A}{2} + N - 2)$ and $s = \sigma + \frac{A}{2}(q-1)$.

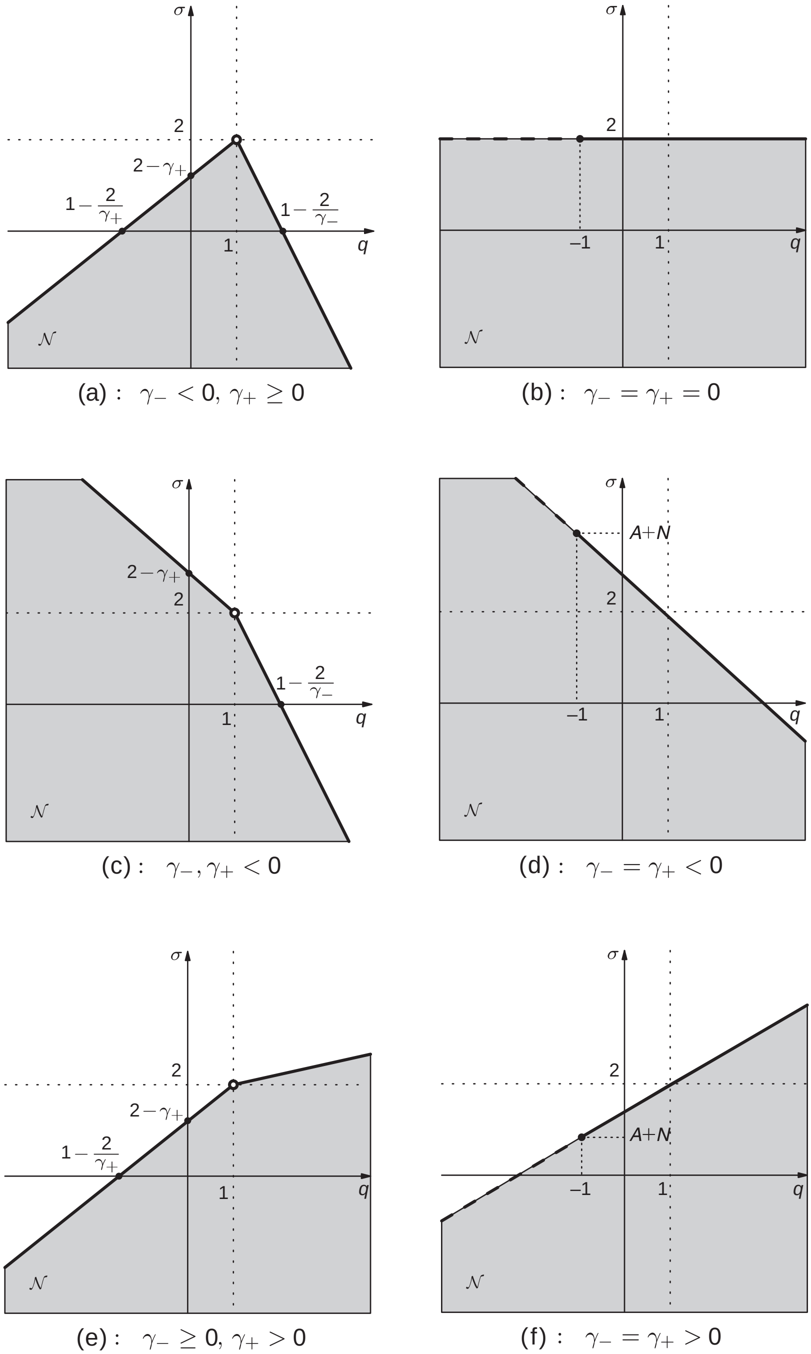

Fig. 5. The nonexistence set $\mathcal{N}$ of equation (4.52) for typical values of γ^- and γ^+.

If $\mu \le C_H + \lambda_1$ then the quadratic equation

$$\alpha(\alpha + N - 2) = \lambda_1 - \mu, \tag{4.55}$$

has real roots, denoted by $\alpha^- \le \alpha^+$. If $\mu = C_H + \lambda_1$ then $\alpha^\pm = \frac{2-N}{2}$. In this case we write $\alpha_* := \frac{2-N}{2}$ for convenience. As before, we introduce the critical line $\sigma = \Lambda(q)$ with

$$\Lambda = \Lambda(q, \mu, \Omega) := \min\{\alpha^-(q-1) + 2, \alpha^+(q-1) + 2\} \quad (q \in \mathbb{R}),$$

and the nonexistence set

$$\mathcal{N} = \{(q, s) \in \mathbb{R}^2 \setminus \{1, 2\} : \textit{equation } (4.54) \textit{ has no positive supersolutions}\}.$$

Theorem 4.30 is a direct consequence of the next theorem on the equation (4.54).

THEOREM 4.32. *The following assertions are valid:*
(i) *Let* $\mu < C_H + \lambda_1$ *Then* $\mathcal{N} = \{\sigma \le \Lambda(q)\}$.
(ii) *Let* $\mu = C_H + \lambda_1$. *Then*
$$\{\sigma < \Lambda(q)\} \cup \{\sigma = \Lambda(q), q \ge -1\} \subseteq \mathcal{N} \subseteq \{\sigma \le \Lambda(q)\}.$$
If $\Omega = S^{N-1}$ *then* $\mathcal{N} = \{\sigma < \Lambda(q)\} \cup \{\sigma = \Lambda(q), q \ge -1\}$.

The proof of part (i) is almost the same as the proof of Theorem 4.24. The part (ii) is proved in the same way as Theorem 3.1(ii) with the use of the estimates for the case of the cone obtained in Theorem 4.16.

REMARK 4.33. In the case of proper domains $\Omega \Subset S^{N-1}$, the existence (or nonexistence) of positive supersolutions to (4.54) with $q < -1$ and $s = \alpha_*(q-1) + 2$ becomes a more delicate issue. This question remains open.

4.2.3. *Divergence-type equations*

In this subsection we study the existence and nonexistence of positive solutions and supersolutions to a semi-linear second-order divergence-type elliptic equation

$$-\nabla \cdot a \cdot \nabla u = u^q \quad \text{in } \mathcal{C}_\Omega^\rho. \tag{4.56}$$

We assume throughout the subsection that the matrix $a = (a_{ij}(x))_{i,j=1}^N$ is measurable and uniformly elliptic, i.e. there exists an ellipticity constant $\nu = \nu(a) > 0$ such that

$$\nu^{-1}|\xi|^2 \le \sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \le \nu|\xi|^2, \quad \text{for all } \xi \in \mathbb{R}^N \text{ and almost all } x \in \mathcal{C}_\Omega. \tag{4.57}$$

We also assume in this section that the matrix a is **symmetric**. This is a technical assumption, and we believe that all the results hold without it. Due to the nature of the cone-like domains there is no possibility to determine the exact values of critical exponents for equation (4.56) for the whole class of uniformly elliptic matrices (even with the same ellipticity constants), let alone the critical lines which are the interplay between q and the exponent $-\sigma$ in the polynomial factor in front of the nonlinear term. That is why we do not put this factor in (4.56). For the same reason a sensible setting is to look for positive solutions at infinity and not in $\mathcal{C}^1_\Omega$. The nonexistence results will then apply to $\mathcal{C}^\rho_\Omega$ with any $\rho > 0$, and the existence will be proved for a sufficiently large ρ.

Recall that the *critical exponents* to equation (4.56) are defined by

$$\begin{aligned} q^* &= q^*(a, \mathcal{C}^\rho_\Omega) \\ &= \inf\{q > 1 : (4.56) \text{ has a positive supersolution at infinity in } \mathcal{C}^\rho_\Omega\}, \\ q_* &= q_*(a, \mathcal{C}^\rho_\Omega) \\ &= \sup\{q < 1 : (4.56) \text{ has a positive supersolution at infinity in } \mathcal{C}^\rho_\Omega\}. \end{aligned}$$

The next proposition whose proof is straightforward shows that the above values are well defined.

PROPOSITION 4.34. *Let $u > 0$ be a supersolution to the inequality*

$$-\nabla \cdot a \cdot \nabla u = u^p \quad \text{in } \mathcal{C}^\rho_\Omega.$$

Then for $q > p > 1$ or $q < p < 1$ the function $v = u^{\frac{p-1}{q-1}}$ is a positive supersolution to

$$-\nabla \cdot a \cdot \nabla u = u^q \quad \text{in } \mathcal{C}^\rho_\Omega.$$

In order to formulate the results we separate the superlinear case $q > 1$, where we follow [40] and the sublinear case which is taken from [56].

Superlinear case $q > 1$. The following theorem shows that the value of the critical exponent depends on the matrix a.

THEOREM 4.35. *Let $\Omega \subset S^{N-1}$ be a domain such that $\lambda_1(\Omega) > 0$. Then for any $q \in (1, \frac{N}{N-2})$ there exists a uniformly elliptic matrix a_q such that $q^*(a_q, \mathcal{C}_\Omega) = q$.*

REMARK 4.36. The matrix a_q can be constructed in such a way that (4.56) either *has* or *has no* positive supersolutions at infinity in $\mathcal{C}_\Omega$ in the critical case $q = q^*(a_q, \mathcal{C}_\Omega)$, see Remark 4.42 for details.

The next theorem, one of the main results of this subsection, asserts that equation (4.56) with arbitrary uniformly elliptic measurable matrix a on a "nontrivial" cone-like domain always admits a "nontrivial" critical exponent.

THEOREM 4.37. *Let $\Omega \subseteq S^{N-1}$ be a domain and a be a uniformly elliptic matrix. Then $q^*(a, \mathcal{C}_\Omega) > 1$. If the interior of $S^{N-1} \setminus \Omega$ is nonempty then $q^*(a, \mathcal{C}_\Omega) < \frac{N}{N-2}$.*

REMARK 4.38. It is not difficult to see that in the case $N = 2$ equation (4.56) has no positive solutions outside a ball for any $q > 1$. However, when $\mathcal{C}_\Omega$ is a "nontrivial" cone-like domain in $\mathbb{R}^2$, that is $S^1 \setminus \Omega \neq \emptyset$, then all the results of the paper remain true with minor modifications of some proofs.

Sublinear case $q < 1$. As in the superlinear case, the value of the critical exponent essentially depends on the matrix a and cannot be explicitly controlled without further restrictions on the properties of a, which is seen from the next theorem.

THEOREM 4.39. *Let $\Omega \Subset S^{N-1}$ be a proper subdomain. Then for any $q \in (-\infty, 1)$ there exists a uniformly elliptic symmetric matrix a_q such that $q_*(a_q, \mathcal{C}_\Omega) = q$.*

The main result of this subsection in the sublinear case says that similarly to the Laplace equations, sublinear divergence-type equations on proper cone-like domain admit a nontrivial critical exponent q_* (the lower critical exponent).

THEOREM 4.40. *Let $\Omega \Subset S^{N-1}$ be a proper subdomain. Then $q_*(a, \mathcal{C}_\Omega) \in (-\infty, 1)$.*

REMARK 4.41. We conjecture that for both superlinear case $1 < q < \frac{N}{N-2}$ and the sublinear case $q < 1$ the following should be true: for a given uniformly elliptic matrix a and a given q there exist Ω and Ω' such that equation (4.56) has positive supersolutions at infinity in $\mathcal{C}_\Omega$ and has no positive supersolutions in $\mathcal{C}_{\Omega'}$.

***Proofs of Theorems* 4.35*,* 4.37*,* 4.39 *and* 4.40**

PROOF OF THEOREM 4.35. Let $\Omega \subset S^{N-1}$ be a domain such that $\lambda_1 = \lambda_1(\Omega) > 0$. Define the operator L_d by

$$L_d = -\frac{\partial^2}{\partial r^2} - \frac{N-1}{r}\frac{\partial}{\partial r} - \frac{d(r)}{\lambda_1}\frac{1}{r^2}\Delta_\omega, \tag{4.58}$$

where $d(r)$ is measurable and squeezed between two positive constants. Then L_d is a divergence-type uniformly elliptic operator $-\nabla \cdot a_d \cdot \nabla$ (see, e.g., [90]).

Set $d(r) \equiv \alpha(\alpha + N - 2)$ with $\alpha < 2 - N$. Following the lines of the proof of Theorem 4.24 we conclude that $q^*(a_d, \mathcal{C}_\Omega) = 1 - 2/\alpha$. Clearly for any given $q \in (1, \frac{N}{N-2})$, one can choose α such that $q^*(a_d, \mathcal{C}_\Omega) = q$. □

REMARK 4.42. In the above proof equation (4.56) has no positive supersolutions at infinity in $\mathcal{C}_\Omega$ in the critical case $q = q^*(a_d, \mathcal{C}_\Omega)$. Next we give an example of equation (4.56) with a positive supersolution at infinity in the critical case.

Let $\Omega \subset S^{N-1}$ be smooth and $L_{\tilde{d}}$ be as in (4.58) with

$$\tilde{d}(r) = \alpha(\alpha + N - 2) + \frac{2 - N - 2\alpha}{\log(r)} + \frac{2}{\log^2(r)},$$

where $\alpha < 2 - N$. For large enough $R \gg 1$ the operator $L_{\tilde{d}} = -\nabla \cdot a_{\tilde{d}} \cdot \nabla$ is uniformly elliptic on $\mathcal{C}_\Omega^R$. Let $\phi_1 > 0$ be the principal Dirichlet eigenfunction of $-\Delta_\omega$, corresponding to λ_1. Direct computation shows that the function

$$v_{\phi_1} := \frac{r^\alpha}{\log(r)}\phi_1$$

is a solution to the equation

$$L_{\tilde{d}} v = 0 \quad \text{in } \mathcal{C}_\Omega^R. \tag{4.59}$$

Since Ω is smooth, the Hopf lemma implies that v_{ϕ_1} is a minimal positive solution to (4.59) in $\mathcal{C}_\Omega^R$. Following the lines of the proof of Theorem 4.23, subcritical case, we conclude that $q^*(a_{\tilde{d}}, \mathcal{C}_\Omega) = 1 - 2/\alpha$. On the other hand, one can readily verify that $u = r^\alpha \phi_1$ is a positive supersolution to (4.56) in the critical case $q = 1 - 2/\alpha$.

Note that the value of the critical exponent for $L_{\tilde{d}}$ is the same as L_d due to the fact that $\lim_{r\to\infty}(\tilde{d}(r) - d(r)) = 0$. However the rate of convergence is not sufficient to guarantee the equivalence of the corresponding minimal positive solutions (see, e.g. [5,66] for the related estimates of Green's functions). This explains the nature of the different behavior of the nonlinear equations (4.56) at the critical value of q.

PROOF OF THEOREM 4.37. The next lemma is again the key tool in our proofs of nonexistence of positive solutions to nonlinear equation (4.56). It is a direct consequence of Theorem 2.13 and the sharpness of the Hardy inequality on cone-like domains (see Section B).

LEMMA 4.43 (Nonexistence lemma). *Let* $0 \le V \in L^1_{\mathrm{loc}}(\mathcal{C}_\Omega^\rho)$ *satisfy*

$$|x|^2 V(x) \to \infty \quad \text{as } x \in \mathcal{C}_{\Omega'}^\rho \text{ and } |x| \to \infty \tag{4.60}$$

for a subdomain $\Omega' \subseteq \Omega$. *Then the equation*

$$-\nabla \cdot a \cdot \nabla u - Vu = 0 \quad \text{in } \mathcal{C}_\Omega^\rho \tag{4.61}$$

has no nontrivial nonnegative supersolutions.

First we show that for any domain $\Omega \subseteq S^{N-1}$ one has $q^*(a, \mathcal{C}_\Omega) > 1$. Then we prove the second part of Theorem 4.37, saying that if the complement of Ω has nonempty interior then $q^*(a, \mathcal{C}_\Omega) < \frac{N}{N-2}$.

The lower bound (4.3) allows us to prove nonexistence of positive solutions to (4.56) exactly by the same argument as was used in the proof of Theorem 4.23 in the subcritical case.

The following two propositions form the proof of Theorem 4.37.

PROPOSITION 4.44 (Nonexistence). *Let* $\Omega \subseteq S^{N-1}$ *be a domain. Then* $q^*(a, \mathcal{C}_\Omega) \ge 1 - 2/\alpha$, *where* $\alpha \le 2 - N$ *is the exponent in the lower bound* (4.3).

PROOF. Assume that $u \ge 0$ is a supersolution to (4.56) in $\mathcal{C}_\Omega^\rho$ with exponent $q < 1 - 1/\alpha$. By Lemma 2.6 and (4.3) we conclude that for any subdomain $\Omega' \subset \Omega$ there exists $c = c(\Omega) > 0$ such that

$$u \ge c|x|^\alpha \quad \text{in } \mathcal{C}_{\Omega'}^{2\rho+2}.$$

Therefore u is a supersolution to

$$-\nabla \cdot a \cdot \nabla u = Vu \quad \text{in } \mathcal{C}_{\Omega'}^{2\rho+2},$$

where $V(x) := u^{q-1}(x)$ satisfies the inequality

$$V(x) \ge c'|x|^{\alpha(q-1)} \quad \text{in } \mathcal{C}_{\Omega'}^{2\rho+2}$$

with $\alpha(q-1) > -2$. Then Lemma 4.43 implies that $u \equiv 0$ in $\mathcal{C}_\Omega^\rho$. Since $\alpha > 0$ does not depend on ρ, we conclude that $q^*(a, \mathcal{C}_\Omega) \geq 1 - 1/\alpha$. □

PROPOSITION 4.45 (Existence). *Let $\Omega \subset S^{N-1}$ be a domain such that $S^{N-1} \setminus \Omega$ has nonempty interior. Then $q^*(a, \mathcal{C}_\Omega) \leq 1 - 2/\beta$, where $\beta < 2 - N$ is from the upper bound (4.10).*

PROOF. Fix $q > q_0 = 1 - 2/\beta$ and set $\delta = q - q_0$. Let $w_\psi > 0$ be a minimal positive solution in $\mathcal{C}_\Omega^1$ to

$$-\nabla \cdot a \cdot \nabla w - \epsilon W w = 0 \quad \text{in } \mathcal{C}_\Omega,$$

where $W(x) = \frac{1}{(|x|^2 \log^2 |x|) \vee 1}$, $\epsilon > 0$ is from Lemma 4.4. Then by (4.10) for some $\bar\tau = \bar\tau(\delta) > 0$ small enough the function $\bar\tau w_\psi$ satisfies

$$(\bar\tau w_\psi)^{q-1} \leq \bar\tau^{q-1}(c|x|^\beta)^{p-1} \leq \frac{\bar\tau^{q-1} c_1}{|x|^{2+\delta|\beta|}} \leq \frac{\epsilon}{|x|^2 \log^2(|x|+2)} = \epsilon W(x) \quad \text{in } \mathcal{C}_\Omega^1.$$

Therefore

$$-\nabla \cdot a \cdot \nabla(\bar\tau w_\psi) = \epsilon W(\bar\tau w_\psi) \geq (\bar\tau w_\psi)^{q-1}(\bar\tau w_\psi) = (\bar\tau w_\psi)^q \quad \text{in } \mathcal{C}_\Omega^1,$$

that is $\bar\tau w_\psi > 0$ is a supersolution to (4.56) in $\mathcal{C}_\Omega^1$. □

PROOF OF THEOREM 4.39. The argument is exactly the same as in the proof of Theorem 4.35. □

PROOF OF THEOREM 4.40 (Nonexistence). We start with the following standard estimate on supersolutions to (4.37).

LEMMA 4.46. *Let $u > 0$ be a supersolution to (4.37). Then for any subdomain $\Omega' \Subset \Omega$ there exists $c = c(\Omega')$ such that*

$$m_R(u, \Omega') \geq c R^{\frac{2}{1-q}} \quad (R \gg 1). \tag{4.62}$$

We omit the proof as the same Keller–Osserman type of estimates was discussed before. Lemma 4.46 shows that a polynomial upper bound on positive supersolutions to

$$-\nabla \cdot a \cdot \nabla w = 0 \quad \text{in } \mathcal{C}_\Omega^\rho \tag{4.63}$$

implies an upper bound on the critical exponent $q_*(a, \mathcal{C}_\Omega^\rho)$.

PROPOSITION 4.47. *Assume there exists a subdomain $\Omega' \subseteq \Omega$ and $\alpha > 0$ such that any supersolution $w > 0$ to (4.63) satisfies*

$$m_R(u, \Omega') \leq c R^\alpha \quad (R \gg \rho). \tag{4.64}$$

Then $q_*(a, \mathcal{C}_\Omega^\rho) \leq 1 - 2/\alpha$.

PROOF. Fix $q > 1 - 2/\alpha$. Assume $u > 0$ is a positive supersolution to (4.37). Hence u is a positive supersolution to (4.63). Then (4.62) is incompatible with (4.63), a contradiction. □

Combining Proposition 4.47 and Lemma 4.1, we conclude that for any subdomain $\Omega \subseteq S^{N-1}$ and a uniformly elliptic matrix a there exists $\alpha = \alpha(a,\Omega) > 0$ such that $q_*(a,\mathcal{C}_\Omega) \leq 1 - 2/\alpha$. This completes the nonexistence part of the proof of Theorem 4.40. □

PROOF OF THEOREM 4.40 (Existence). Consider the linear equation

$$-\nabla \cdot a \cdot \nabla w - \frac{\epsilon}{|x|^2 \log^2 |x|} w = 0 \quad \text{in } \mathcal{C}^\rho_\Omega, \tag{4.65}$$

where $\epsilon > 0$ will be specified later. We show that a lower bound on positive supersolutions to (4.65) implies an upper bound on the critical exponent $q_*(a,\mathcal{C}^\rho_\Omega)$.

PROPOSITION 4.48. *Suppose there exists $\epsilon > 0$ and $\rho > 1$ such that there exists a supersolution $w > 0$ to equation (4.65) satisfying*

$$w \geq c|x|^\beta \quad \textit{in } \mathcal{C}^\rho_\Omega, \tag{4.66}$$

with $\beta > 0$. Then $q_*(a,\mathcal{C}^1_\Omega) \geq 1 - 2/\beta$.

PROOF. Fix $q < q_0 = 1 - 2/\beta$ and set $\delta = q_0 - q$. Let $w > 0$ be a supersolution to (4.65) that satisfies (4.66). Then one can choose $\tau = \tau(\delta) > 0$ such that

$$(\tau w)^{q-1} \leq \tau^{q-1}(c|x|^\beta)^{q-1} \leq \frac{(c\tau)^{q-1}}{|x|^{2+\delta|\beta|}} \leq \frac{\epsilon}{|x|^2 \log^2 |x|} \quad \text{in } \mathcal{C}^\rho_\Omega.$$

Therefore

$$-\nabla \cdot a \cdot \nabla(\tau w) = \frac{\epsilon}{|x|^2 \log^2 |x|}(\tau w) \geq (\tau w)^{q-1}(\tau w) = (\bar{\tau} w)^q \quad \text{in } \mathcal{C}^\rho_\Omega,$$

that is $\tau w > 0$ is a supersolution to (4.56) in $\mathcal{C}^\rho_\Omega$. □

4.3. *Discussion*

(1) In Section 3 we studied the existence and nonexistence of positive supersolutions in exterior domains, and in this section—in cone-like domains. Since we are concerned with supersolutions without prescribing some boundary conditions, it is clear that the existence of positive supersolutions in a domain implies the existence in a subdomain, and vice versa, the nonexistence on a subdomain implies the nonexistence on the domain. From this simple observation we can make conclusions on domains different from cones or exterior domains. For instance, since a paraboloid-type domain (cut the vertex like in the cone) can be embedded in any cone-like domain, one easily concludes that the equation $-\Delta u = u^q$ had positive supersolutions in a paraboloid-type domain for any $q > 1$. So in a way the discussed problem becomes trivial on a paraboloid. This is also true if one prescribes the homogeneous Dirichlet boundary conditions on the boundary of the paraboloid. However, the situation changes if one prescribes the Neumann boundary conditions on the boundary of the paraboloid.

Let $x = (x', x_N) \in \mathbb{R}^N$, $x' \in \mathbb{R}^{N-1}$. Let $\mathcal{P} = \{x \in \mathbb{R}^N : x_N = |x'|^\lambda,\ x_N \ge 1\}$, $\lambda > 1$. Consider the following problem

$$\begin{cases} -\Delta u = u^q \text{ if } x_N = \max\{|x'|^\lambda, 1\}, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 \qquad \text{on } \mathcal{P}. \end{cases} \tag{4.67}$$

The following assertion holds (unpublished).

THEOREM 4.49. *Let $q > 1$. If $\lambda \ge N - 1$ then (4.67) has no nontrivial positive supersolutions. If $\lambda < N - 1$ then (4.67) has nontrivial positive supersolutions if and only for $q > \frac{N-1+q}{N-1-q}$.*

The ideas of the proof are the same as for the cone-like domains, but one should note different scaling for the paraboloid.

(2) The program described in Section 4.2.3 should be possible to realize for $\mathcal{L}$ with nonsymmetric uniformly elliptic matrix and with lower-order terms. This issue is open at the moment. It is also of interest to answer the following question: what are the conditions on the matrix a so that the qualitative picture of the existence of positive supersolutions to $-\nabla \cdot a \cdot \nabla u = u^q$ on the cone-like domains is the same as for the equation $-\Delta u = u^q$? Currently the authors have the affirmative answer to this question (unpublished) for $q > 1$ on cones with Lipschitz boundary and $a \to Id$ (identity matrix) as $x \to \infty$, and the convergence is faster than a Dini function at infinity. A similar question on the comparison of two equations $-\nabla \cdot a_1 \cdot \nabla u = u^q$ and $-\nabla \cdot a_2 \cdot \nabla u = u^q$ with uniformly elliptic matrices a_1 and a_2 such that the difference $a_1 - a_2 \to 0$ as $x \to \infty$, seems to be much more difficult.

5. Exterior domains. Nondivergence-type equations

In this section we study the existence and nonexistence of positive supersolutions to a semi-linear second-order nondivergence-type elliptic equation

$$-a \cdot \partial^2 u = u^q \quad \text{in } G, \tag{5.1}$$

for the full range of the parameter $q \in (-\infty, +\infty)$. Here $G \subset \mathbb{R}^N$ $(N \ge 2)$ is an exterior domain (i.e., complement to a closed ball) and $-a \cdot \partial^2 u := -\sum_{i,j=1}^N a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}$ is a second-order nondivergence-type elliptic expression. We assume throughout the section that the matrix $a = (a_{ij}(x))_{i,j=1}^N$ is symmetric measurable and uniformly elliptic, i.e. there exists an ellipticity constant $\nu > 0$ such that

$$\nu^{-1}|\xi|^2 \le \sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \le \nu|\xi|^2, \quad \text{for all } \xi \in \mathbb{R}^N \text{ and almost all } x \in G. \tag{5.2}$$

We will discuss the existence and nonexistence of positive supersolutions to (5.1) without any smoothness assumptions on the coefficients. In this generality it is known that the linear Dirichlet problem is not well posed [62], and there is no potential theory.

This makes strong restrictions on the available techniques. Since in the case of smooth coefficients the asymptotic behavior at infinity of the Green's function of the operator $a \cdot \partial^2$ is relatively well understood [5,66] one can expect that the value of the critical exponent of (5.1) will depend on the behavior of the coefficients at infinity. Indeed, we will see in the main result of this section that it is the case, and in a "generic" situation equation (5.1) has one critical exponent which, unlike in the case of the divergence-type equations, can move to the sublinear region ($q < 1$) if the ratio of the maximal and minimal eigenvalues of the matrix a is large enough. This phenomenon has been already observed in cone-like domains, in which case it was related to the presence of the harmonic growing at infinity. Here we meet the same situation. The only difference is that we cannot hope to obtain precise asymptotic behavior of subsolutions at infinity.

In order to set the framework we need to define strong solutions at infinity to (5.1).

We say that u is a *solution* (*supersolution, subsolutions*) to equation (5.1) *at infinity* if there exists a closed ball $\bar{B}_\rho$ centered at the origin such that if $u \in W^{2,N}_{\text{loc}}(\bar{B}^c_\rho)$, $\bar{B}^c_\rho = \mathbb{R}^N \setminus \bar{B}_\rho$ and $-a \cdot \partial^2 u = (\geq, \leq)\, u^p$ a.e. on $\mathbb{R}^N \setminus \bar{B}_\rho$.

We refer the reader to [28, Chapter 9] for properties of strong (super-)solutions to the linear equation $-a \cdot \partial^2 u = 0$.

Recall the definition *critical exponents* to equation (5.1):

$$\begin{aligned} q^* &= \inf\{p > 1 : (5.1) \text{ has a positive supersolution at infinity}\}, \\ q_* &= \sup\{p < 1 : (5.1) \text{ has a positive supersolution at infinity}\}. \end{aligned} \tag{5.3}$$

The next proposition, whose proof is straightforward, shows that the above values are well defined.

PROPOSITION 5.1. *Let $u > 0$ be a solution to the inequality*

$$a \cdot \partial^2 u + u^p \leq 0 \quad \text{on } \bar{B}^c_R.$$

Then for $q > p > 1$ or $q < p < 1$ the function $v = u^{\frac{p-1}{q-1}}$ is a positive solution to

$$a \cdot \partial^2 v + v^q \leq 0 \quad \text{on } \bar{B}^c_R.$$

This proposition shows that in order to establish the existence of the critical exponent in the superlinear case, one need to find a value $q_1 > 1$ such that (5.1) has no positive supersolutions at infinity with $q = q_1$, and the value $q_2 > q_1$ such that a positive supersolution to (5.1) at infinity with $q = q_2$. Then it will follow that there exists a critical value q^* between q_1 and q_2. The sublinear case $q < 1$ is similar. Under the general assumptions, apart from the fact of existence of a critical exponent, it is not possible to say more. One can expect that the numerical value of the critical exponent will become available if the matrix of coefficients $a = (a_{ij}(x))^N_{i,j=1}$ tends to a constant matrix $a^0 = (a^0_{ij})^N_{i,j=1}$ as $|x|$ tends to infinity, and the predicted value of the critical exponent is the same as in the case of the Laplacian. This is indeed true as we will see below. It turns out that the quantity responsible for the qualitative picture as well as for the numerical value of the critical exponent is the following function

$$\Psi_a(x) := \frac{Tr(a)}{\frac{(ax,x)}{|x|^2}} \tag{5.4}$$

which was introduced in [54] where it was called the "effective dimension" for the equation $a \cdot \partial^2 u = 0$. It is the stabilization at infinity of this function that gives the exact numerical value of the critical exponent to (5.1), and the rate of its stabilization determines whether equation (5.1) with the critical value of p has positive supersolutions at infinity. The standard condition on the rate of convergence of the variable coefficients to the constant coefficients in the theorems on proximity of the Green's functions corresponding to the linear equation $a \cdot \partial^2 u = 0$ is the Dini condition at infinity (see [5,66]). We provide the conditions of the absence of positive supersolutions in the critical case, which are more general and show the sharpness by an example.

First, observe that Ψ_a is invariant under orthogonal transformations but not invariant under affine transformations. Note also that

$$1 < 1 + (N-1)\nu^{-2} \le \Psi_a(x) \le 1 + (N-1)\nu^2. \tag{5.5}$$

Let g be a nondegenerate matrix $\det g \neq 0$. Making the change of variables $y = gx$ in (5.1) one obtains

$$-\sum_{i,j=1}^{N} \big(a_g(y)\big)_{ij} \frac{\partial^2 v}{\partial y_i \partial y_j} = v^p,$$
$$\text{with } a_g(y) = g\, a(g^{-1}y)\, g^*,\ v(y) = u(g^{-1}y). \tag{5.6}$$

It is clear that the fact of the existence of positive supersolutions to (5.1) does not depend on the change of variables, and the critical exponents q_* and q^* are the same for (5.1) and for (5.6). In order to formulate the main result we introduce the following quantities.

$$\begin{aligned} \overline{\Psi_a} &:= \inf_g \limsup_{|x|\to\infty} \Psi_{a_g}(x), \\ \underline{\Psi_a} &:= \sup_g \liminf_{|x|\to\infty} \Psi_{a_g}(x). \end{aligned} \tag{5.7}$$

Now we are ready to formulate the main result of this section.

THEOREM 5.2. *Let $\underline{\Psi_a}$, $\overline{\Psi_a}$ be defined in (5.7). Then the following assertions hold.*

(i) *If $\underline{\Psi_a} \ge 2$ then $q_* = -\infty$ and $1 + \frac{2}{\overline{\Psi_a}-2} \le q^* \le 1 + \frac{2}{\underline{\Psi_a}-2}$.*

(ii) *If $\overline{\Psi_a} \le 2$ then $q^* = \infty$ and $1 - \frac{2}{2-\overline{\Psi_a}} \le q_* \le 1 - \frac{2}{2-\underline{\Psi_a}}$.*

(iii) *If $\overline{\Psi_a} = \underline{\Psi_a} = A \neq 2$ and there exists $g \in GL_N$ such that the function*

$$\delta(r) := \sup_{|x|\ge r} |\Psi_{a_g}(x) - A| \to 0 \quad (r \to \infty)$$

and

$$\int^{\infty} \exp\left\{-\frac{2}{|A-2|}\int_{r_0}^{r} \delta(s)\frac{ds}{s}\right\}\frac{dr}{r} = \infty \quad \textit{for some } r_0 > 0, \tag{5.8}$$

then (5.1) has no positive supersolutions at infinity for the critical value $q = 1 + \frac{2}{A-2}$.

(iv) *If* $\overline{\Psi_a} = \underline{\Psi_a} = 2$ *then* $q_* = -\infty$ *and* $q^* = \infty$, *so that* (5.1) *has no positive supersolutions at infinity for any* $q \in \mathbb{R}$.

REMARK 5.3. (a) It is interesting to note that equation (5.1) has no positive supersolutions for all $q \in [-1, 1]$, due to (5.5).

(b) Theorem 5.2 describes three distinct cases in existence of positive supersolutions to (5.1). Using the matrix

$$a_{ij}(x) = \delta_{ij} + \gamma(r)\frac{x_i x_j}{|x|^2}, \tag{5.9}$$

one can show that for the case $\underline{\Psi_a} < 2 < \overline{\Psi_a}$ each of the three situations is possible, and for $\underline{\Psi_a} < 2 = \overline{\Psi_a}$ cases (ii) and (iv) are possible. Similarly, for $\underline{\Psi_a} = 2 < \overline{\Psi_a}$ cases (i) and (iv) are possible.

(c) If δ is a Dini function at infinity then it satisfies condition (5.8).

(d) The next example shows that the conditions of Theorem 5.2 (iii) are optimal. Let $A \neq 2$, $\varkappa \in \mathbb{R}$. Set

$$a_{ij}(x) = \delta_{ij} + \gamma(r)\left[\delta_{ij} - \frac{x_i x_j}{|x|^2}\right], \qquad \gamma(r) = \frac{A - N + \frac{\varkappa}{\log r}}{N-1}.$$

Then

$$\Psi(x) = N + \gamma(r)(N-1) = A + \frac{\varkappa}{\log r}, \qquad \delta(r) = \frac{|\varkappa|}{\log r} \to 0 \quad \text{as } r \to \infty.$$

For $|\varkappa| \le \frac{|A-2|}{2}$ it follows from Theorem 5.2 (iii) that equation (5.1) has no positive supersolutions for $p = \frac{A}{A-2}$. On the other hand, one can readily verify that there exists $R > 1$ such that the function

$$u(x) = c|x|^{2-A}(\log|x|)^{\frac{2-A}{2}},$$

satisfies the inequality

$$a \cdot \partial^2 u + u^{\frac{A}{A-2}} \le 0$$

if, for $A > 2$, $\varkappa > \frac{|A-2|}{2}$ with sufficiently small $c > 0$, and for $A < 2$, $\varkappa < -\frac{|A-2|}{2}$ with sufficiently large $c > 0$.

COROLLARY 5.4. *Let there exist* $a^0 \in GL_N$ *such that*

$$\delta(r) := \max_{1 \le i,j \le N} \sup_{|x| \ge r} |a_{ij}(x) - a^0_{ij}| \to 0 \quad \textit{as } r \to \infty.$$

Then $q_* = -\infty$ *and* $q^* = 1 + \frac{2}{N-2}$. *Moreover, if* $N \ge 3$ *and* δ *satisfies* (5.8) *(in particular, if* δ *is a Dini function at infinity) then* (5.1) *has no positive supersolutions at infinity for* $q = q^*$.

The detailed proof of Theorem 5.2 can be found in [39]. Here we only give the ideas.

Similarly to the case of the divergence-type equations (Section 3) the superlinear case $q > 1$ is based on the maximum principle, which gives a priori estimates for supersolutions. This is achieved by constructing explicit subsolutions (barriers). Subsequent linearization of the equation leads to a contradiction with the next lemma, the proof of which is based on a result from [10] and a scaling argument.

LEMMA 5.5 (Nonexistence lemma). *Let* $0 \le V \in L^{\infty}_{\mathrm{loc}}(\bar{B}^c_R)$ *satisfy*

$$|x|^2 V(x) \to \infty \quad \text{as } |x| \to \infty. \tag{5.10}$$

Then the equation

$$-a \cdot \partial^2 u - Vu = 0 \quad \text{in } \bar{B}^c_R \tag{5.11}$$

has no nontrivial nonnegative supersolutions.

The proof in the sublinear case is based on an a priori estimate for supersolutions to (5.1) similar to Lemma 1.4 which however is proved in a different way.

PROPOSITION 5.6. *Let* $q < 1$. *Let* $R_0 > 0$. *Let u be a positive supersolution to*

$$a \cdot \partial^2 u + u^q \le 0 \quad \text{in } \bar{B}^c_{R_0}.$$

Then there exists $c > 0$ *such that*

$$u(x) \ge c|x|^{\frac{2}{1-q}}. \tag{5.12}$$

A proof of Proposition 5.6 can be based on the next proposition which is nowadays well-known (see, e.g. [37]) and a scaling argument.

LEMMA 5.7. *Let* $q > 1$. *Let* v *be a positive solution to*

$$a \cdot \partial^2 v \ge v^q \quad \text{in } A_{1,4}.$$

Then there is a constant $c = c(v, N, q)$ *such that*

$$v(x) \le c \quad \text{for all } x \in A_{2,3}.$$

The existence part of the proof consists of producing explicit supersolutions to (5.1).

The proof of part (iii) of Theorem 5.2 is based on a careful construction of precise one-dimensional barriers. We refer the reader to [39] for the details.

Discussion. (1) It should be possible to extend the result of Theorem 5.2 to the case of the equations with first and zero-order terms. It seems also possible to extend the results to the case of nonuniformly elliptic operators with locally bounded coefficients.

(2) Another interesting direction is the study of positive supersolutions to semi-linear nondivergence-type equations on cone-like domains. First results in this direction were obtained in [85] for the superlinear case $q > 1$ and for sufficiently smooth cones.

6. Quasi-linear equations

In this section we discuss in brief some results on the existence and nonexistence of positive supersolutions to the equation $\mathcal{L}u = c_0|x|^{-\sigma}u^q$ in exterior domains with $\mathcal{L}$ being a quasi-linear operator.

One of the simplest cases of the quasi-linear operator in divergence form is the so-called p-Laplacian, $p > 1$, which is formally given by

$$\Delta_p u := \nabla(|\nabla u|^{p-2}\nabla u).$$

And the first quasi-linear equation for which the existence of positive supersolutions in an exterior of a ball we are going to discuss is of the form

$$-\Delta_p u \geq c_0|x|^{-\sigma}u^q, \qquad c_0 > 0, \quad x \in \bar{B_1}^c. \tag{6.1}$$

In view of the scaling invariance of the equation the results on the existence of positive supersolutions in $\bar{B_\rho}^c$ are the same for different $\rho > 0$.

The first result on (6.1) is due to Bidaut-Véron [11] (see also [12,13]).

THEOREM 6.1. *Let $1 < p \leq N$ and $q > 0$. Then there are no positive nontrivial supersolutions to equation* (6.1) *if and only if*

$$\sigma \leq \min\left\{p, \frac{(q-p+1)(p-N)}{p-1}\right\},$$

so $p - 1 < q^* = \frac{(q-p+1)(p-N)}{p-1}$ and $q_* = -\infty$ for $\sigma \leq p$.

The technique of proving nonexistence in [11,12] in the case $q > p - 1$ is based on the reduction to the radially symmetric supersolutions and then studying the corresponding one-dimensional problem. For $0 < q < p - 1$ the Keller–Osserman a priori estimate is combined with some integral estimates on supersolutions obtained by a clever choice of test functions.

We will show in the next subsection that the ideas developed in the previous sections for semi-linear equations are fruitful for (6.1) and even more general equations involving homogeneous perturbations. We will be also able to remove the restrictions $p \leq N$ and $q > 0$.

6.1. *p-Laplacian and Hardy potential in exterior domains*

In this subsection we briefly describe the results from [49] on the existence and nonexistence of positive (super) solutions to nonlinear p-Laplace equation with Hardy potential

$$-\Delta_p u - \frac{\mu}{|x|^p}u^{p-1} = \frac{c_0}{|x|^\sigma}u^q \quad \text{in } B_\rho^c, \tag{6.2}$$

$1 < p < \infty$, $c_0 > 0$, $\mu \in \mathbb{R}$, $(q, \sigma) \in \mathbb{R}^2$ and $G = \bar{B}_\rho^c$ is the exterior of the ball in $\mathbb{R}^N$, with $N \geq 2$. We say that $u \in W^{1,p}_{\mathrm{loc}}(G) \cap C(G)$ is a *supersolution* to equation (6.2) in G if for all $0 \leq \varphi \in W^{1,p}_c(G) \cap C(G)$ the following inequality holds

$$\int_G \nabla u |\nabla u|^{p-2} \nabla \varphi \, dx - \int_G \frac{\mu}{|x|^p} u^{p-1} \varphi \, dx \geq \int_G \frac{c_0}{|x|^\sigma} u^q \varphi \, dx.$$

Recall that $W^{1,p}_c(G) := \{u \in W^{1,p}_{\mathrm{loc}}(G),\ Supp(u) \Subset G\}$. The notions of a subsolution and solution are defined similarly, by replacing "$\geq$" with "$\leq$" and "$=$", respectively. It follows from the Harnack inequality (cf. [77]) that any nontrivial nonnegative supersolution to (6.2) in G is strictly positive in G.

Let us explore the impact of the potential on the value of the critical exponents q^* and q_*. Consider the equation of the form

$$-\Delta_p u - \frac{\mu}{|x|^{p+\epsilon}} u^{p-1} = u^q \quad \text{in } B_\rho^c, \tag{6.3}$$

here $\mu \in \mathbb{R}$ and $\epsilon \in \mathbb{R}$. One can readily show (see the next section and [50]) that if $\epsilon > 0$ then (6.3) *has the same critical exponents* q^* and q_* as in Theorem 6.1. This was proved in [50] for $p \geq 2$ and even for more general perturbations of the type $\frac{\mu(x)}{|x|^2}$ with $\mu(x)$ being a Dini function at infinity. For (6.3) this can be easily obtained by arguments similar to those used in Section 3.2.2. Notice also that the value of the critical exponent q^* in this case does not depend on the value of the parameter μ as one can see, e.g. by scaling.

On the other hand, one can verify directly that *if* $\epsilon < 0$ *and* $\mu < 0$*, then* (6.3) *admits positive solutions for all* $q \in \mathbb{R}$, while *if* $\epsilon < 0$ *and* $\mu > 0$ *then* (6.3) *has no positive supersolutions for any* $q \in \mathbb{R}$. The latter follows immediately from Theorem B.2.

In the borderline case $\epsilon = 0$ the critical exponents q^* and q_* depend explicitly on the value of the constant μ, as we already observed this in a particular case in Section 3.1.

The improved Hardy Inequality (Theorem B.2) plays a crucial role in the analysis of equation (6.2) in the critical case $\mu = C_H$.

To formulate the main result of [49] we assume that $\mu \leq C_H$, otherwise (6.2) has no positive supersolutions (by the optimality of the Hardy inequality, see Appendix B). When $\mu \leq C_H$, the scalar equation

$$-\gamma|\gamma|^{p-2}\left(\gamma(p-1) + N - p\right) = \mu$$

has two real roots $\gamma_- \leq \gamma_+$. Note that if $\mu = C_H$ then $\gamma_- = \gamma_+ = \frac{p-N}{p}$. For $\mu \leq C_H$ we introduce the critical line $\sigma = \Lambda^*(q, \mu)$ for equation (6.2) on the (q, σ)-plane defined by

$$\Lambda_*(q, \mu) := \min\{\gamma_-(q - p + 1) + p, \gamma_+(q - p + 1) + p\} \quad (q \in \mathbb{R}),$$

and the nonexistence set

$$\mathcal{N} = \{(q, \sigma) \in \mathbb{R}^2 \setminus (p-1, p) : (6.2) \textit{ has no positive supersolutions in } B_\rho^c\}.$$

THEOREM 6.2. *The following assertions are valid.*

(i) *If* $\mu < C_H$ *then* $\mathcal{N} = \{\sigma \leq \Lambda_*(q)\}$.

(ii) *If* $\mu = C_H$ *then* $\mathcal{N} = \{\sigma < \Lambda_*(q)\} \cup \{\sigma = \Lambda_*(q),\ q \geq -1\}$.

REMARK 6.3. (i) Using sub- and supersolution techniques one can show that if (1.1) has a positive supersolution in B_ρ^c then it has a positive solution in B_ρ^c. Thus for any $(q, \sigma) \in \mathbb{R}^2 \setminus \mathcal{N}$ equation (6.2) admits positive solutions.
(ii) Figure 6 shows the qualitative pictures of the set $\mathcal{N}$ for typical values of γ^-, γ^+ and different relations between p and the dimension $N \geq 2$.

The proof of the nonexistence for the case $q > p - 1$ is again based on the comparison and the maximum principle. However, the presence of the negative "potential" V makes these tools much more delicate. We refer the reader to [72,81] (see also [49]) for the corresponding maximum and comparison principle. The proof in the case $q < p - 1$ again as in the previous sections is based on the a priori estimate of the Keller–Osserman type and a Phragmèn–Lindelöf type comparison arguments. The difficulty in comparison with the semi-linear cases arises when one needs a subsolution to the homogeneous equation, with zero on the boundary of the ball and a given asymptotic at infinity (compare how it was done in the case of a linear equation). In [49] this has been overcome by means of a generalized Prüfer transformation. The most delicate case when $\mu = C_H$ and $1 \leq q < 1$ was treated using the improved Hardy inequality (see Appendix B). We refer the reader to [49] for the details of the proofs.

Discussion. There are still many open issues on the existence and nonexistence of positive solutions to the p-Laplace equations. One of the problems would be to extend the nonexistence results described in this section to a more general notion of a supersolution. The desirable conditions on the supersolutions are $C({B_\rho}^c) \cap W^{1,p-1}_{\mathrm{loc}}({B_\rho}^c)$. The problem of the existence and nonexistence of positive solutions to the equation $-\Delta_p u + V u^{p-1} = c_0|x|^{-\sigma} u^q$ for $1 < p < 2$ with V weaker than the Hardy potential, satisfying conditions in the spirit of Section 3.2, is still waiting to be explored.

6.2. *General quasi-linear equations in exterior domains*

Here we discuss the results on the nonexistence of positive supersolutions to general quasi-linear elliptic equations in exterior domains.

$$\mathcal{L}u := -\operatorname{div} \mathbf{A}(x, u, \nabla u) + a_0(x, \nabla u, u) \geq c_0|x|^{-\sigma} u^q. \tag{6.4}$$

We suppose standard conditions on the main term on the left-hand side of (6.4). Namely, we suppose that the functions $\mathbf{A} : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$, $a_0 : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are such that $\mathbf{A}(\cdot, \xi, u)$, $a_0(\cdot, u, \xi)$ are Lebesgue measurable for all $\xi \in \mathbb{R}^N$ and all $u \in \mathbb{R}$, and $\mathbf{A}(x, u, \cdot)$, $a_0(x, u, \cdot)$ are continuous for almost all $x \in \mathbb{R}^N$, $u \in \mathbb{R}$.

We also assume that the following structural conditions on $\mathbf{A}$ are fulfilled for all $x, \xi, \eta \in \mathbb{R}^N$, with $p > 1$:

$$\begin{aligned}
&(\mathbf{A}(x, u, \xi) - \mathbf{A}(x, u, \eta)) \cdot (\xi - \eta) > 0, \quad \xi \neq \eta, \\
&\mathbf{A}(x, u, \xi) \cdot \xi \geq |\xi|^p - g_1(x)|u|^p, \\
&|\mathbf{A}(x, u, \xi)| \leq \nu_1 |\xi|^{p-1} + g_2(x)|u|^{p-1}, \\
&|a_0(x, u, \xi)| \leq g_3(x)|\xi|^{p-1} + g_4(x)|u|^{p-1},
\end{aligned} \tag{6.5}$$

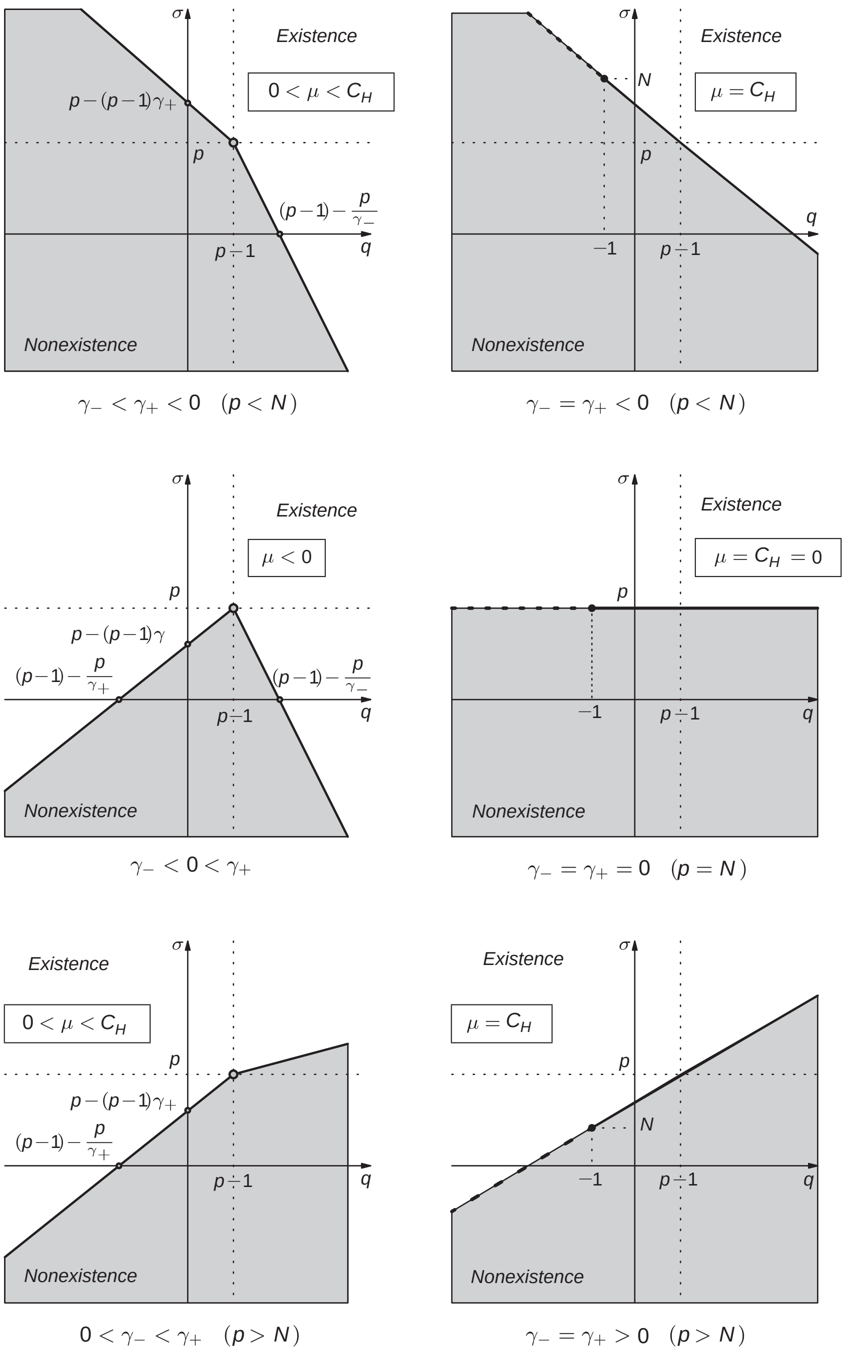

Fig. 6. The nonexistence set $\mathcal{N}$ of equation (1.1) for typical values of γ_- and γ_+.

where g_1, g_2, g_3, g_4 are locally bounded (one can allow for mild singularities, but we will not discuss this issue here).

Under the above conditions it is well known by now (see [77]) that positive supersolutions to $\mathcal{L}u = 0$ satisfy the weak Harnack inequality, and solutions are Hölder-continuous.

The next theorem which gives the nonexistence of positive solutions in the whole of $\mathbb{R}^N$ was proved in [13].

THEOREM 6.4. *Let* (6.5) *be fulfilled with* $g_i = 0$, $i = \overline{1,4}$. *Assume that* $1 < p \le N$ *and* $q > p-1$. *If* $q \le \frac{(N-\sigma)(p-1)}{N-p}$ *then there are no nontrivial positive solutions to* (6.4) *in* $\mathbb{R}^N$.

If instead of (6.5) one assumes that $\mathbf{A}(x, u, \eta) = A(|\eta|)\eta$, $\eta \in \mathbb{R}^N$, where $A \in C([0, \infty), \mathbb{R}) \cap C^1((0, \infty), \mathbb{R})$ $t \mapsto A(t)t$ is nondecreasing and there exists $M > 0$ such that

$$\begin{cases} A(t) \le M t^{p-2} & \text{for any } t > 0, \\ A(t) \ge M^{-1} t^{p-2} & \text{for small } t > 0, \end{cases} \tag{6.6}$$

then the following nonexistence result in exterior domains holds (see [13,12]).

THEOREM 6.5. *Let* $1 < p \le N$ *and* $q > p - 1$. *Suppose the above condition on* $\mathbf{A}$ *is fulfilled. Then* (6.4) *has no nontrivial positive solutions in* $\bar{B}_1^{\,c}$ *if*

$$q \le \frac{(N-\sigma)(p-1)}{N-p}.$$

Theorems 6.4 and 6.5 and some generalizations are extensively discussed in [13,55]. However, the lower-order terms g_i, $i = \overline{1,4}$ are absent in these results. Using the methods of the mentioned works based on a wise choice of test functions for (6.4), it seems fairly straightforward to include some lower-order terms in proving the nonexistence theorems up to the critical value of q (but not inclusive). Even without the lower-order terms under conditions (6.5) the results in [13,55] do not include the critical value of q.

In order to include the lower-order terms in a nonexistence result on exterior domains we make the following additional assumption:

$$\begin{gathered} \text{there exist } \delta \in (0,1) \text{ and a positive nonincreasing function } G : \mathbb{R} \to \mathbb{R} \\ \text{which is Dini at infinity, such that} \end{gathered}$$

$$\begin{aligned} g_1(x) &\le \left[\frac{N-p}{p}\right]^p (1-\delta)\,\frac{1}{|x|^p}, \\ g_3(x) &\le \frac{G(|x|)}{|x|}, \\ g_4(x) &\le \frac{G(|x|)}{|x|^p} \end{aligned} \tag{6.7}$$

hold for $|x| > 1$.

The result below is based on the comparison principle. So we require an additional combined assumption on $\mathbf{A}$ and a_0. We assume that for every $u, v \in H^1_c(B_1^c) \cap L^\infty_c(B_1^c)$

$$\int_{B_1^c} [\mathbf{A}(x, u, \nabla u) - \mathbf{A}(x, v, \nabla v)] \cdot (\nabla u - \nabla v)]dx$$
$$+ \int_{B_1^c} [a_0(x, u, \nabla u) - a_0(x, v, \nabla v)](u - v)dx \tag{6.8}$$
$$\geq \nu_0 \int_{B_1^c} |\nabla u - \nabla v|^p dx. \tag{6.9}$$

The next theorem is the main result of [50].

THEOREM 6.6. *Let conditions* (6.5)*,* (6.8)*,* (6.7) *be fulfilled. Let*

$$p - 1 \leq q \leq \frac{(N - \sigma)(p - 1)}{N - p}, \quad \sigma < p.$$

Then equation (6.4) *has no nontrivial positive solutions in* $\bar{B}_1{}^c$.

REMARK 6.7. (1) Theorem 6.6 because of the condition (6.8) implicitly assumes that $p \geq 2$. The corresponding result for $1 < p < 2$ is an open problem.
(2) Recently under the conditions of Theorem 6.6 the existence of a weak solution to $\mathcal{L}u = 0$ in $\mathbb{R}^N$, which is squeezed between two positive constants, was proved in [51]. This solution can be used to prove nonexistence of positive solutions to (6.6) for $0 < q < 1$. However, in comparison with the case of a semi-linear equation, the case $q < 0$ remains an open question.

Acknowledgments

The authors are grateful to Vitaly Moroz for providing unpublished material [56] which helped substantially in writing Section 4. We would also like to thank Michael Perelmuter and Aubrey Truman for help in reading the manuscript.

A. Appendix. Extended Dirichlet space

Let $\mathcal{E}$ be a symmetric bilinear form defined by

$$\mathcal{E}(u, v) := \int_G \nabla u \cdot a \cdot \nabla v \, dx + \int_G V uv \, dx \quad (u, v \in H^1_c(G) \cap L^\infty_c(G)),$$

where $G \subseteq \mathbb{R}^N$ is a domain, a is a symmetric locally elliptic matrix and $V \in L^\infty_{\mathrm{loc}}(G)$ a potential. In this section we briefly discuss several facts concerning the relations between the positivity of the form $\mathcal{E}$ and the existence of positive (super) solutions to the linear equation

$$(-\nabla \cdot a \cdot \nabla + V)v = f \quad \text{in } G, \tag{A.1}$$

associated with $\mathcal{E}_V$, where $f \in L^\infty_{\mathrm{loc}}(G)$ (see some more details in [48]). As usual, a supersolution to (A.1) is a function $u \in H^1_{\mathrm{loc}}(G)$ such that

$$\int_G \nabla u \cdot a \cdot \nabla\varphi\, dx + \int_G V u\varphi\, dx \geq \int_G f\varphi\, dx, \quad \forall\, 0 \leq \varphi \in H^1_c(G) \cap L^\infty_c(G). \tag{A.2}$$

Assume that the form $\mathcal{E}$ is *positive definite*, that is

$$\mathcal{E}(u,u) > 0, \quad \forall 0 \neq u \in H^1_c(G) \cap L^\infty_c(G). \tag{A.3}$$

Following Fukushima [26, p. 35–36], denote by $\mathcal{D}(\mathcal{E}, G)$ the family of measurable a.e. finite functions $u : G \to \mathbb{R}$ such that there exists an $\mathcal{E}$-Cauchy sequence $(u_n) \subset H^1_c(G) \cap L^\infty_c(G)$ that converges to u a.e. in G. This sequence (u_n) is called an *approximating sequence* for $u \in \mathcal{D}(\mathcal{E}, G)$. Then the limit $\mathcal{E}(u,u) := \lim_{n\to\infty} \mathcal{E}(u_n, u_n)$ exists and is independent of the choice of the approximating sequence. Thus $\mathcal{E}$ is extended uniquely to a nonnegative definite bilinear form on $\mathcal{D}(\mathcal{E}, G)$. The family $\mathcal{D}(\mathcal{E}, G)$ is called the *extended Dirichlet space* of $\mathcal{E}$. It is easy to see that $\mathcal{D}(\mathcal{E}, G)$ is linear and invariant under the standard truncations: if $u, v \in \mathcal{D}(\mathcal{E}, G)$ then $u^+ = u \vee 0 \in \mathcal{D}(\mathcal{E}, G)$ and $u \vee v, u \wedge v \in \mathcal{D}(\mathcal{E}, G)$.

Following [3,4], we say that the form $\mathcal{E}$ satisfies the *λ-property* if there exists a function $0 < \lambda \in L^1_{\mathrm{loc}}(G)$ such that $\lambda^{-1} \in L^\infty_{\mathrm{loc}}(G)$ and

$$\mathcal{E}(u,u) \geq \int_G u^2 \lambda(x)\, dx, \quad \forall\, u \in H^1_c(G) \cap L^\infty_c(G). \tag{A.4}$$

If $\mathcal{E}$ satisfies the λ-property then the extended Dirichlet space $\mathcal{D}(\mathcal{E}, G)$ is a Hilbert space with the inner product $\mathcal{E}(\cdot,\cdot)$ and the corresponding norm $\|\cdot\|_{\mathcal{D}} = \sqrt{\mathcal{E}(\cdot,\cdot)}$. Clearly

$$H^1_c(G) \cap L^\infty_c(G) \subset \mathcal{D}(\mathcal{E}, G) \subset H^1_{\mathrm{loc}}(G) \quad \text{and} \quad \mathcal{D}(\mathcal{E}, G) \subset L^2(G, \lambda\, dx).$$

By $\mathcal{D}'(\mathcal{E}, G)$ we denote the space of linear continuous functionals on $\mathcal{D}(\mathcal{E}, G)$. The following lemma is a standard consequence of the Riesz Representation Theorem.

LEMMA A.1. *Assume that $\mathcal{E}$ satisfies the λ-property. Let $l \in \mathcal{D}'(\mathcal{E}, G)$. Then there exists a unique $\phi_* \in \mathcal{D}(\mathcal{E}, G)$ such that*

$$\mathcal{E}(\phi_*, \varphi) = l(\varphi), \quad \forall\, \varphi \in \mathcal{D}(\mathcal{E}, G). \tag{A.5}$$

It is easy to see that

$$L^2(G, \lambda^{-1}\, dx) \subset \mathcal{D}'(\mathcal{E}_V, G).$$

Consider the homogeneous equation

$$(-\nabla \cdot a \cdot \nabla + V)u = 0 \quad \text{in } G. \tag{A.6}$$

The following maximum and comparison principles can be proved much in the same way as Lemmas 2.5 and 2.6.

LEMMA A.2. *Assume that $\mathcal{E}$ satisfies the λ-property. Let $w \in H^1_{\mathrm{loc}}(G)$ be a supersolution to (A.6) such that $w^- \in \mathcal{D}(\mathcal{E}, G)$. Then $w \geq 0$ in G.*

LEMMA A.3. *Assume that $\mathcal{E}$ satisfies the λ-property. Let $0 \leq w \in H^1_{\mathrm{loc}}(G)$, $v \in \mathcal{D}(\mathcal{E}, G)$ and $w - v$ be a supersolution to equation (A.6). Then $w \geq v$ in G.*

B. Appendix. Improved Hardy's inequality

The standard Hardy inequality has the form

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \geq \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} \, dx \quad \text{for all } u \in H^1_c(\mathbb{R}^N). \tag{B.1}$$

It is meaningless in the dimension $N = 2$. Instead, the following inequality holds on the exterior of the unit ball

$$\int_{\mathbb{R}^2} |\nabla u|^2 \, dx \geq \frac{1}{4} \int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2 \log^2 |x|} \, dx \quad \text{for all } u \in H^1_c(\mathbb{R}^2 \setminus \bar{B}_1). \tag{B.2}$$

In the main body we use the following extension of (B.2) to the N-dimensional case:

$$\int_{\mathbb{R}^N} |x|^{2-N} |\nabla u|^2 \, dx \geq \frac{1}{4} \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^N \log^2 |x|} \, dx \quad \text{for all } u \in H^1_c(\mathbb{R}^N \setminus \bar{B}_1). \tag{B.3}$$

The above inequalities can be easily verified using Theorem 2.13 by choosing appropriate supersolutions to homogeneous equations with potentials. For instance, (B.3) follows from Theorem 2.13 since the function $v(x) = (\log |x|)^{1/2}$ solves the equation $-\nabla \cdot |x|^{2-N} \cdot \nabla v - \frac{1}{4|x|^N (\log |x|)^2} v = 0$.

In the interior of a ball or the exterior of a ball the Hardy inequality (B.1) can be improved [1,25]. We give an improved Hardy-type inequality on cone-like domains, which is appropriate for our purposes (see e.g. [48] for a proof). Recall that, for $0 \leq \rho < R \leq +\infty$, we denote

$$\mathcal{C}_\Omega^{(\rho,R)} := \{(r,\omega) \in \mathbb{R}^N : \omega \in \Omega, r \in (\rho, R)\}, \quad \mathcal{C}_\Omega^\rho := \mathcal{C}_\Omega^{(\rho,+\infty)}, \quad \mathcal{C}_\Omega = \mathcal{C}_\Omega^0,$$

where $\Omega \subseteq S^{N-1}$ is a subdomain of $S^{N-1} = \{x \in \mathbb{R}^N : |x| = 1\}$. By λ_1 we denote the first eigenvalue of the Dirichlet Laplace–Beltrami operator on Ω.

THEOREM B.1. *The following inequality holds:*

$$\int_{\mathcal{C}_\Omega^\rho} |\nabla v|^2 dx \geq (C_H + \lambda_1) \int_{\mathcal{C}_\Omega^\rho} \frac{v^2}{|x|^2} dx + \frac{1}{4} \int_{\mathcal{C}_\Omega^\rho} \frac{v^2}{|x|^2 \log^2 |x|} dx, \quad \forall\, v \in H^1_c(\mathcal{C}_\Omega^\rho) \cap L^\infty(\mathcal{C}_\Omega^\rho), \tag{B.4}$$

where $C_H := \left(\frac{N-2}{2}\right)^2$. *The constants* $C_H + \lambda_1$ *and* $\frac{1}{4}$ *are optimal in the sense that the inequality*

$$\int_{\mathcal{C}_\Omega^\rho} |\nabla v|^2 dx \geq \mu \int_{\mathcal{C}_\Omega^\rho} \frac{v^2}{|x|^2} dx + \epsilon \int_{\mathcal{C}_\Omega^\rho} \frac{v^2}{|x|^2 \log^2 |x|} dx, \quad \forall\, v \in H^1_c(\mathcal{C}_\Omega^\rho) \cap L^\infty(\mathcal{C}_\Omega^\rho), \tag{B.5}$$

fails in any of the following two cases:

(i) $\mu = C_H + \lambda_1$ *and* $\epsilon > 1/4$,
(ii) $\mu > C_H + \lambda_1$ *and* $\forall \epsilon \in \mathbb{R}$.

An improved version of the Hardy inequality on exterior domains is also valid for the case of p-Laplacian. In the form below it is taken from [49].

THEOREM B.2 (Improved Hardy Inequality for p-Laplace). *For every* $p > 1$ *there exists* $\rho \geq 1$ *such that*

$$\int_{B_\rho^c} |\nabla v|^p dx \geq C_H \int_{B_\rho^c} \frac{|v|^p}{|x|^p} dx + \varepsilon \int_{B_\rho^c} \frac{|v|^p}{|x|^p \log^{m_*} |x|} dx,$$
$$v \in W_c^{1,p}(B_\rho^c) \cap L^\infty(B_\rho^c), \tag{B.6}$$

with

$$C_H := \left| \frac{N-p}{p} \right|^p, \qquad \gamma_* := \frac{p-N}{p},$$
$$\varepsilon \leq C_* := \begin{cases} \frac{p-1}{2p} \left| \frac{N-p}{p} \right|^{p-2}, & N \neq p, \\ \left(\frac{N-1}{N} \right)^N, & N = p, \end{cases} \qquad m_* := \begin{cases} 2, & N \neq p, \\ N, & N = p. \end{cases} \tag{B.7}$$

The constants C_H *and* C_* *are sharp in the sense that the inequality fails if* $\varepsilon > C_*$.

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CHAPTER 4

Symmetry of Solutions of Elliptic Equations via Maximum Principles

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Abstract

We review here some results in the study of symmetry properties of solutions of elliptic nonlinear differential equations, obtained using maximum principles.

Keywords: Maximum principle, Moving-plane method, Comparison principle, Elliptic equations, Foliated Schwarz symmetry, Degenerate elliptic equations

AMS Subject Classifications: 35B50, 35B99, 35J60, 35J70

Contents

HANDBOOK OF DIFFERENTIAL EQUATIONS
Stationary Partial Differential Equations, volume 6
Edited by M. Chipot

1. Introduction

To establish symmetry properties of solutions of differential equations is obviously an important goal in mathematical analysis, both from a theoretical point of view and for the applications. This question received new impulse after Serrin's paper [46] about an over determined problem where he proved that if a function u satisfies $-\Delta u = 1$ in a smooth, bounded, connected domain Ω, together with the boundary conditions

$$u = 0, \qquad \frac{\partial u}{\partial \nu} = \text{constant on } \partial\Omega$$

then Ω is a ball and u is radially symmetric. To prove this result he introduced in the differential equations framework the method of moving planes which had been previously used in differential geometry by A.D. Alexandroff to study surfaces of constant mean curvature. This device consists in moving parallel hyperplane up to a critical position, while comparing the solution with its reflection with respect to the hyperplane, in order to show that the solution is symmetric about the moving hyperplane. The main ingredients to carry on this procedure are comparison principles which, for linear operators, are equivalent to maximum principles. Years later, the same method was employed by Gidas, Ni and Nirenberg to obtain in [31] fundamental monotonicity and symmetry results for positive solutions of nonlinear elliptic equations under very general assumptions on the nonlinearities.

After that many other interesting results followed: different operators, different boundary conditions, different geometries were studied. However though the Gidas–Ni–Nirenberg results and their developments apply in very general situations, there are cases when break of symmetry occurs, as a consequence of the failure of some of the hypotheses necessary to apply the moving-plane method.

Some of these are described in Section 3. In these cases it is reasonable to expect, for certain nonlinearities, or for certain types of solutions that at least part of the symmetry of the domain is inherited by the solution. This is a new line of research that recently has got some attention and which we describe in Section 4.

Another context where the moving-plane method does not apply in a straightforward manner, is when the differential equation under consideration involves operators which are not uniformly elliptic. A typical case is p-Laplace equations where the elliptic operator is

$$\Delta_p u = \text{div}\,(|Du|^{p-2} Du), \quad p > 1$$

which is singular, if $p < 2$ or degenerate, if $p > 2$. The main difficulty here is that comparison principles, which are essential to apply the moving-plane method, are not available in the same form as for $p = 2$. Indeed there are counterexamples both to the validity of comparison principles and the symmetry results, as described in Section 5.1. Nevertheless it is possible to extend the moving-plane method under suitable assumptions and get some symmetry and monotonicity results for positive solutions, [22,23,25–28,47].

In this paper we survey some results in bounded domains or in the whole space in the directions we have outlined, mainly partial symmetry and p-Laplace equations. The common feature of the results we present here is that they rely on maximum or comparison principles, in different forms.

We start by reviewing the moving-plane method and the Gidas–Ni–Nirenberg results in Section 2. In Section 3 we describe different counterexamples to radial symmetry. Section 4 is devoted to partial symmetry results, while in Section 5 we review the developments concerning p-Laplace equations.

2. Moving-plane method

Let us denote by H_λ, the hyperplane, whose normal is along the direction e in $\mathbb{R}^N$, with $|e| = 1$:

$$H_\lambda = \{x \in \mathbb{R}^N : x \cdot e = \lambda\}, \quad \lambda \in \mathbb{R}.$$

Let Ω be a smooth domain in $\mathbb{R}^N$. The cap that is cut off from Ω by this hyperplane is

$$\Sigma_\lambda = \{x \in \Omega : x \cdot e > \lambda\}.$$

We move the hyperplane H_λ from a minimal position to a maximal position till the reflected cap with respect to H_λ lies within Ω. The comparison of the solution u with its reflection $\tilde{u}$ with respect to the hyperplane is used to derive the symmetry properties. Hence various forms of maximum principles are used depending on whether Ω is bounded or not. We describe here some main results in each case.

2.1. *Bounded domains*

Let Ω be a bounded, regular domain in $\mathbb{R}^N$, $N \geq 2$. For a direction e in $\mathbb{R}^N$, with $|e| = 1$, we define

$$a(e) = \sup_{x \in \Omega} x \cdot e.$$

We start moving the hyperplane H_λ from $\lambda = a(e)$ to a maximal position. In this case, strong maximum principle, Hopf maximum principle at a boundary point and its refinement due to Serrin at corner points, were quite crucial in the early works. This restricted the use of the method to smooth domains only. For simplicity, we fix the direction e to be along the x_1 axis. A typical symmetry result in [31] is:

THEOREM 2.1. *Let u be a positive solution in the ball of radius R in $\mathbb{R}^N$ of*

$$\begin{aligned} &\Delta u + f(u) = 0 \quad \text{in } B_R, \\ &u = 0 \quad \text{on } \partial B_R. \end{aligned} \tag{2.1}$$

Let $f \in C^1$. Then u is radially symmetric and the radial derivative u_r satisfies

$$u_r < 0 \quad \text{for } 0 < r < R. \tag{2.2}$$

□

In fact a more general result is true [31].

THEOREM 2.2. *Let Ω be a C^2 domain in $\mathbb{R}^N$, convex in the x_1-direction and symmetric with respect to the plane $x_1 = 0$. Let u be a positive $C^2(\overline{\Omega})$ solution of*

$$\begin{aligned} &\Delta u + f(u) = 0 \quad \text{in } \Omega, \\ &u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.3}$$

with $f \in C^1$. Then u is symmetric with respect to x_1, and

$$u_{x_1} < 0 \quad \text{in } \{x \in \Omega : x_1 > 0\}. \tag{2.4}$$

□

But the corresponding question for simple domains as cubes remained open because of the presence of corners. Later Berestycki and Nirenberg introduced the maximum principle in narrow domains in this context (see [9]) and that simplified the proofs and widened the applicability of this method to arbitrary domains. A typical result from [9] is:

THEOREM 2.3. *Let Ω be an arbitrary domain in $\mathbb{R}^n$, convex in the x_1-direction and symmetric with respect to the plane $x_1 = 0$. Let u be a positive solution of (2.3) belonging to $W^{2,n}_{loc}(\Omega) \cap C(\overline{\Omega})$ and assume that f is Lipschitz-continuous. Then u is symmetric with respect to x_1 and (2.4) holds.*

PROOF. Let $a = \sup_{x\in\Omega} x_1$. Consider the hyperplane $H_\lambda = \{x \in \mathbb{R}^N : x_1 = \lambda\}$ and Σ_λ, the cap from Ω, cut off by H_λ :

$$\Sigma_\lambda = \{x \in \Omega : x_1 > \lambda\}.$$

Define in Σ_λ,

$$u_\lambda(x_1, y) = u(x_1 - 2\lambda, y).$$

Let us define the set

$$\Lambda := \{\lambda \in (0, a) : \forall a > \mu > \lambda, u \geq u_\mu \text{ in } \Sigma_\mu\}.$$

The proof consists of these main steps:

(1) Show that Λ is nonempty.
(2) Show that if $\lambda_0 := \inf \Lambda$, then $\lambda_0 = 0$ and $u \equiv u_{\lambda_0}$ in Σ_{λ_0}.

Step 1: Define in Σ_λ,

$$w(x, \lambda) = u(x) - u_\lambda(x)$$

w satisfies the equation

$$\begin{aligned} &\Delta w + C(x, \lambda)w = 0 \quad \text{in } \Sigma_\lambda \\ &w = 0 \quad \text{on } \partial\Sigma_\lambda, \end{aligned} \tag{2.5}$$

where

$$C(x, \lambda) = \frac{f(u(x)) - f(u_\lambda(x))}{u(x) - u_\lambda(x)}.$$

Since f is Lipschitz continuous, there exists some constant $b > 0$ such that

$$|C| \leq b \quad \forall\, x \in \Sigma_\lambda \quad \forall\, \lambda.$$

For $a - \lambda$ small, the domain Σ_λ is narrow in the x_1-direction. Hence we can apply the following maximum principle in narrow domains to conclude $w \le 0$ on Σ_λ, for λ sufficiently close to a. □

PROPOSITION 2.1 ([9]). *Consider a second-order elliptic operator in a bounded domain $\Omega \subset \mathbb{R}^N$:*

$$L = M + c = a_{ij}\partial_{ij} + b_i(x)\partial_i + c(x)$$

with L^∞ coefficients and uniformly elliptic:

$$c_0|x|^2 \le a_{ij}(x)x_i x_j \le C_0|\xi|^2, \quad c_0, C_0 > 0\ \forall x \in \mathbb{R}^N,$$

and

$$\sqrt{\Sigma b_i^2} < b, \quad |c| \le b.$$

Assume that the diameter of $\Omega \le d$. There exists $\delta > 0$ depending only on N, d, c_0 and b such that whenever

$$\begin{aligned} Lw \ge 0 &\quad \text{on } \Omega, \\ w \le 0 &\quad \text{on } \partial\Omega. \end{aligned} \tag{2.6}$$

We have $w \le 0$ on Ω, provided

$$\textit{measure } (\Omega) = |\Omega| < \delta. \tag{2.7}$$

□

We give the proof of this proposition after proving the theorem. It is clear that one can find ε such that

$$a - \varepsilon < \lambda < a \Rightarrow |\Sigma_\lambda| < \delta.$$

Proposition 2.1 then implies that

$$w_\lambda \le 0 \quad \text{on } \Sigma_\lambda$$

for these λ's : $a - \varepsilon < \lambda < a$. Thus we have

$$u(x_\lambda) \ge u(x) \quad \text{on } \Sigma_\lambda \text{ for } a - \varepsilon < \lambda < a. \tag{2.8}$$

Thus the set Λ is nonempty.

Step 2: Let μ be the infimum of λ's for which equation (2.8) holds. We would like to show that $\mu = 0$. We suppose that $\mu > 0$ and argue to get a contradiction.

By continuity, $w_\mu \le 0$ on Σ_μ. Since $\mu > 0$, the reflection of the cap is strictly inside the ball where $u > 0$. Thus we have $w_\mu < 0$ on $\overline{\Sigma}_\mu \cap (\partial\Omega \backslash H_\lambda)$. Hence it follows by strong maximum principle, that $w_\mu < 0$ on Σ_μ. We now show that $w_{\mu-\varepsilon}(x) \le 0$ on $\Sigma_{\mu-\varepsilon}$ for all small positive $\varepsilon < \mu$.

Let $\delta > 0$ be the constant given by Proposition 2.1.

$$\Sigma_\mu = K \cup (\Sigma_\mu \setminus K),$$

where K is a compact proper subset of Σ_μ and

$$|\Sigma_\mu \setminus K| < \frac{\delta}{2}. \tag{2.9}$$

The idea is to get the nonpositivity of $w_{\mu-\varepsilon}$, for small ε, in K and $(\Sigma_{\mu-\varepsilon} \setminus K)$ separately by different arguments, leading to a contradiction with the fact μ is the infimum of such λ's.

By compactness of K, for some small $\theta > 0$

$$w_\mu < -\theta \quad \text{on } K.$$

Again by continuity, for small ε, $0 < \varepsilon < \varepsilon_0$,

$$|\Sigma_{\mu-\varepsilon} \setminus K| < \delta.$$

Proposition 2.1 then implies that

$$w_{\mu-\varepsilon} \le 0 \quad \text{on } \Sigma_{\mu-\varepsilon} \setminus K$$

for $0 < \varepsilon < \varepsilon_0$. Combining the two inequalities for $w_{\mu-\varepsilon}$, it follows that

$$w_{\mu-\varepsilon} \le 0 \quad \text{on } \Sigma_{\mu-\varepsilon}$$

for $0 < \varepsilon < \varepsilon_0$, which is a contradiction to the definition of μ. Thus $\mu = 0$.

Now for each λ, $0 < \lambda < a$, we have by strong maximum principle

$$w_\lambda < 0 \quad \text{on } \Sigma_\lambda.$$

Hopf maximum principle then implies that

$$-\frac{\partial w}{\partial x_1} = -\frac{\partial u}{\partial x_1} - \frac{\partial u}{\partial x_1} > 0 \quad \text{on } H_\lambda \cap \Omega.$$

Thus $u_{x_1} < 0$ for $x_1 > 0$. □

PROOF OF PROPOSITION 2.1. We rewrite the equation as

$$a_{ij}u_{ij} + b_i(x)u_i - c^-(x)u = -c^+(x)u$$

and apply the lemma of Alexandroff, Bakelman and Pucci (see [33], Chapter 9) in the following form:

If $c \le 0$ and w satisfies $Lw \ge f$ and $w \le 0$ on $\partial\Omega$, then

$$\sup_\Omega w \le C\|f\|_{L^N},$$

where C depends only on N, c_0, b and d. □

REMARK 2.1. In the above theorems, we can allow the nonlinearity f to depend on $r = |x|$ also, provided that the r dependence has the right monotonicity. In particular, in equation (2.5) for $w = u - u_\lambda = u - v$ in Σ_λ, if

$$\Delta w + C(x, \lambda) w \geq 0$$

then again the maximum principle will be applicable. For that we need

$$f(v, |x_\lambda|) \geq f(v, |x|)$$

so that

$$f(u, |x|) - f(v, |x_\lambda|) \leq f(u, |x|) - f(v, |x|).$$

As $|x| > |x_\lambda|$ in Σ_λ, we require that f should be nonincreasing in the r-variable.

In a similar way, we can allow $f(u)$ to be $f_1 + f_2$ with $f_1(u)$ Lipschitz and $f_2(u)$ with the right monotonicity to allow the application of the maximum principle in the above equation for w. In particular, we require f_2 to be nondecreasing in u.

2.2. *Symmetry results in R^N*

If Ω is the whole space, then the moving-plane procedure has to start in a different way because neither Hopf maximum principle nor maximum principle for narrow domains is available as before. Let us consider positive solutions of the following problem.

$$\begin{aligned} &\Delta u(x) + f(u) = 0 \quad \text{in } \mathbb{R}^N, \quad N \geq 2 \\ &u(x) \to 0 \quad \text{as } |x| \to \infty. \end{aligned} \tag{2.10}$$

In general, to start the procedure, it is important to have the asymptotic decay rate of the solutions near infinity. In the case $f(0) = 0$ and $f'(0) < 0$, all positive solutions can be shown to decay exponentially and are indeed radially symmetric, as the following theorem from [32] shows:

THEOREM 2.4. *Let u be a positive C^2 solution of* (2.10) *with*

$$f(u) = -u + g(u),$$

where g is continuous and $g(u) = O(u^\alpha)$ near $u = 0$ for some $\alpha > 1$. On the interval, $0 \leq s \leq u_0 = \max u$, assume that

$$g(s) = g_1(s) + g_2(s)$$

with g_2 nondecreasing and $g_1 \in C^1$, satisfying, for some $C > 0$, $p > 1$,

$$|g_1(s) - g_1(t)| \leq \frac{C|s - t|}{|\log \min(s, t)|^p}$$

for all $0 \leq s, t \leq u_0$. Then u is radially symmetric about some point x_0 in $\mathbb{R}^N$ and $u_r < 0$ for $r = |x - x_0| > 0$. Furthermore

$$\lim_{r \to \infty} r^{\frac{N-2}{2}} e^r u(r) = \mu > 0. \tag{2.11}$$

Observe that this theorem applies to the case of a smooth nonlinearity f, with $f(0) = 0$ and $f'(0) < 0$. An improvement of the above theorem for the case $f'(s) \leq 0$ for sufficiently small s, is obtained in [39]. In this case, equation (2.10) may have solutions, which decay slower than (2.11), for example, power decay or even logarithmic decay. See example 1, in [39].

THEOREM 2.5. *Suppose that $f'(s) \leq 0$ for sufficiently small $s > 0$. Then all positive solutions of (2.10) must be radially symmetric about the origin up to translation and*

$$u_r < 0 \quad \textit{for } r = |x| > 0.$$

Both these theorems do not apply to the case when $f(u) = u^\alpha$, for some $\alpha > 0$. More generally, when $f(u) \geq 0$ and $f(u) = o(u)$ near $u = 0$, the situation is more complicated because solutions may have different asymptotic behaviours at ∞. The symmetry conclusion can be drawn only for solutions with "fast" decay at ∞. Theorem $1'$ in [32] covers certain such situations. This was later improved in [38]. There it is actually proved for a more general case of fully nonlinear equations. Let us mention here a version for the Laplace equation:

THEOREM 2.6. *Suppose there exist $s_0, \alpha > 0$ such that for all u, v such that $0 < u < v < s_0$ we have*

$$\frac{f(v) - f(u)}{v - u} \leq C(u + v)^\alpha. \tag{2.12}$$

Let u be a positive C^2 solution of (2.10) satisfying

$$u(x) = O\left(\frac{1}{|x|^m}\right) \quad \textit{as } |x| \to \infty \tag{2.13}$$

with $m\alpha > 2$. Then u is radially symmetric and strictly decreasing about some point.

The original proof of Theorem 2.4 was by using the Green's function to write down the solution as

$$u(x) = c \int_{R^N} \frac{f(y)}{|x - y|^{N-2}} dy$$

for some $c > 0$ and then using decay estimates at infinity. The proof in [39] uses the maximum principle but without any decay estimates. We indicate here a simple proof for the last two theorems, following [26], avoiding pointwise arguments and hence extendable directly to the degenerate case of p-laplacian (see section 3 in [26]).

PROOF OF THEOREM 2.5. Let us introduce the half space:

$$\Sigma_\lambda^\nu = \{x \in \mathbb{R}^N : x \cdot \nu < \lambda\},$$

for a direction ν in $\mathbb{R}^N$, i.e. $|\nu| = 1$ and for $\lambda \in \mathbb{R}$. Define the hyperplane

$$H_\lambda^\nu = \{x \in \mathbb{R}^N : x \cdot \nu = \lambda\}$$

and the reflection R_λ^ν through H_λ^ν, i.e.

$$R_\lambda^\nu(x) = x_\lambda^\nu = x + 2(\lambda - x \cdot \nu)\nu,$$

for any $x \in \mathbb{R}^N$. Let

$$(\Sigma_\lambda^\nu)' = R_\lambda^\nu(\Sigma_\lambda^\nu).$$

For a function $u \in C^1(\mathbb{R}^N)$, let the reflected function

$$u_\lambda(x) = u(x_\lambda^\nu) \quad \forall\, x \in \mathbb{R}^N.$$

Let us define the set

$$\Lambda := \{\lambda \in \mathbb{R}\ \forall \mu > \lambda, u \geq u_\mu \text{ in } \Sigma_\mu^\nu\}.$$

As before, the proof consists of these main steps:

(1) Show that Λ is nonempty and bounded from below.

(2) Show that if $\lambda_0 := \inf \Lambda$, then $u \equiv u_{\lambda_0}$ in $\Sigma_{\lambda_0}^\nu$.

Then the radial symmetry and monotonicity of u will follow with standard arguments.

Step 1. Let us put $v = u_\lambda$ and observe that v satisfies the same equation as u in Σ_λ^ν. Since $u \to 0$ when $x \to \infty$, the function $(v - u - \varepsilon)^+$, for $\varepsilon > 0$ has compact support. Further $u = v$ on H_λ^ν. Hence, we can use the test function $(v - u - \varepsilon)^+$ in the equations for v and u in Σ_λ^ν. Subtracting the equations we get

$$\int_{\Sigma_\lambda^\nu \cap [v \geq u+\varepsilon]} |D(v - u - \varepsilon)|^2 = \int_{\Sigma_\lambda^\nu \cap [v \geq u+\varepsilon]} [f(v) - f(u)](v - u - \varepsilon)^+. \quad (2.14)$$

Here we have used the fact that $D(v - u - \varepsilon)^+ = D(v - u)^+ \chi_{\{v \geq u+\varepsilon\}}$. Since $u \to 0$ as $|x| \to \infty$, there exists R such that for x outside B_R, $u(x) < \epsilon_0$. By the assumption, $f'(s) \leq 0$ for $s < \epsilon_0$, for $\lambda > R$, we have $v < \epsilon$ on Σ_λ^ν and hence on the set $\Sigma_\lambda^\nu \cap \{x : v \geq u + \epsilon\}$ the function f' is nonpositive and hence $f(v) - f(u) \leq 0$. Thus it follows from (2.14) that

$$\int_{\Sigma_\lambda^\nu} |D(v - u - \epsilon)^+|^2 \leq 0.$$

As $\epsilon \to 0$, we get by monotone convergence theorem

$$\int_{\Sigma_\lambda^\nu} |D(v - u - \epsilon)^+|^2 \leq 0.$$

Since $(v - u)^+$ vanishes on H_λ^ν, we conclude that it vanishes a.e. on Σ_λ^ν. Thus $v \leq u$ on Σ_λ^ν for all $\lambda > R$ and $\Lambda \neq \phi$. The same argument shows that Λ corresponding to $-\nu$ is also nonempty. Hence for the direction ν, this set is bounded below.

Step 2. Let us put $\lambda_0 := \inf \Lambda$. By continuity of u with respect to λ, $u \geq u_{\lambda_0}$ in $\Sigma_{\lambda_0}^\nu$. Suppose by contradiction that $u \not\equiv u_{\lambda_0}$ in $\Sigma_{\lambda_0}^\nu$. Since $w = w_{\lambda_0} = u - u_{\lambda_0}$ satisfies in $\Sigma_{\lambda_0}^\nu$ the linear equation $-\Delta w = c_{\lambda_0}(x) w$, where $c_{\lambda_0}(x) = \frac{f(u(x)) - f(u_{\lambda_0}(x))}{u(x) - u_{\lambda_0}(x)} \in L_{loc}^\infty(\mathbb{R}^N)$, by the strong maximum principle we get $u > u_{\lambda_0}$ in $\Sigma^{\nu}_{\lambda_0}$. We will show that this implies that the inequality $u \geq u_\lambda$ in Σ^{ν}_{λ} continues to hold when $\lambda < \lambda_0$ is close to λ_0, contradicting the definition of λ_0.

Consider $\lambda < \lambda_0$, close to λ_0. Define P to be the projection of the origin on H^{ν}_{λ}. Let

$$\tilde{B}_\lambda = \Sigma^{\nu}_{\lambda} \cap B(P, R),$$

where $B(P, R)$ is the ball centred at P and has radius R, chosen so that $u(x) < \epsilon_0$, for x outside $B(P, R)$. We split Σ^{ν}_{λ} as follows:

$$\Sigma^{\nu}_{\lambda} = \widetilde{B} U(\Sigma^{\nu}_{\lambda} \setminus \widetilde{B}).$$

As before, we take the test function $(v - u - \epsilon)^+$ in Σ^{ν}_{λ} for the equation for u and v. Subtracting the equations, we get

$$\begin{aligned}\int_{\Sigma^{\nu}_{\lambda}} |D(v-u-\epsilon)^+|^2 &= \int_{\widetilde{B}} (f(v) - f(u))(v-u-\epsilon)^+ \\ &\quad + \int_{\Sigma^{\nu}_{\lambda}\setminus\widetilde{B}} (f(v) - f(u))(v-u-\epsilon)^+ \\ &\leq \int_{\widetilde{B}} (f(v) - f(u))(v-u-\epsilon)^+. \end{aligned} \tag{2.15}$$

This is because $f(v) - f(u) \leq 0$ on $\Sigma^{\nu}_{\lambda} \setminus \widetilde{B}$, since $u(x) < v(x) < \epsilon_0$ on that set.

Recall that for functions in $H^1(\widetilde{B})$ vanishing on $\partial\widetilde{B} \cap H^{\nu}_{\lambda}$, we have the Poincaré's inequality (see for example, Lemma 2.1 in [26]):

$$\|w\|_{2,\widetilde{B}} \leq C|\widetilde{B} \cap \text{ supp } (w)^{1/2N} ||Dw||_{2,\widetilde{B}},$$

C being a constant depending only on N.

Using the Lipschitz constant L, for f on the compact set $\widetilde{B}$,

$$\int_{\Sigma^{\nu}_{\lambda}} |D(v-u-\epsilon)^+|^2 \leq L \int_{\widetilde{B}} |v-u||v-u-\epsilon|^+.$$

Passing to the limit as $\epsilon \to 0$, we get

$$\begin{aligned}\int_{\widetilde{B}} |D(v-u)^+|^2 &\leq \int_{\widetilde{B}} (|v-u|^+)^2 \\ &\leq C|\widetilde{B} \cap \text{ supp}(v-u)^+|^{1/N} \int_{\widetilde{B}} |D(v-u)^+|^2. \end{aligned} \tag{2.16}$$

Since $(u - u_{\lambda_0}) > 0$ on any compact subset K of $\Sigma^{\nu}_{\lambda_0}$, we have by continuity of u, $(u - u_\lambda) > 0$ on $K \subset \Sigma^{\nu}_{\lambda}$, for λ close to λ_0. Thus the supp $(v - u)^+$ is small in $\widetilde{B}$ for $\lambda < \lambda_0$, λ close to λ_0, we conclude that for such λ, when

$$C|\widetilde{B} \cap \text{ supp}(v-u)^+|^{1/N} < 1$$

the function $D(v - u)^+ = 0$ a.e. Since $(v - u)^+ = 0$ on H^{λ}_{ν}, it follows that $(v - u)^+ = 0$ a.e. on $\widetilde{B}$.

Using (2.15) and (2.16), it now follows that $D(v-u)^+ = 0$ a.e. on $\Sigma_\lambda^\nu \setminus \widetilde{B}$ also and hence $v \le u$ on Σ_λ^ν. But this contradicts the minimality of λ_0. Thus the only possibility is $u \equiv u_{\lambda_0}$ on $\Sigma_{\lambda_0}^\nu$. Repeating the argument in N-orthogonal directions, we can conclude that u is radially symmetric with respect to O, the common point of intersection of these planes. □

PROOF OF THEOREM 2.6. This is also along the same lines as the above proof except that we deal differently with the integrals on the unbounded sets. Since $f(u)$ need not be decreasing for small u as in the earlier case, we use the assumption (2.12) on f and then Hardy's inequality to estimate these integrals. To be able to use the test function $(v-u-\varepsilon)^+$, we need to know that

$$\int_{\Sigma_\lambda^\nu} [f(v) - f(u)]^+ (v-u)^+ < \infty.$$

Let B_R be chosen such that $u(x) < s_0$ outside this ball. By our assumption (2.12), we have for $\lambda > R$, $v(x) < s_0$ on Σ_λ^ν and

$$[f(v) - f(u)]^+ [(v-u)^+]^t \le C v^{\alpha+t+1}.$$

By our assumption (2.13), $v^{\alpha+t+1} = O(\frac{1}{|x|^{m(\alpha+t+1)}})$ at infinity. Hence we can find $t \ge 1$ such that $m(\alpha+t+1) > N$ and then

$$\int_{\Sigma_\lambda^\nu} [f(v) - f(u)]^+ [(v-u)^+]^t < \infty.$$

Let us now take $[(v-u-\varepsilon)^+]^t$ as the test function in the equation for u and v. Subtracting the equations we get

$$\begin{aligned} t \int_{\Sigma_\lambda^\nu \cap [v \ge u+\varepsilon]} & [(v-u-\varepsilon)^+]^{t-1} |D(v-u)|^2 \\ &= \int_{\Sigma_\lambda^\nu \cap [v \ge u+\varepsilon]} [f(v) - f(u)][(v-u-\varepsilon)^+]^t \\ &\le \int_{\Sigma_\lambda^\nu \cap [v \ge u+\varepsilon]} [f(v) - f(u)]^+[(v-u-\varepsilon)^+]^t. \end{aligned}$$

As $\varepsilon \to 0$ we get

$$\int_{\Sigma_\lambda^\nu} [(v-u)^+]^{t-1} |D(v-u)^+|^2 \le C \int_{\Sigma_\lambda^\nu} [f(v) - f(u)]^+ [(v-u)^+]^t < \infty.$$

Let $w := [(v-u)^+]^{\frac{t+1}{2}}$. Then this can be written as

$$\int_{\Sigma_\lambda^\nu} |Dw|^2 \le C_0 \int_{\Sigma_\lambda^\nu} [f(v) - f(u)]^+ [(v-u)^+]^t < \infty. \tag{2.17}$$

If $\lambda > R$, then $v = u_\lambda < s_0$ and $\frac{f(v)-f(u)}{v-u} \le C\, v^\alpha$, so we get

$$\int_{\Sigma_\lambda^\nu} |Dw|^2 \le C \int_{\Sigma_\lambda^\nu} v^\alpha w^2 \le C \int_{\Sigma_\lambda^\nu} \frac{1}{|x_\lambda^\nu|^{m\alpha}} w^2. \tag{2.18}$$

Notice that if $\lambda > 0$ then $|x_\lambda^\nu| > |x|$ and by Hardy's inequality,

$$\int_{\Sigma_\lambda^\nu} |Dw|^2 \le C \int_{\Sigma_\lambda^\nu} \frac{1}{|x_\lambda^\nu|^{m\alpha-2}} \frac{w^2}{|x|^2} \le C_1 \left(\sup_{\Sigma_\lambda^\nu} \frac{1}{|x_\lambda^\nu|^{m\alpha-2}} \right) \int_{\Sigma_\lambda^\nu} |Dw|^2. \tag{2.19}$$

As $m\alpha > 2$,

$$\sup_{\Sigma_\lambda^\nu} \frac{1}{|x_\lambda^\nu|^{m\alpha-2}} \to 0 \quad \text{as } \lambda \to \infty.$$

Hence there exists λ' such that if $\lambda \ge \lambda'$ then $\int_{\Sigma_\lambda^\nu} |Dw|^2 = 0$ i.e. w is constant. This implies that $w = 0$, since $w = 0$ on H_λ^ν. Thus $u \ge u_\lambda^\nu$ in Σ_λ^ν and Λ is nonempty.

Step 2. Arguing as in the proof of Step 1 of the earlier theorem but with the test function $[(v - u - \varepsilon)^+]^t$, we arrive at

$$\begin{aligned} t \int_{\Sigma_\lambda^\nu} &[(v - u - \varepsilon)^+]^{t-1} |D(v - u - \epsilon)^+|^2 \\ &= \int_{\widetilde{B}} (f(v) - f(u))[(v - u)^+]^t + \int_{\Sigma_\lambda^\nu \setminus \widetilde{B}} (f(v) - f(u))[(v - u)^+]^t. \end{aligned} \tag{2.20}$$

There exists $R_0 > 0$, $C > 0$, such that for all λ in a neighbourhood of λ_0 we have $u(x) \le \frac{C}{|x|^m}$ if $|x| > R_0$. Then if $R \ge R_0$ is big enough,

$$\begin{aligned} \int_{\Sigma_\lambda^\nu \setminus \widetilde{B}} [f(v) - f(u)]^+ [(v - u)^+]^t &\le C_1 \left(\sup_{\Sigma_\lambda^\nu \setminus \widetilde{B}} \frac{1}{|x_\lambda^\nu|^{m\alpha-2}} \right) \int_{\Sigma_\lambda^\nu \setminus \widetilde{B}} |Dw|^2 \\ &\le \frac{1}{2} \int_{\Sigma_\lambda^\nu \setminus \widetilde{B}} |Dw|^2 \end{aligned} \tag{2.21}$$

for all λ in a neighbourhood of λ_0.

Using the Lipschitz continuity of f in $\widetilde{B}$ and the Poincaré's inequality, we have

$$\begin{aligned} \int_{\widetilde{B}} [f(v) - f(u)]^+ [(v - u)^+]^t &\le C \int_{\widetilde{B}} |w|^2 \\ &\le C \left| B_\lambda^\nu \cap \text{ supp } (u_\lambda - u)^+ \right|^{\frac{1}{N}} \int_{\widetilde{B}} |Dw|^2 \\ &\le \frac{1}{2} \int_{\widetilde{B}} |Dw|^2. \end{aligned} \tag{2.22}$$

Combining (2.21) and (2.22) we get that if λ is close to λ_0 we have

$$\int_{\Sigma_\lambda^\nu} |Dw|^2 \le \frac{1}{2} \int_{\Sigma_\lambda^\nu} |Dw|^2$$

i.e. $Dw = 0$ and $u \ge u_\lambda^\nu$. This contradicts the definition of λ_0 and shows that $u \equiv u_{\lambda_0}$ in $\Sigma_{\lambda_0}^\nu$.

3. Break of radial symmetry

In this chapter we will analyse different cases when the solution of a semi-linear elliptic equation of the type considered in the previous chapter fails to be radial, though both the domain and the data of the problem are spherically symmetric. More precisely we will consider solutions of the equation

$$-\Delta u = f(|x|, u) \quad \text{in } B, \tag{3.1}$$

where B will be either a ball or an annulus in $\mathbb{R}^N$, $N \geq 2$, while the assumptions on the nonlinearity and the boundary conditions will be made precise in each case.

3.1. *Non-Lipschitz nonlinearity*

It is natural to ask if Lipschitz condition on the nonlinearity can be dropped in these symmetry theorems. The example given in [31] shows that in that case, there may be compactly supported positive solutions. If $p > 2$

$$w(x) = (1 - |x|^2)^p, \quad |x| \leq 1,$$
$$w = 0 \quad \text{in } |x| > 1$$

satisfies

$$\Delta w + f(w) = 0 \tag{3.2}$$

with the nonlinearity, a Hölder continuous function with exponent $(1 - 2/p)$

$$f(w) = -2p(p-2)w^{1-2/p} + 2p(n+2p-2)w^{1-1/p}.$$

Then the function

$$u(x) = w(x) + w(x - x_0)$$

with some fixed x_0 satisfying $|x_0| = 3$, satisfies (3.2) in $|x| < 5$ with the same f but u is not radially symmetric, but a combination of two "bumps". In fact, this is typically the situation for non-Lipschitz nonlinearity, as the following theorem from [17] shows:

THEOREM 3.1. *Assume that*
(a) $f(0) \leq 0$, f *continuous in* $[0, \infty)$, *locally Lipschitz in* $(0, \infty)$;
(b) *there exists* $a > 0$ *such that* f *is strictly decreasing in* $[0, a]$.
Let u *be a classical solution of*

$$\Delta u + f(u) = 0 \quad \text{in } \mathbb{R}^N, \quad N > 1$$
$$u \geq 0.$$

Then we have
(i) *Either the support of* u *is the whole of* $\mathbb{R}^N$ *and* u *is radially symmetric with respect to a point.*
(ii) *Or any connected component of the set* $\{x : u(x) > 0\}$ *is a ball and* u *restricted to this ball is radially symmetric with respect to its centre.*

A typical example is

$$f(u) = u^p - u^q$$

for $0 < q < 1 < p < \frac{N+2}{N-2}$ and $N \geq 3$. Note that $f(0) = 0$ and $f < 0$ near origin and f is non-Lipschitz (see [18]).

3.2. *Henon equation*

Let us assume that B is a ball in $\mathbb{R}^N$, $N \geq 2$ and $f(|x|, u) = |x|^\alpha u^p$ with $\alpha > 0$ and $p \in (1, +\infty)$ if $N = 2$ or $p \in \left(1, \frac{N+2}{N-2}\right)$ if $N \geq 3$. Using a constrained minimization method it is quite easy to prove the existence of a positive solution of (3.1) satisfying the boundary condition

$$u = 0 \quad \text{on } \partial B. \tag{3.3}$$

Indeed, defining

$$S_\alpha = \inf_{\substack{v \in H_0^1(B) \\ v \neq 0}} \frac{\int_B |\nabla v|^2 dx}{\left(\int_B |x|^\alpha |v|^{p+1} dx\right)^{2/p+1}},$$

since the embedding $H_0^1(B) \hookrightarrow L^{p+1}(B)$ is compact $\left(p < \frac{N+2}{N-2} \text{ if } N \geq 3\right)$, we have that S_α is achieved by a function v which can be assumed nonnegative and that, after suitable rescaling, solves (3.1)–(3.3). By the strong maximum principle it follows that $v > 0$ in B.

Analogously we can consider another infimum

$$Z_\alpha = \inf_{\substack{v \in H_{0,r}^1(B) \\ v \neq 0}} \frac{\int_B |\nabla v|^2 dx}{\left(\int_B |x|^\alpha |v|^{p+1} dx\right)^{2/p+1}}, \qquad \alpha \geq 0,$$

where $H_{0,r}^1(B)$ is the subspace of the radial functions in $H_0^1(B)$. Obviously, also Z_α is achieved by a radial function which, after rescaling, gives rise to a positive radial solution of (3.1)–(3.3).

Obviously $S_\alpha \leq Z_\alpha$ and the natural question is to ask whether S_α and Z_α coincide, i.e. if the ground state solution of (3.1)–(3.3) is radial or not. The following result is proved in [48].

THEOREM 3.2. *There exists $\alpha^* > 0$ such that for any $\alpha > \alpha^*$, $S_\alpha < Z_\alpha$.*

This implies that a symmetry result as the one described in the previous chapter, does not hold for this type of problem. Comparing with the nonlinearities considered in the Gidas–Ni–Nirenberg theorem, we observe that the function $f(|x|, u) = |x|^\alpha u^p$ does not have the right monotonicity in the x-variable, because it is monotonically increasing.

3.3. *Sign-changing solutions*

Let us start by observing that it is very easy to construct sign-changing solutions which break the symmetry in symmetric domains, in particular those which are not radial in spherically symmetric domains. Indeed it suffices to consider the second eigenfunction of the laplacian operator in the ball which is an antisymmetric function with respect to a hyperplane passing through the origin. More generally if $f(u)$ is an odd nonlinearity and u is a positive solution of the problem

$$\begin{cases} -\Delta u = f(u) & \text{in } B^- \\ u = 0 & \text{on } \partial B^- \end{cases} \tag{3.4}$$

in the half ball $B^- = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N, x_1 < 0\}$, $N \geq 2$, then, by reflecting u by oddness in $B^+ = \{x = (x_1, \dots, x_N), x > 0\}$, we get a sign-changing solution of (3.1)–(3.3) in B which is obviously not radial. On the other hand if B is a ball or an annulus it is often easy, by using the associated ordinary differential equation, to also find radial solutions of (3.1)–(3.3) which change sign. Hence a more subtle question is how to distinguish radial or nonradial solutions, in particular how to prove whether "the least energy" nodal solution is radial or not.

At this point it is useful to observe that for semi-linear elliptic Dirichlet problems of the type

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial \Omega \end{cases} \tag{3.5}$$

when Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, several multiplicity results have been obtained using a variety of variational methods. However only quite recently it has been proved that sign-changing solutions exist, by studying the associated energy functional

$$J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \int_\Omega F(u) dx,$$

where F is a primitive of the nonlinearity f, on the set

$$M = \{u \in H_0^1(\Omega), u^+ \neq 0, u^- \neq 0, (J'(u), u^+) = (J'(u), u^-) = 0)\}$$

(see [14] and [6]). In particular, the existence of a sign-changing solution which minimizes the functional J on M is shown and it is therefore called least energy nodal solution. In particular this solution has Morse index 2 and precisely two nodal regions.

In [3] some results about the break of symmetry of this solution, as well as of solutions of (3.5) with low Morse index are proved. Let us describe them briefly.

Let us assume that $\Omega = B$, i.e. is either a ball or annulus in $\mathbb{R}^N$, $N \geq 2$ and f is a $C^{1,\alpha}$ nonlinearity. We have

THEOREM 3.3. *Any radial sign-changing solution of* (3.5) *has Morse index greater than or equal to* $N + 1$.

In particular, from this theorem it follows that a least energy nodal solution (3.5) is not radial, because it has Morse index equal to 2.

A more general result on the geometry of a symmetric, but nonradial solution, is also obtained in [3].

THEOREM 3.4. *If B is a ball and u is a nodal solution which is even in k-variables, with Morse index less than or equal to k, then its nodal set N intersects the boundary.*

We recall that the nodal set is defined as

$$N = \overline{\{x \in B, u(x) = 0\}}.$$

It can be useful to sketch the idea behind these results. It relies on the study of the sign of some particular eigenvalues of the linearized operator $L = -\Delta - f'(u)$ at the solution u. Indeed let us consider the hyperplanes $T_i = \{x = (x_1, \dots, x_N) \in B, x_i = 0\}$ and the half domains $B_i^- = \{x \in B, x_i < 0\}, i = 1, \dots, N$. Then we denote by μ_i the first eigenvalue of L in B_i. It is easy to see that, by the symmetry of a radial function u in each x_i-variable, the numbers μ_i are also eigenvalues of L in the whole B, with a sign changing corresponding eigenfunction, obtained by odd extension to B of the original eigenfunction in B_i^-. In [3] it is proved, by using the maximum principle, that all μ_i are negative, providing the negative eigenvalues for the linearized operator L in B. Since also the first eigenvalue must be negative we get that the Morse index of a radial solution is greater than or equal to $N + 1$.

Let us observe that the results described hold for solutions of autonomous equations like (3.5). Break of symmetry has also been proved for nonautonomous problems. Some interesting results in this context have recently been obtained in [30] (see also [45]) for a class of superlinear Ambrosetti–Prodi-type problems in the ball. More precisely, in [30] the following semi-linear elliptic problem is studied.

$$\begin{cases} -\Delta u = u^2 - t\varphi_1 & \text{in } B \\ u = 0 & \text{on } \partial B, \end{cases} \tag{3.6}$$

where t is a real parameter and φ_1 is the first eigenfunction of the laplacian in B, with zero Dirichlet boundary data. By comparing the Morse index of solutions in $H_0^1(B)$ or in the subspace $H_{0,r}^1(B)$ of the radial functions, the authors are able to show that the solutions of (3.6) with Morse index one (which always exist by the Mountain Pass Theorem) change sign and are not radially symmetric, for $t > 0$ sufficiently large.

3.4. *Nonconvex domains*

It is easy to understand that the moving-plane method is a continuous procedure which requires the convexity of the domain in the direction of the displacement of the hyperplanes. When the convexity fails it is possible to prove break of symmetry. In the case of spherically symmetric domains this means showing that in some cases there are positive solutions in an annulus which are not radial. An interesting counterexample of this type was given in the classical paper [10] in the presence of a critical nonlinearity. Let us consider the problem

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}} + \lambda u & \text{in } B \\ u > 0 & \text{in } B \\ u = 0 & \text{on } \partial B, \end{cases} \tag{3.7}$$

where B is an annulus in $\mathbb{R}^N$, $N \geq 4$ and λ is a positive real parameter.

In [10] it is proved that for any $\lambda \in (0, \lambda_1)$, λ_1 being the first eigenvalue of $-\Delta$ in $H_0^1(B)$, there is a solution of the problem analogous to (3.7) in general smooth bounded domains Ω. This is achieved by proving that the infimum

$$S_\lambda = \inf_{\substack{v \in H_0^1(\Omega) \\ \|v\|_{L^{2^*}(\Omega)}=1}} \int_\Omega |\nabla v|^2 dx - \lambda \int_\Omega |u|^2 dx, \quad 2^* = \frac{2N}{N-2} \tag{3.8}$$

is achieved, for any $\lambda \in (0, \lambda_1)$, by a positive function v which, up to a suitable rescaling, solves (3.7). In the case of a spherically symmetric domain B, as in Section 3.1, we can also consider the following infimum

$$Z_\lambda = \inf_{\substack{v \in H_{0,r}^1(B) \\ \|v\|_{L^{2^*}(B)}=1}} \int_B |\Delta v|^2 dx - \lambda \int_\Omega |v|^2 dx \tag{3.9}$$

which is achieved if $\lambda > 0$. However, when B is annulus, thanks to the fact that the embedding $H_{0,r}^1(B) \hookrightarrow L^{2^*}(B)$ is compact, we have that the infimum Z_0 defined in (3.9) is achieved even when $\lambda = 0$, while the infimum S_0 for $\lambda = 0$ defined in (3.8) is never achieved in any bounded domain. This remark is the crucial point in [10] to show that, for λ close to zero, $S_\lambda < Z_\lambda$ and hence the least energy solution of (3.7) in an annulus is not radial.

However it is also interesting to remark that the symmetry of all positive solutions of some problems of type (3.1) may be preserved sometimes under symmetric perturbations of the initial domain, even if these perturbations destroy the convexity properties of the domain. More precisely, the following type of results is proved in [35].

Let Ω be a domain with the required convexity properties to apply the moving-plane method and let Ω_n be a sequence of suitable approximating domains which are still symmetric but not convex anymore. A typical example is obtained by making one or more holes in Ω, i.e. $\Omega_n = \Omega \setminus \cup_{i=1}^k B_i$, where B_i are small balls in Ω, whose radius tends to zero, as n tends to ∞. In [35], it is shown that for any given nonlinearity $f(u)$ in a certain class, all positive solutions of a problem of type (3.1) in Ω_n are symmetric if Ω_n is sufficiently close to Ω. In particular, they are radial if Ω_n is an annulus with a small inner radius. Note that the "closeness" of Ω_n to Ω, for which the symmetry is preserved depends on the assigned nonlinearity and hence this result is not in contradiction with the counterexample in the annulus given before.

We conclude this section mentioning that some other interesting symmetry results in nonconvex domains are proved in [36] using again the maximum principle and a variant of the sliding method of [9].

3.5. *Other boundary conditions*

When the Dirichlet boundary conditions are replaced by other kinds of conditions, again radial symmetry breaks even for positive solutions. A typical case is when we deal with homogeneous Neumann boundary conditions and an example can be given by taking a critical nonlinearity.

Let us consider the problem

$$\begin{cases} -\Delta u + \lambda u = u^{\frac{N+2}{N-2}} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \upsilon} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.10}$$

where λ is a positive real parameter and Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$, while υ denotes the outer normal to $\partial\Omega$.

In [1] it was proved that there exists $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$ the problem (3.10) admits a solution which minimizes the functional

$$Q_\lambda(u) = \frac{\int_\Omega |\nabla u|^2 dx + \lambda \int_\Omega u^2 dx}{(\int_\Omega |u|^{2^*})^{2/2^*}}, \quad u \neq 0$$

in the space $H^1(\Omega)$. Subsequently in the paper [2], the authors proved that, for every domain Ω, this solution attains its maximum at only one point belonging to $\partial\Omega$, if λ is sufficiently large. Moreover in higher dimensions this point must belong to the set of the points of maximal mean curvature of the boundary. In the case of the ball, this result implies, obviously, that the least energy solution is not radial. It also suggests that, for Neumann boundary conditions, the ball is not the "right" domain to require that solutions would preserve the symmetry, even if they are positive. Another result in this direction is a spherical symmetry result for positive solutions of a semi-linear equation with mixed boundary conditions in a sector, obtained in [8]. This result indicates that the mixed Neumann–Dirichlet boundary conditions prescribed on suitable parts of the boundary are the right conditions in a convex sector to get the spherical symmetry of the solution. More precisely we consider the semi-linear elliptic problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Sigma(\alpha, R) \\ u = 0 & \text{on } \Gamma_0 \\ \frac{\partial u}{\partial \upsilon} = 0 & \text{on } \Gamma_1, \end{cases} \tag{3.11}$$

where $\Sigma(\alpha, R)$ is the spherical sector of radius $R > 0$ and amplitude $\alpha \in (0, 2\pi)$ defined by

$$\Sigma(\alpha, R) = \{x \in \mathbb{R}^N, 0 < |x| = \rho < R, \theta_i \in (0, \pi), \\ i = 1, .N-2, \theta_{N-1} \in (0, \alpha)\}$$

$(\rho, \theta_1, \ldots, \theta_{N-2}, \theta_{N-1})$ being the polar coordinates in $\mathbb{R}^N$. Consequently, $\partial\Sigma(\alpha, R) = \Gamma_0 \cup \Gamma_1$, where

$$\Gamma_0 = \{x \in \partial\Sigma(\alpha, R), |x| = R\},$$
$$\Gamma_1 = \{x \in \partial\Sigma(\alpha, R), \theta_{N-1} = 0 \text{ or } \theta_{N-1} = \alpha\}$$

The main result proved in [8] is the following.

THEOREM 3.5. *Let* $0 < \alpha \leq \pi$ and $f \in C^1(\mathbb{R})$. *If* $u \in C^2(\overline{\Sigma(\alpha, R)})$ *is a positive solution of* (3.11), *then* u *is spherically symmetric and* $\frac{\partial u}{\partial \rho} < 0$ *for* $0 < \rho < R$.

This result is also optimal with respect to the amplitude of the sector in the sense that for $\alpha \in (\pi, 2\pi)$ it is possible to prove that for some power nonlinearity there are positive solutions of (3.11) which are not spherically symmetric.

4. Partial symmetry results

In the previous chapter we have analyzed several cases when the full symmetry of the solution breaks i.e., in the case of a ball or an annulus, this means that solutions are not radial. Nevertheless, for some nonlinearities, or for certain types of solutions, it is natural to expect that the solution inherits at least part of the symmetry of the domain.

In this chapter we will describe some recent results in this direction in spherically symmetric domains obtained with the aid of the maximum principle, exploiting the Morse index of the solution. The type of symmetry proved is an axial symmetry with a monotonicity in the angular coordinate which is often referred as foliated Schwarz symmetry or codimension 1 symmetry. In the last section we will also indicate some axial symmetry results for solutions of some asymptotic problems.

4.1. *Preliminaries on foliated Schwarz symmetry*

Let us consider a semi-linear elliptic problem of the type:

$$\begin{cases} -\Delta u = f(|x|, u) & \text{in } B \\ u = 0 & \text{on } \partial B, \end{cases} \tag{4.1}$$

where B is either a ball or an annulus centred at zero in $\mathbb{R}^N$, $N \geq 2$, and $f : \overline{B} \times \mathbb{R} \to \mathbb{R}$ is a locally $C^{1,\alpha}$ function. Now we recall the definition of foliated Schwarz symmetry.

DEFINITION 4.1. We say that a function $v \in C(\overline{B})$ is foliated Schwarz symmetric if there is a unit vector $p \in \mathbb{R}^N$ such that $v(x)$ only depends on $r = |x|$ and $\theta := \arccos(\frac{x}{|x|} \cdot p)$, and u is nonincreasing in θ.

Let S be the unit sphere in $\mathbb{R}^N$, $S = \{x \in \mathbb{R}^N : |x| = 1\}$. For a unit vector $e \in S$ we consider the hyperplane $H(e) = \{x \in \mathbb{R}^N : x.e = 0\}$ and the open half domain $B(e) = \{x \in B : x.e > 0\}$. We write $\sigma_e : B \to B$ for the reflection with respect to $H(e)$, that is $\sigma_e(X) = x - 2(x.e)e$ for every $x \in B$. Note that

$$H(-e) = H(e) \quad \text{and} \quad B(-e) = \sigma_e(B(e)) = -B(e) \quad \text{for every } e \in S.$$

In the sequel we will use a variant of the moving-plane method which is very suitable to find symmetry hyperplanes in spherically symmetric domains. It consists in rotating hyperplanes, rather than translating them parallel, starting from an original position where a comparison between a solution u of (4.1) and its reflection holds. Again the main ingredient to carry on the procedure is the maximum principle.

LEMMA 4.1. *Let $u \in H_0^1(B) \cap C(\overline{B})$ be a weak solution of (4.1) and assume that for a unit vector $e \in S$ we have*

$$u(x) \geq u(\sigma_e(x)) \quad \textit{for any } x \in B(e). \tag{4.2}$$

Then for any rotation axis $\overline{\tau}$ belonging to $H(e)$ and passing through the origin there exists a $\overline{\theta} \in [0, \pi)$ such that u is symmetric with respect to the hyperplane $H(e_{\overline{\theta}})$, where $e_{\overline{\theta}}$ is the unit vector belonging to the hyperplane through the origin orthogonal to $\overline{\tau}$ and forming with e an angle $\overline{\theta}$.

PROOF. Let us define the function

$$w_e(x) = u(x) - u(\sigma_e(x)), \quad x \in B(e)$$

which satisfies the equation

$$-\Delta w_e(x) = V_e(x) w_e(x) \quad \text{in } B(e), \tag{4.3}$$

where

$$V_e(x) = \begin{cases} \frac{f(|x|,u(x)) - f(|x|,u(\sigma_e(x)))}{w_e(x)} & \text{if } w_e(x) \neq 0 \\ 0 & \text{if } w_e(x) = 0 \end{cases} \tag{4.4}$$

and obviously, $V_e(x) \in L^\infty(B)$.

By (4.2) we have that $w_e(x) \geq 0$ in $B(e)$ and hence, by (4.3) and the strong maximum principle, either $w_e \equiv 0$ in $B(e)$ or $w_e > 0$ in $B(e)$. In the first case the assertion is proved, taking $\bar{\theta} = 0$, in the second case we start the rotating-plane procedure in the following way. Without loss of generality we set $e = (0, 0, \dots, 0, 1)$,

$$e_\theta = (\sin\theta, 0, \dots, 0, \cos\theta), \quad \theta \geq 0$$

and define

$$\overline{\theta} = \sup\{\theta \in [0, \pi) : w_{e_\theta} > 0 \text{ in } B(e_\theta)\}.$$

Because of (4.2) and by continuity we have $w_{e_{\overline{\theta}}}(x) \geq 0$, which, by the strong maximum principle implies that either $w_{e_{\overline{\theta}}} \equiv 0$ in $B(e_{\overline{\theta}})$ which proves the assertion, or $w_{e_{\overline{\theta}}} > 0$ in $B(e_{\overline{\theta}})$. Exactly as in the classical moving-plane procedure, described in chapter 2, this second possibility is excluded, by the maximality of $\overline{\theta}$, using the maximum principle in domains of small measure. □

Now we prove some sufficient conditions for the foliated Schwarz symmetry (see also [13]).

LEMMA 4.2. *Let u be a solution of (4.1) and assume that for every unit vector $e \in S$ either $u(x) \geq u(\sigma_e(x))$ for all $x \in B(e)$ or $u(x) \leq u(\sigma_e(x))$ for all $x \in B(e)$. Then u is foliated Schwarz symmetric.*

PROOF. Let $l_1 = (1, 0 \ldots, 0)$ and $l_2 = (0, 1, 0, \ldots, 0)$ and denote $x \in \mathbb{R}^N$, by (x_1, x_2, x''), $x'' \in R^{N-2}$. We introduce the polar coordinates (z, θ) in the plane (x_1, x_2), by

$$x_1 = z\cos\theta, \quad x_2 = z\sin\theta, \quad \theta \in [0, 2\pi] \quad z \geq 0.$$

Applying Lemma 4.2 starting with the hyperplane $H(l_2)$ and choosing as rotation axis the one defined by $x_1 = x_2 = 0$ we get that u is symmetric with respect to $H(e_{\overline{\theta}})$, for $e_{\overline{\theta}} = (-\sin\overline{\theta}, \cos\overline{\theta}, 0, \ldots, 0)$, for some $\overline{\theta} \in [0, \pi)$. Let us define $e_1 = e_{\overline{\theta}}$ and let S_1 be the hyperplane orthogonal to e_1. In S_1 we fix two mutually orthogonal vectors e_3 and e_4 and, repeating the same procedure we find a vector e_2 in S_1 such that u is symmetric with respect to $H(e_2)$. Then we define the subspace S_2 orthogonal to e_1 and e_2 and repeat again the same procedure. In this way we find $N-1$ orthogonal unit vectors $\{e_1, \ldots, e_{N-1}\}$ such that

$$u(x) = u(\sigma_{e_i}(x)) \quad \text{for any } x \in B(e_i), i = 1, \ldots, N-1.$$

Now let us prove that u is axially symmetric around the axis e determined by one of the two unit vectors orthogonal to the $span\{e_1, \ldots, e_{N-1}\}$. We denote by e one of these two vectors, (the other one being $-e$) and by

$$H(l) = \{x \in \mathbb{R}^N : x.l = 0\}, \qquad l \in span\{e_1, \ldots, e_{N-1}\},$$

any hyperplane containing e. We have to prove that

$$u(x) = u(\sigma_e(x)) \quad \text{for any } x \in B(e).$$

By the symmetry of u with respect to $H(e_i)$, $i = 1 \ldots N-1$, it is easy to see that

$$u(\sigma_e(te)) = u(te) \quad \text{for any } t \in \mathbb{R}.$$

On the other hand, by hypothesis we have

$$u(x) \geq u(\sigma_e(x)) \quad (\text{or } u(x) \leq u(\sigma_e(x)) \quad \text{for any } x \in B(e).$$

Hence, by the strong maximum principle, we must have $u(x) \equiv u(\sigma_e(x))$ for any $x \in B(e)$ and the axial symmetry follows.

The monotonicity of u with respect to the angular coordinate follows by applying the maximum principle to the function $\frac{\partial u}{\partial \theta}$ which solves the linearized equation, after a suitable choice of coordinates, as shown in [42], Proposition 2.3 (see also Lemma 4.3 below). This yields that either u is radial or is never symmetric with respect to a hyperplane which does not pass through the symmetry axis. □

We now give a simple criterion for the foliated Schwarz symmetry. Let us consider the linearized operator.

$$\begin{aligned} &L : H^2(B) \cap H_0^1(B) \subset L^2(B) \to L^2(B) \\ &Lv : -\Delta v - V_u(X)v, \end{aligned} \tag{4.5}$$

where $V_u(x) := f'(|x|, u(x)) = \frac{\partial f}{\partial u}(|x|, u(x))$. Since, by elliptic regularity theory, $u \in C^{3,\alpha}(\overline{B})$ the function $V_u(x)$ is continuous in $\overline{B}$. Then the operator L is selfadjoint and its spectrum consists of a sequence of eigenvalues $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \to \infty$.

We denote instead by $\lambda_k(e, V_u)$ the eigenvalues of the operator L in the half domain $B(e)$, with homogeneous Dirichlet boundary conditions. □

LEMMA 4.3. *Let u be a solution of (4.1) and assume that there exists $e \in S$ such that u is symmetric with respect to the hyperplane $H(e)$ and such that $\lambda_1(e_1, V_u) \geq 0$. Then u is foliated Schwarz symmetric.*

PROOF. Since the proof is quite long and technical we only sketch it, all details can be found in the proof of Proposition 2.3 of [42]. After a rotation, we may assume that $e_2 = (0, 1, 0, \dots, 0)$, hence $H(e) = \{x \in \mathbb{R}^N : x_2 = 0\}$. We want to apply Lemma 4.2. So we consider an arbitrary unit vector $e' \in S$ different from $\pm e$. After another orthogonal transformation which leaves e_2 and $H(e_2)$ invariant, we may assume that $e' = (\cos\theta_0, \sin\theta_0, 0, \dots, 0)$ for some $\theta_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Now we choose new coordinates, replacing x_1, x_2 by polar coordinates r, θ with $x_1 = r\cos\theta, x_2 = r\sin\theta$ and leaving $x'' := (x_3, \dots, x_N)$ unchanged. Differentiating the equation in (4.1) with respect to θ, we obtain $-\Delta u_\theta = f'(|x|, u)u_\theta = V_u(x)u_\theta$ in B, where u_θ stands for $\frac{\partial u}{\partial\theta}$. Moreover, $u_\theta = 0$ on ∂B, because $u = 0$ on ∂B. Hence u_θ is an eigenfunction of L corresponding to the eigenvalue 0. Moreover, since u is symmetric with respect to $H(e)$, it is easy to see that u_θ is antisymmetric with respect to $H(e)$. We claim that u_θ does not change sign in $B(e)$, otherwise its restriction to $B(e)$ would be a sign-changing Dirichlet eigenfunction, corresponding to the zero eigenvalues, of the operator L in $B(e)$. This contradicts the assumption $\lambda_1(e, V_u) \geq 0$. Hence u is monotone in $B(e)$ (and also in $B(-e)$), with respect to θ. This allows to prove the comparison assumption of Lemma 4.2, which gives the foliated Schwarz symmetry of u. □

4.2. *Symmetry of Morse index one solutions*

In this section we will prove that solutions of (4.1) with Morse index less than or equal to 1 are foliated Schwarz symmetric if the nonlinearity is convex in the second variable. This result was proved in [43] assuming the strict convexity of the nonlinearity $f(u)$. The proof we present here is slightly different from that given in [43] with the aim of unifying the arguments of this chapter and of relaxing the hypothesis of strict convexity made in [43].

Let us recall that the Morse index of a solution u of (4.1) is the number of negative eigenvalues of the linearized operator L.

We start with a preliminary result which emphasizes the role of the eigenvalue $\lambda_1(e, V_u)$ in the foliated Schwarz symmetry.

PROPOSITION 4.1. *Let B be a ball or an annulus and is a solution of (4.1) with $f(|x|, u)$ convex in the second variable. If there exists a direction $e \in S$, such that $\lambda_1(e, V_u) \geq 0$ then there exists a possibly different direction e' such that u is symmetric with respect to $H(e')$ and $\lambda_1(e', V_u) \geq 0$.*

PROOF. Let us consider again the function

$$w_e(x) = u(x) - u(\sigma_e(x)), \quad x \in B(e)$$

and observe that, by the convexity of $f(|x|, u)$ in the second variable we have

$$f(|\sigma_e(x)|, u(\sigma_e(x))) - f(|x|, u(x)) \geq f'(|x|, u(x))w_e(x).$$

Hence w_e satisfies

$$\begin{cases} -\Delta w_e \geq f'(|x|, u)w_e & \text{in } B(e) \\ w_e = 0 & \text{on } \partial B(e). \end{cases} \tag{4.6}$$

We claim that w_e does not change sign in $B(e)$. Indeed, if $w_e < 0$ in a connected component D, strictly contained in $B(e)$, we would have, by (4.6)

$$\int_D |Dw_e|^2 - \int_D f'(|x|, u)w_e^2 \leq 0$$

which implies that the first eigenvalue of L in D is less than or equal to zero. Hence, by monotonicity, $\lambda_1(e, V_u) < 0$, against the hypothesis. So w_e does not change sign in B_e and we can assume that $w_e \geq 0$ in B_e. By the strong maximum principle either $w_e \equiv 0$ in B_e from which the assertion would follow, or $w_e > 0$ in B_e. In this last case we are in a position to apply Lemma 4.1 and we find by applying the rotating-plane method, a direction $e_{\tilde{\theta}}$ (see the notations in the proof of Lemma 4.2) such that u is symmetric with respect to the hyperplane $H(e_{\tilde{\theta}})$,

$$\tilde{\theta} = \sup\{\theta \in [0, \pi) : w_{e_\theta} > 0 \text{ in } B(e_v)\}. \tag{4.7}$$

In addition we have that, for every $\theta \in [0, \pi)$, w_{e_θ} solves the linear problem

$$\begin{cases} -\Delta w_{e_\theta} = V_{e_\theta}(x)w_{e_\theta} & \text{in } B(e_\theta) \\ w_{e_\theta} = 0 & \text{on } \partial B(e_\theta), \end{cases}$$

where V_{e_θ} is defined as in (4.4).

Hence w_{e_θ} is an eigenfunction for the operator $-\Delta - V_{e_\theta}(x)$ in $B(e_\theta)$, corresponding to the zero eigenvalue which is the first one, whenever $w_{e_\theta} > 0$. Then, by (4.7), we have, by continuity, that the first eigenvalue $\lambda_1(e, V_{e_\theta})$ of the operator $-\Delta - V_{e\tilde{\theta}}(x)$ in $B(e_{\tilde{\theta}})$ is zero. Because of the symmetry of u with respect to $H(e_{\tilde{\theta}})$ we have that $V_{e_{\tilde{\theta}}} = f'(|x|, u(x))$, hence $\lambda_1(e', V_u) = 0$ for $e' = e_{\tilde{\theta}}$. □

We can now prove the symmetry result for solution of Morse index one.

THEOREM 4.1. *Let B be either a ball or an annulus and assume that u is a solution of* (4.1) *with $f(|x|, u)$ convex in the second variable. If the Morse index of u is equal to one then u is foliated Schwarz symmetric.*

PROOF. Since the solution u has Morse index one the second eigenvalue of the linearized operator in B, $\lambda_2 = \lambda_2(L, B)$ is nonnegative. Thus, using the variational characterization of the second eigenvalue it is easy to prove that for any direction $e \in S$, at least one among $\lambda_1(e, V_u)$ and $\lambda_1(-e, V_u)$ must be nonnegative. By first using Proposition 4.1 and then Lemma 4.3, we get that u is foliated Schwarz symmetric. □

REMARK 4.1. If $f(|x|, u)$ is convex in the second variable and the solution u of (4.1) has Morse index zero it is easy to see that u is radial. Indeed in this case the first eigenvalue of the linearized operator in the whole B is nonnegative. This implies that, for any $e \in S$, both eigenvalues $\lambda_1(e, V_u)$ and $\lambda_1(-e, V_u)$ are positive, i.e. the maximum principle holds for the operator $L = -\Delta - f'(|x|, u)$ in $B(e)$ and $B(e)$ and $B(-e)$. Then by (4.6), we get that $w_e \geq 0$ in $B(e)$ and, analogously, $w_e \geq 0$ in $B(-e)$. Hence $w_e \equiv 0$, for any direction $e \in S$.

REMARK 4.2. Theorem 4.1 applies to solutions of a large variety of problems. Indeed in many cases a solution is found by using the Mountain Pass theorem of Ambrosetti and Rabinowitz and it is well known that these solutions have Morse index one. In particular it can be applied to some of the problems considered in chapter 2 when the radial symmetry is broken. In particular it shows that the ground state solution of the Henon equation (see Section 3.1) is foliated Schwarz symmetric. This was also proved later in [49] using a symmetrization method which applies to minimization problems. From Theorem 4.1 it also follows that the solutions of (3.5) considered in [30] are foliated Schwarz symmetric, as well as the least energy solution of problem (3.6) in the annulus.

REMARK 4.3. Concerning solutions of Neumann problems let us mention that the result of Theorem 4.6 can be extended by using the first eigenvalues of the linearized operator L in the half domains $B(e)$ with respect to mixed boundary conditions. This is proved, among analogous results in unbounded domains, in [41] and it shows, in particular, that the least energy solution of (3.10) (as well as that of the analogous subcritical problem) is foliated Schwarz symmetric.

4.3. *Symmetry of solutions with higher Morse index*

In the previous section we have proved a symmetry result for solutions of Morse index one of (4.1) when the nonlinearity $f(|x|, u)$ is convex in u. Though this result, as indicated at the end of the section, applies to a large number of problems, there are cases when it is not applicable, in particular, for some problems with sign-changing solutions. Indeed, often sign-changing solutions have Morse index greater than one or the nonlinearity $f(|x|, u)$, when considered on the whole real line, is not convex in the second variable. This is, for example, the case of the simple nonlinearity given by $f(s) = |s|^{p-1}s$, $p > 1$. A first result, concerning variational problems where minimizers are sign-changing functions has been obtained in [7] where, using symmetrization techniques a foliated Schwarz symmetry result is obtained.

In this section we go back to the approach of [43], based on the maximum principle and Morse index information, and prove general symmetry results for solutions of (4.1) having higher Morse index, in the case where the nonlinearity f has its first derivative, with respect to the second variable, convex in the second variable.

We start with a preliminary result which needs some further notations.

For a solution u of (4.1) and a direction $e \in S$. We denote by V_{es} the even part of the potential $V_u(x) = f'(|x|, u(x))$ relative to the reflection with respect to the hyperplane

$H(e)$, i.e. we define

$$V_{es}(x) = \frac{1}{2}\left[f'(|x|, u(x)) + f'(|x|, u(\sigma_e(x)))\right]. \tag{4.8}$$

Then we denote by $\lambda_k(e, V_{es})$ the eigenvalues of the operator $(-\Delta - V_{es})$ in the half domain $B(e)$, with homogeneous boundary conditions.

PROPOSITION 4.2. *Assume that $f'(|x|, u)$ is convex in the second variable for every $x \in B$, and let u be a solution of (4.1) and $e \in S$ a direction such that $\lambda_1(e, V_{es}) \geq 0$. Then*

(i) *if either $\lambda_1(e, V_{es}) > 0$ or $f'(|x|, s)$ is strictly convex in S, then u is symmetric with respect to the hyperplane $H(e)$ and hence $\lambda_1(e, V_u) = \lambda_1(e, V_{es}) \geq 0$*

(ii) *there exists a possibly different direction $e' \in S$ such that u is symmetric with respect to the hyperplane $H(e')$ and $\lambda_1(e', V_u) \geq 0$.*

Before proving this proposition let us observe that combining it with Lemma 4.3 we immediately get:

COROLLARY 4.1. *Under the assumption of Proposition 4.2 the solution u is foliated Schwarz symmetric.*

PROOF OF PROPOSITION 4.2. As usual we denote by w_e the difference between u and its reflection with respect to the hyperplane $H(e)$. It is easy to see that w_e solves the linear problem

$$\begin{cases} -\Delta w_e - V_e(x) w_e = 0 & \text{in } B(e) \\ w_e = 0 & \text{on } \partial B(e), \end{cases} \tag{4.9}$$

where

$$V_e(x) = \int_0^1 f'(|x|, tu(x) + (1-t)u(\sigma_e(x))dt, \qquad x \in B(e).$$

Since f' is convex in the second variable, we have

$$\begin{aligned} V_e(x) &\leq \int_0^1 \left[tf'(|x|, u(x)) + (1-t)f'(|x|, u(\sigma_e(x))\right] dt \\ &= \frac{1}{2}\left[f'(|x|, u(x)) + f'(|x|, u(\sigma_e(x)))\right] = V_{es}(x) \end{aligned} \tag{4.10}$$

for any $x \in B$. Here the strict inequality holds if f' is strictly convex and $u(x) \neq u(\sigma_e(x))$. Hence, denoting by $\lambda_k(e, V_e)$ the eigenvalues of the linear operator $-\Delta - V_e(x)$ in $B(e)$ with homogeneous Dirichlet boundary conditions, we have, by (4.10), that $\lambda_k(e, V_e) \geq \lambda_k(e, V_{es})$. In particular $\lambda_1(e, V_e) \geq \lambda_1(e, V_{es}) \geq 0$ by hypothesis. If $\lambda_1(e, V_e) > 0$, then $w_e \equiv 0$ because it satisfies (4.9) and hence we get (i). If $\lambda_1(e, V_e) = 0$, then $V_e = V_{es}$ in B by the strict monotonicity of eigenvalues with respect to the potential. In the case where f' is strictly convex, this implies again that $u(x) \equiv u(\sigma(x))$ for any $x \in B$ and hence (i) holds. It remains to consider the case where f' is only convex and $\lambda_1(e, V_e) = 0$. In this case either $w_e \equiv 0$ or w_e does not change sign in $B(e)$ and, by the strong maximum principle, $w_e > 0$ or $w_e < 0$ in $B(e)$. Thus we are in the condition to apply the rotating-plane method, exactly as in the proof of Proposition 4.1 and we get (ii). □

REMARK 4.4. By Proposition 4.2 it follows that any solution (4.1) of Morse index zero is radial if f' is convex in the second variable. Indeed, in this case the first eigenvalue λ_1 of the linearized operator, $-\Delta - V_u(x)$ in B is nonnegative. Moreover it is easy to see that $\lambda_1(B, V_{es}) \geq \lambda_1 \geq 0$ for any unit vector $e \in S$, denoting by $\lambda_1(B, V_{es})$ the first Dirichlet eigenvalue of the operator $-\Delta - V_{es}$ in the whole domain B. Hence $\lambda_1(e, V_{es}) > 0$ for any direction e, which yields, by Proposition 4.2, the symmetry of the solution with respect to any hyperplane passing through the origin.

Now we can prove the main symmetry result.

THEOREM 4.2. *Let B be either a ball or an annulus in $\mathbb{R}^N$, $N \geq 2$ and $f'(|x|, u)$ convex in the second variable, for every $x \in B$. Then every solution of* (4.1) *with Morse index $j \leq N$ is foliated Schwarz symmetric.*

PROOF. For functions $v, w \in H_0^1(B)$ we define

$$\langle v, w\rangle = \int_B v(x)w(x)dx$$

$$Q_u(v, w) = \int_B \nabla v(x)\nabla w(x)dx - \int_B V_u(x)v(x)w(x)dx,$$

$$Q_{es}(v, w) = \int_B \nabla v(x)\nabla w(x)dx - \int_B V_{es}(x)v(x)w(x)dx.$$

Assume that the solution u has Morse index $m(u) = j \leq N$. Thus, for the Dirichlet eigenvalues λ_k of the linearized operator $L = -\Delta - V_u(x)$ in B we have

$$\lambda_1 < 0, \ldots, \lambda_j < 0 \quad \text{and} \quad \lambda_{j+1} \geq 0. \tag{4.11}$$

We first assume that $j \leq N - 1$ and show how to obtain the foliated Schwarz symmetry of u applying Corollary 4.1 in a simple way. For any direction $e \in S$, let us denote by $g_e \in H_0^1(B)$ the odd extension in B of the positive L^2-normalized eigenfunction of the operator $-\Delta - V_{es}(x)$ in the half domain $B(e)$ corresponding to $\lambda_1(e, V_{es})$. It is easy to see that g_e depends continuously on e in the L^2-norm. Moreover, $g_{-e} = -g_{-e} = -g_e$ for every $e \in S$. Now we denote by $\varphi_1, \varphi_2, \ldots, \varphi_j \in H_0^1(B)$ the L^2-orthonormal eigenfunctions of L corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_j$. It is well known that

$$\inf_{\substack{v \in H_0^1(B)\setminus\{0\} \\ \langle v,\varphi_1\rangle = \cdots = \langle v,\varphi_j\rangle = 0}} \frac{Q_u(v, N)}{\langle v, v\rangle} = \lambda_{j+1} \geq 0. \tag{4.12}$$

We consider the map

$$h : S \to \mathbb{R}^j, \qquad h(e) = \left[\langle [g_e, \varphi_1\rangle, \ldots, \langle g_e, \varphi_j\rangle\right]. \tag{4.13}$$

Since h is an odd continuous map defined on the unit sphere $S \subset \mathbb{R}^N$ and $j \leq N - 1$, h must have a zero by the Borsuk–Ulam Theorem. This means that there is a direction $e \in S$

such that g_e is L^2-orthogonal to all eigenfunctions $\varphi_1, \ldots, \varphi_j$. Thus $Q_u(g_e, g_e) \geq 0$ by (4.12). But, since g_e is an odd function

$$Q_u(g_e, g_e) = Q_{es}(g_e, g_e) = 2\lambda_1(e, V_{es})$$

which yields that $\lambda_1(e, V_{es}) \geq 0$. Having obtained a direction for which $\lambda_1(e, V_{es})$ is nonnegative, Corollary 4.1 applies and yields the foliated Schwarz symmetry of u.

Now we turn to the more difficult case when the Morse index $m(u)$ is equal to N. The main difficulty in this case is that the map h, considered in (4.13), now goes from the $(N-1)$-dimensional sphere S into $\mathbb{R}^N$, so that the Borsuk–Ulam theorem does not apply. We therefore use a different and less direct argument to find a symmetry hyperplane $H(e')$ for u with $\lambda_1(e', V_u) \geq 0$, so that Lemma 4.3 can be applied. We start defining

$$S_* = \{e \in S : w_e = 0 \text{ in } B \text{ and } \lambda_1(e, V_u) < 0\}$$

which is a symmetric set. We observe that either $S_* = \phi$ or in S_* there exists at most k-orthogonal direction $e_i, \ldots, e_k$, with $k \in \{1, \ldots, N-1\}$. Indeed each eigenfunction g_{e_i}, corresponding to $\lambda_1(e_{i,V_u})$, $i = 1, \ldots, k$ can be reflected by oddness to the whole B, producing k negative eigenvalues of the linearized operator L in B, whose corresponding eigenfunctions change sign. Hence, considering also the first eigenvalue of L in B, there are $k+1$ negative eigenvalues and $k+1 \leq N$ implies $k \leq N-1$. The same argument shows that if we denote by L_0 the selfadjoint operator which is the restriction of the operator L to the symmetric space V_0, i.e.

$$\begin{aligned} &L_0 : H^2(B) \cap H_0^1(B) \cap V_0 \to V_0 \\ &L_0 v = -\Delta v - V_u(x) v \end{aligned}$$

and we denote by μ_0 the number of the negative eigenvalues of L_0 (counted with multiplicity), we have

$$1 \leq \mu_0 \leq N - k. \tag{4.14}$$

Note that the inequality on the left just follows from the fact that $\varphi_1 \in V_0$ is an eigenfunction of L relative to a negative eigenvalue. Now we first assume that $S_* \neq \phi$ and consider $S^* = S \cap H(e_1) \cap \cdots \cap H(e_k)$. By the maximality of k, we have $S^* \cap S_* = \phi$. If we prove that there exists a direction $e \in S^*$ such that w_e does not change sign in $B(e)$, for example

$$w_e(x) \geq 0 \quad \text{for every } x \in B(e) \tag{4.15}$$

then the assertion of the theorem is proved. Indeed if $w_e \equiv 0$, because $e \in S^*$ and $S^* \cap S_* = \phi$ we have that $\lambda_1(e, V_u) \geq 0$ so that the foliated Schwarz symmetry follows from Lemma 4.3. If instead (4.15) holds and $w_e \not\equiv 0$ then, by the strong maximum principle, $w_e > 0$ in $B(e)$ and we can start rotating the hyperplanes as in the proof of Proposition 4.1, reaching a direction e' such that $w_e' = 0$ and $\lambda(e', V_u') = 0$. Thus Lemma 4.3 can be again applied. If $S_* = \phi$, we take $S^* = S$ and repeat exactly the same procedure. Hence our aim is to prove (4.15).

Arguing by contradiction we assume that w_e changes sign in $B(e)$ for every $e \in S^*$ and consider the functions.

$$w_e^1 = w_e^+ \chi_{B(e)} - w_e^- \chi_{B(-e)}, \qquad w_e^2 = -w_e^- \chi_{B(e)} + w_e^+ \chi_{B(-e)},$$

where $w^+ = \max\{w, 0\}$, $w^- = \min\{w, 0\}$ and χ_Ω denotes the characteristic function of a set Ω. Since $u \in V_0$, we find that $w_e^1, w_e^2 \in V_0 \cap H_0^1(B) \setminus \{0\}$, and both functions are nonnegative and symmetric with respect to σ_e. Moreover,

$$w_{-e}^1 = w_e^2 \quad \text{and} \quad w_{-e}^2 = w_e^1 \quad \text{for every } e \in S^*. \tag{4.16}$$

By the definition of $V_e(x)$, since w_e satisfies the equation in (4.9) in the whole domain B, with $w_e = 0$ on ∂B, multiplying by $w_e^+ \chi_{B(e)} + w_e^- \chi_{B(-e)}$ and integrating over B we obtain

$$\begin{aligned}
0 &= \int_B \nabla w_e \nabla (w_e^+ \chi_{B(e)} + w_e^- \chi_{B(-e)}) dx \\
&\quad - \int_B V_e(x) w_e (w_e^+ \chi_{B(e)} + w_e^- \chi_{B(-e)}) dx \\
&= \int_B (|\nabla (w_e^+ \chi_{B(e)})|^2 + |\nabla (w_e^- \chi_{B(-e)})|^2 dx \\
&\quad - \int_B V_e(x) [(w_e^+ \chi_{B(e)})^2 + (w_e^- \chi_{B(-e)})^2] dx \\
&= \int_B |\nabla w_e^1|^2 dx - \int_B V_e(x) (w_e^1)^2 dx.
\end{aligned}$$

Now we can use the comparison between the potential $V_e(x)$ and $V_{es}(x)$, given by (4.10), obtaining

$$\begin{aligned}
0 &\geq \int_B |\nabla w_e^1|^2 dx - \int_B V_{es}(x) (w_e^1)^2 dx = Q_{es}(w_e^1, w_e^1) \\
&= Q_u(w_e^1, w_e^1),
\end{aligned} \tag{4.17}$$

because w_e^1 is a symmetric function with respect to σ_e. Similarly we can show

$$Q_u(w_e^2, w_e^2) \leq 0. \tag{4.18}$$

Now, for every $e \in S^*$, we let $\psi_e \in V_0 \cap H_0^1(B)$ be defined by

$$\psi_e(x) = \left(\frac{\langle w_e^2, \varphi_1 \rangle}{\langle w_e^1, \varphi_1 \rangle} \right)^{1/2} w_e^1 - \left(\frac{\langle w_e^1, \varphi_1 \rangle}{\langle w_e^2, \varphi_1 \rangle} \right)^{1/2} w_e^2(x).$$

Using (4.16), it is easy to see that $e \to \psi_e$ is an odd and continuous map from S^* to V_0 by construction, $\langle \psi_e, \varphi_1 \rangle = 0$ for all $e \in S^*$. Moreover, since w_e^1 and w_e^2 have disjoint supports (4.17) and (4.18) imply

$$Q_u(\psi_e, \psi_e) \leq 0 \quad \text{for all } e \in S^*. \tag{4.19}$$

Recalling (4.14) we now distinguish the following cases:

Case 1. $\mu_0 \geq 2$. Then let $\lambda_1, \overline{\lambda_2}, \ldots, \overline{\lambda_{\mu_0}}$ be the negative eigenvalues of the operator L_0 in increasing order, and let $\varphi_1, \overline{\varphi_2}, \ldots, \overline{\varphi_{\mu_0}} \in V_0$ be the corresponding L^2-orthonormal eigenfunctions. Similar to that in (4.12) we have

$$\inf_{\substack{v \in H_0^1(B) \cap V_0, v \neq 0 \\ \langle v, \varphi_1 \rangle = \langle v, \overline{\varphi_2} \rangle, \ldots, \langle v, \overline{\varphi_{\mu_0}} \rangle = 0}} \frac{Q_u(v, v)}{\langle v, v \rangle} \geq 0. \tag{4.20}$$

We now consider the map $h : S^* \to \mathbb{R}^{\mu_0 - 1}$ defined by $h(e) = [\langle \psi_e, \overline{\varphi_2} \rangle, \ldots, \langle \psi_e, \overline{\varphi_{\mu_0}} \rangle]$. Since h is an odd continuous map defined on a $(N - k - 1)$-dimensional sphere and $\mu_0 \leq N - k$, h must have a zero by the Borsuk–Ulam Theorem. Hence there is $e \in S$ such that $\langle \psi_e, \overline{\varphi_k} \rangle = 0$ for $k = 2, \ldots, \mu_0$ and, by construction, $\langle \psi_e, \varphi_1 \rangle = 0$. Since $\psi_e \in V_0$ and $Q_u(\psi_e, \psi_e) \leq 0$, the function ψ_e is a minimizer for the quotient in (4.20). Consequently, it must be an eigenfunction of the operator L_0 corresponding to the eigenvalue zero. Hence $\psi_e \in C^2(B)$, and ψ_e solves $-\Delta \psi_e - V_u(x)\psi_e = 0$ in B. Moreover, $\psi_e = 0$ on $H(e)$ by the definition of ψ_e, and $\partial_e \psi_e = 0$ on $H(e)$, since ψ_e is symmetric with respect to the reflection at $H(e)$. From this it is easy to deduce that the function $\hat{\psi}_e$ defined by

$$\hat{\psi}_e(x) = \begin{cases} \psi_e(x), & x \in B(e), \\ 0, & x \in B(-e), \end{cases}$$

is also a (weak) solution of $-\Delta \hat{\psi}_e - V_u(x)\hat{\psi}_e = 0$. This however contradicts the unique continuation theorem for this equation.

Case 2. $\mu_0 = 1$. Then, for any $e \in S^*$, since $\psi_e \in V_0$, $\langle \psi_e, \varphi_1 \rangle = 0$ and $Q_u(\psi_e, \psi_e) \leq 0$, the function ψ_e must be a Dirichlet eigenfunction of the operator L_0 corresponding to the eigenvalue zero. This leads to a contradiction as in Case 1. Since in both cases we reached a contradiction, there must be a direction $e \in S^*$ such that w_e does not change sign on $B(e)$. Hence (4.15) holds either for e or for $-e$ and the assertion is proved.

By combining this result with the proof of Theorem 3.3 in [42] also the following geometric result is proved. □

THEOREM 4.3. *If B is a ball in $\mathbb{R}^N$, $N \geq 2$, $f = f(u)$ does not depend on x and $f'(u)$ is convex, then the nodal set of any sign-changing solution of* (4.1) *with Morse index less than or equal to N intersects the boundary of B.*

Let us conclude this section recalling that the assumption of Theorem 4.2 are, in particular, satisfied by the power nonlinearities of the type $f(s) = |s|^{p-1}s$, $p \geq 2$.

4.4. *Axial symmetry of solutions of an asymptotic problem*

In the previous section we have proved that low Morse index solutions are foliated Schwarz symmetric under some convexity assumption on the nonlinear term $f(|x|, s)$. Let us remark that this kind of symmetry has more properties than just the axial symmetry because it carries also information on the monotonicity with respect to the angular coordinate and implies that all critical points of the solutions lie on the symmetry axis.

We believe that this kind of symmetry is peculiar to solutions with low Morse index because it is somehow equivalent to the nonnegativity of the first eigenvalues of the linearized operator in the half domains determined by the symmetry hyperplane. Hence solutions with Morse index greater than N, are in general not foliated Schwarz symmetric. However they could be only axially symmetric.

It is indeed an interesting open question to find the right hypotheses to get axial symmetry or just symmetry hyperplanes for the solutions of (4.1). In this section we present some symmetry results for a specific problem related to the critical Sobolev exponent. The techniques used to get these results rely also somehow on the maximum principle but, more than this, exploit a careful blow-up analysis which is typical of these kinds of problems. Therefore we just outline these results, without giving many details. Let us consider the problem

$$\begin{cases} -\Delta u = N(N-2)u^{2^*-1-\varepsilon} & \text{in } A \\ u > 0 & \text{in } A \\ u = 0 & \text{on } \partial A, \end{cases} \tag{4.21}$$

where A is an annulus centred at the origin in $\mathbb{R}^N$, $N \geq 3$, $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent and $\varepsilon > 0$ is a small parameter. It is well known that the study of (4.21) is strictly related to the limiting problem ($\varepsilon = 0$) which exhibits a lack of compactness and gives rise to solutions of (4.11) which blow up and concentrate in a finite number of points, as $\varepsilon \to 0$.

It is obvious that the solutions of (4.21) which concentrate in a finite number of points cannot be radial. Nevertheless it is natural to expect a partial symmetry of the solutions, as well as a symmetric location of the limiting blow-up points. To be more precise we need some notations. We say that a family of solutions $\{u_\varepsilon\}$ of (4.21) has $k \geq 1$ concentration points $P_\varepsilon^1, P_\varepsilon^2, \ldots, P_\varepsilon^k$ in A, if the following holds.

$$\begin{aligned} &P_\varepsilon^i \neq P_\varepsilon^j;\ i \neq j \text{ and each } P_\varepsilon^i \text{ is a strict local maximum for } u_\varepsilon \\ &u_\varepsilon \to 0 \text{ as } \varepsilon \to 0 \text{ locally uniformly in } A \setminus \{P_\varepsilon^1, P_\varepsilon^2, \ldots, P_\varepsilon^k\} \\ &u_\varepsilon(P_\varepsilon^i) \to \infty \text{ as } \varepsilon \to 0. \end{aligned}$$

The results obtained in [15,16] concern solutions with one or two concentration points.

THEOREM 4.4. *Let u_ε be a family of solutions of* (4.21) *with one concentration point $P_\varepsilon \in A$. Then for ε small, u_ε is foliated Schwarz symmetric and its symmetry axis is the one which connects the origin with the point P_ε.*

THEOREM 4.5. *Let $\{u_\varepsilon\}$ be a family of solutions of* (4.21) *with two concentration points, P_ε^1 and P_ε^2 belonging to A. Then, for ε small, the points P_ε^i lay on the same axis passing through the origin and u_ε is axially symmetric with respect to this axis.*

REMARK 4.5. If the solution u_ε with one concentration point has Morse index one then its foliated Schwarz symmetry follows from Theorem 4.1. However solutions with one concentration point may have index higher than one, since it is related to the Morse index of the Robin function which is the regular part of the green function. If instead the solution u_ε has two concentration points then from Theorem 4.5 we get its axial symmetry, but it is clear that if the two concentration points are, as we conjecture, on opposite sides with respect to the origin, then u_ε cannot be foliated Schwarz symmetric and hence its Morse index must be greater than N, otherwise Theorem 4.2 would apply.

We conclude this section with another symmetry result for solutions which concentrate in more than two points.

THEOREM 4.6. *Let u_ε be a family of solutions of (4.21) which concentrates at k points $P_\varepsilon^j \in A$, $j = 1, \dots, k$, $k \geq 3$ and $k \leq N$. Then, for ε small, the points P_ε^j lie on the same $(k-1)$-dimensional hyperplane $\prod_k$ passing through the origin and u_ε is symmetric with respect to any $(N-1)$-dimensional hyperplane containing $\prod_k$.*

5. Symmetry of solutions of p-Laplace equations

In all previous chapters we have analyzed different kinds of symmetry results for solutions of semi-linear elliptic problems of type (3.1). Most of these results could be extended easily to more general types of equations where a general uniformly elliptic operator takes the place of the laplacian or the nonlinearity depends also on the gradient of the solutions making obviously suitable assumptions.

The situation changes completely if we deal with differential equations involving degenerate or singular operators which may also be nonlinear. Indeed the main ingredient of the results analyzed so far are maximum principles or, equivalently, comparison principles which, in general are not available for elliptic operators of the general type. In this chapter we will consider the case of elliptic equations involving the so-called p-Laplace operator and review some recent symmetry results for positive solutions of Dirichlet problems.

5.1. *Preliminaries*

Let us consider the problem

$$\begin{cases} -\Delta_p u = f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where Ω is a smooth domain in $\mathbb{R}^N$, $N \geq 2$, f is a real function whose regularity will be specified later and Δ_p denotes the p-Laplace operator, $\Delta_p u = \operatorname{div}(|Du|^{p-2}Du)$, $p > 1$. We are interested in studying monotonicity and symmetry properties of the solutions in dependence on the geometry of the domain Ω.

Note that if $p \neq 2$, solutions (5.1) can only be considered in a weak sense because, generally, they belong to the space $C^{1,\alpha}(\Omega)$ ([29,50]). Anyway this is not a difficulty because the moving-plane method can be adapted to weak solutions of strictly elliptic problems in divergence form (see [21] and [19]). The real difficulty with problem (5.1), for $p \neq 2$, is that the p-Laplace operator is degenerate or singular in the critical points of the solutions, so that comparison principles (which could substitute the maximum principles in dealing with nonlinear operators) are not available in the same form as for $p = 2$.

Actually there are counterexamples both to the validity of comparison principles and to the symmetry results ([34,12]). Let us describe one.

For $p > 2$ and $s > 2$ let us define

$$w(x) = \begin{cases} (1 - |x|^2)^s & \text{if } |x| \le 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$

and

$$v(x) = \begin{cases} 1 & \text{if } |x| < 5 \\ 1 - ((|x|^2 - 25)/11)^s & \text{if } 5 \le |x| \le 6. \end{cases}$$

We choose two points x_1 and x_2 with $|x_i| < 4$, $i = 1, 2$ $|x_1 - x_2| \ge 2$ and set

$$u(x) = v(x) + w(x - x_1) + w(x - x_2)$$

for any $x \in B_6 = \{x \in \mathbb{R}^N, |x| < 6\}$.

It is easy to see that u is not radially symmetric though it is a solution of (5.1) in the ball B_6 with

$$f(u) = \begin{cases} (2s/11)^{p-1}(25 + 11(1-u)^{1/s})^{(p/2)-1}(1-u)^{p-(p/s)-1} & \\ \cdot\{50/11)(p-1)(s-1) + (2ps - 2s - p + n)(1-u)^{1/s}\} & \text{if } 0 \le u \le 1 \\ (2s)^{p-1}(1 - (u-1)^{1/s}))^{(p/2)-1}(u-1)^{p-(p/s)-1} & \\ \cdot\{-2(s-1)(p-1) + (2ps - 2s - p + n)(u-1)^{1/s}\} & \text{if } 1 \le u \le 2. \end{cases}$$

Note that the nonlinearity f changes sign and belongs to $C^2([0, 2])$ if $s > p/p - 2$. If $1 < p < 2$ analogous counterexamples hold for less regular nonlinearities. In view of this, it is clear that some extra assumptions are needed to get monotonicity and symmetry results. The main progress have been made recently assuming $p \in (1, 2)$ or $p > 2$ and $f > 0$ and requiring in both cases f to be locally Lipschitz-continuous. These results will be described in the following sections.

Let us now review briefly some other existing results. When Ω is a ball, a first partial result is obtained in [5], where it is proved that, if f is locally Lipschitz-continuous, the solutions are radially symmetric, assuming that their gradient vanishes only at the origin. In this case, the solutions are of class C^2 in $\Omega \setminus \{0\}$ and there the equation is uniformly elliptic, therefore the application of the moving-plane method, as in [31] does not present much difficulty. In [34] it is shown, by an approximation procedure, that isolated solutions with nonzero index, in suitable function spaces, are symmetric.

A different approach was used in [37] where, combining symmetrization techniques and the Pohozaev identity (as done in [40]) it is proved that if $p = N, \Omega$ is a ball and f is merely continuous, but $f(s) > 0$ for $s > 0$ then u is radially symmetric and strictly decreasing. Using a new rearrangement technique, called continuous Steiner symmetrization, spherical symmetry results in the ball have also been obtained in [11] and [12]. This technique which does not use comparison principles, works also in the case of problems in the whole $\mathbb{R}^N$.

In the following sections we will describe some monotonicity and symmetry results that have been obtained extending the method of moving planes to the case of the p-laplacian. As already observed, this extension is not at all straightforward and relies on weak and strong comparison results which have recently been proved in [19,27,28] and [44]. We will first consider the case of bounded domains and then we will treat solutions of equations in the entire space. We will also distinguish between the case $p \in (1, 2)$ and $p > 2$. Indeed in the first case complete results have been obtained in [22,23] (see also [24]), regardless of the sign of the nonlinearity which is assumed to be Lipschitz-continuous. Instead when $p > 2$ complete results concern the case when f is positive, as expected in view of the counterexample described above (see [27]). Moreover in [27] and [44], some very interesting results on the structure of the critical set of the solution are provided, in particular in some cases the authors show that it consists of only one point P, so that the solution is of class C^2 in $\Omega \setminus \{P\}$.

5.2. *Bounded domains: The case* $1 < p < 2$

In this section we mainly describe the results of [22] and [23] (see also [24]). We will consider solutions of problem (5.1) where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$ and f satisfies the following hypothesis:

(H) f is locally Lipschitz-continuous in $(0, +\infty)$ and either $f \geq 0$ in $[0, +\infty)$ or there exist $s_0 > 0$ and a continuous nondecreasing function $\beta : [0, s_0] \to \mathbb{R}$ satisfying

$$\beta(0) = 0, \qquad \beta(s) > 0 \quad \text{for } s > 0, \qquad \int_0^{s_0} (\beta(s)s)^{-1/p} dx = \infty$$

such that $f(s) + \beta(s) \geq 0 \; \forall s \in [0, s_0]$.

Note that any nonlinearity $f(s) = g(s) - cs^q$ satisfies (H) if g is locally Lipschitz-continuous in $(0 + \infty)$, $g(s) \geq 0, c > 0$ and $q \geq p - 1$. Moreover the hypothesis (H) also ensures that any nonnegative solution is positive, by the maximum principle of Vazquez [51].

To state the monotonicity and symmetry results we need some notations which will be similar to the one used in Chapter 4. For a unit vector $e \in S = \{x \in \mathbb{R}^N : |x| = 1\}$ and a real number λ we define

$$\begin{aligned} H_\lambda(e) &= \{x \in \mathbb{R}^N : x \cdot e = \lambda\} \\ \Omega_\lambda(e) &= \{x \in \Omega : x \cdot e < \lambda\} \\ x_\lambda^e &= x + 2(\lambda - x \cdot e)e, \quad x \in \mathbb{R}^N. \end{aligned}$$

(i.e. x_λ^e is the reflection of x through the hyperplane $H_\lambda(e)$.)

$$a(e) = \inf_{x \in \Omega} x \cdot e.$$

If $\lambda > a(e)$, then $\Omega_\lambda(e)$ is nonempty; thus we set

$$\Omega'_\lambda(e) = \{x + 2(\lambda - x \cdot e)e, x \in \Omega_\lambda(e)\}.$$

As in Chapter 2 we have that if Ω is smooth and $\lambda > a(e)$, λ close to $a(e)$, then the reflected cap $\Omega'_\lambda(e)$ is contained in Ω and will remain in it, at least until one of the following occurs.

(i) $\Omega'_\lambda(e)$ becomes internally tangent to $\partial\Omega$ at some point not on $H_\lambda(e)$.
(ii) $H_\lambda(e)$ is orthogonal to $\partial\Omega$ at some point.

Let $\Lambda_1(e)$ be the set of those $\lambda > a(e)$ such that for each $\mu \in (a(e), \lambda]$ neither (i) nor (ii) holds, and define

$$\lambda_1(e) = \sup \Lambda_1(e).$$

Note that $\lambda_1(e)$ is a lower-semicontinuous function whenever $\partial\Omega$ is smooth.

The main result proved in [23] (which extends a previous result of [22] is the following monotonicity theorem.

THEOREM 5.1. *Let $u \in C^1(\overline{\Omega})$ be a weak solution of (5.1) with f satisfying (H) and $1 < p < 2$. Then, for any direction $e \in S$ and for λ in the interval $(a(e), \lambda_1(e))$ we have*

$$u(x) < u(x^e_\lambda), \quad \forall x \in \Omega_\lambda(e).$$

Moreover,

$$\frac{\partial u}{\partial e}(x) > 0, \quad \forall x \in \Omega_{\lambda_1(e)}(e) \setminus Z,$$

where $Z = \{x \in \Omega : Du(x) = 0\}$, and u is strictly increasing in the e-direction in the set $\Omega_{\lambda_{1(e)}}(e)$.

From this the following symmetry result follows

COROLLARY 5.1. *If, for a direction e, the domain Ω is symmetric with respect to the hyperplane $H_0(e)$ and $\lambda_1(e) = 0$, then $u(x) = u(x^e_0)$ for any $x \in \Omega$. Moreover u is strictly increasing in the e-direction in the set $\Omega_0(e)$ with $\frac{\partial u}{\partial e} > 0$ in $\Omega_0(e) \setminus Z$. In particular if Ω is a ball, then u is radially symmetric and $\frac{\partial u}{\partial r} < 0$ in $\Omega \setminus \{0\}$.*

The proofs of these results rely on the moving-plane method which needs to be modified to overcome the difficulties arising from the singularity of the operator Δ_p. The new idea, introduced in [22] consists in moving simultaneously hyperplanes orthogonal to directions close to a fixed direction e_0. To ensure continuity (with respect to the directions) in this procedure we assume that Ω is smooth. The proof of Theorem 5.1 and Corollary 5.1 is long and technically very complicated, hence we only indicate the main steps, referring the reader to [22] and [23] for the complete details. Let us finally remark that other crucial ingredients of the proofs are the weak and strong comparison principles obtained in [20].

Sketch of the proof of Theorem 5.1. Let u be a $C^1(\overline{\Omega})$ solution of 5.1. For any direction e, let us set

$$u^e_\lambda(x) = u(x^e_\lambda), \quad x \in \Omega_\lambda(e)$$
$$Z^e_\lambda = Z^e_\lambda(u) = \{x \in \Omega_\lambda(e) : Du(x) = Du^e_\lambda(x) = 0\}$$
$$\Lambda_0(e) = \{\lambda \in (a(e), \lambda_1(e)] : u \le u^e_\mu \text{ in } \Omega_\mu(e) \text{ for any } \mu \in (a(e), \lambda]\}.$$

If $\Lambda_0(e) \neq \phi$, we set

$$\lambda_0(e) = \sup \Lambda_0(e).$$

A preliminary result which is crucial to prove Theorem 5.1 is the following proposition which gives a useful information on how the set Z of the critical point of a solution u of (5.1) can intersect the cap $\Omega_{\lambda_0(e)}(e)$.

PROPOSITION 5.1. *Suppose that u is a $C^1(\overline{\Omega})$-weak solution of* (5.1), *with* $1 < p < 2$. *For any direction e the cap $\Omega_{\lambda_0(e)}(e)$ contains an open subset of $H_{\lambda_0(e)}(e)$ (relatively to the induced topology).*

The proof of this proposition relies on a careful use of the Hopf's lemma.

To prove Theorem 5.1 we show that $\Lambda_0(e) \neq \phi$ and $\lambda_0(e) = \lambda_1(e)$. The last one will be proved by showing that if $\lambda_0(e) < \lambda_1(e)$ then there exists a "small" set Γ of critical points of u in the cap $\Omega_{\lambda_0(e)}(e)$ on which u is constant and whose projection on the hyperplane $H_{\lambda_0(e)}(e)$ contains on open subset of $H_{\lambda_0(e)}(e)$. Of course this would be in contradiction to the statement of Proposition 5.1.

We now state another result which is a different formulation and an extension of Theorem 1.5 in [20]. It essentially asserts that, once we start the moving place procedure we must necessarily reach the position $H^{(e)}_{\lambda_1(e)}$ unless the set Z of the critical points of u creates a connected component C of the set, where $Du \neq 0$ which is symmetric with respect to the hyperplane $H^{(e)}_{\lambda_0(e)}$ and where u coincides with the symmetric function $u^e_{\lambda_0(e)}$.

PROPOSITION 5.2. *For any direction e we have that $\Lambda_0(e) \neq \phi$ and, if $\lambda_0(e) < \lambda_1(e)$, then there exists at least one connected component C^e of $\Omega^{(e)}_{\lambda_0(e)} \backslash Z^e_{\lambda_0(e)}$ such that $u \equiv u^e_{\lambda_0(e)}$ in C^e. For any such component C^e we get*

$$\begin{aligned} &Du(x) \neq 0 \quad \forall\, x \in C^e \\ &Du(x) = 0 \quad \forall\, x \in \partial C^e \setminus (H_{\lambda_0(e)}(e) \cup \partial\Omega). \end{aligned}$$

Moreover for any λ with $a(e) < \lambda < \lambda_0(e)$ we have

$$u < u^e_\lambda \quad \text{in } \Omega_\lambda(e) \setminus Z^e_\lambda$$

and finally

$$\frac{\partial u}{\partial e}(x) > 0 \quad \forall x \in \Omega_{\lambda_0(e)}(e) \setminus Z.$$

Now, for any direction e, let F_e be the collection of the connected components C^e of $\Omega_{\lambda_0(e)}(e) \setminus Z^e_{\lambda_0(e)}$ such that $u \equiv u^e_{\lambda_0(e)}$ in $C^e, Du \neq 0$ in $C^e, Du = 0$ on $\partial C^e \backslash (H_{\lambda_{0(e)}}(e) \cup \partial\Omega)$. If $\lambda_0(e) < \lambda_1(e)$ we deduce from Proposition 5.2 that $F_e \neq \phi$. If this is the case and $C^e \in F_e$ we also have that $u \equiv u^e_{\lambda_0(e)}$ in $\overline{C}^e$ so that $(\overline{C}^e \cap \partial\Omega) \backslash H_{\lambda_0(e)}(e) = \phi$ since $u = 0$ on $\partial\Omega$ while $u^e_{\lambda_0(e)} > 0$ in $\overline{C}^e \setminus H^{(e)}_{\lambda_0(e)}$, because by the definition of $\lambda_1(e)$, we have that $\overline{\Omega}_{\lambda_0(e)}(e) \setminus H_{\lambda_o(e)}(e) \subset \Omega$.

Hence there are two alternatives: either $Du(x) = 0$ for all $x \in \partial C^e$, in which case we define $\tilde{C}^e = C^e$, or there are points $x \in \partial C^e \cup H_{\lambda_0(e)}(e)$ such that $Du(x) \neq 0$. In this last case we define $\tilde{C}^e = C^e \cup C^e_1 \cup C^e_2$, where C^e_1 is the reflection of C^e with respect to the hyperplane $H_{\lambda_0(e)}(e)$ and $C^e_2 = \{x \in \partial C^e \cap H_{\lambda_0(e)}(e) : Du(x) \neq 0\}$. It is easy to see that $\tilde{C}^e$ is open and connected with $Du \neq 0$ in $\tilde{C}^e$, $Du = 0$ on $\partial\tilde{C}^e$. Let us finally denote by $\tilde{F}_e$ the collection $\{\tilde{C}^\nu : C^\nu \in F_\nu\}$ and by $I_s(e)$ the set

$$I_\delta(e) = \{\mu \in \mathbb{R}^N : |\mu| = 1, |\mu - e| < \delta\}.$$

As already observed, Theorem 5.1 will be proved if we show that $\lambda_0(e) = \lambda_1(e)$ for any direction on e. Therefore suppose that e_0 is a direction such that $\lambda_0(e) < \lambda_1(e)$. Then from

Proposition 5.2, it follows that $F_{e_0} \neq 0$ and since $\mathbb{R}^N$ is a separable metric space and every component is open, F_{e_0} contains at most countably many components of $\Omega_{\lambda_0(e_0)} \setminus Z^{(e_0)}_{\lambda_0(e_0)}$, so

$$F_{e_0} = \{C_i^{e_0}, i \in I \subseteq \mathbb{N}\}.$$

The remaining part of the proof can be summarized in the following three steps whose proofs are omitted (see [22,23]).

Step 1. *The function* $\lambda_0(e)$ *is continuous. Moreover there exists* $\delta_0 > 0$ *such that for any* $e \in I_{\delta_0}(e_0)$ *there exists* $i \in I$ *with* $\tilde{C}_i^{e_0} \in \tilde{F}_e$.

The second part of this statement asserts that for any direction e in a suitable neighbourhood $I_{\delta_0}(e_0)$, there exists a set $\tilde{C}_i^{e_0}$ in the collection F_{e_0} which also belongs to $\tilde{F}_e$.

Step 2. *There exists a direction* $e_1 \in I_{\delta_0(e_0)}$, *a neighbourhood* $I_{\delta_1}(e_1)$ *and an index* $i_1 \in \{1, \dots, n_0\}$ *such that for any* $e \in I_{\delta_1}(e_1)$ *the set* $\tilde{C}_{i_1}^{e_0}$ *belongs to the collection* $\tilde{F}_e$.

From this we deduce that in $\tilde{C}_{i_1}^{e_0}$ the function u is symmetric with respect to all hyperplanes $H_{\lambda_0(e_1)}(e)$ with $e \in I_{\delta_1}(e_1)$. It is this symmetry property which is exploited in the next step to conclude the proof of Theorem 5.1.

Step 3. *Let* e_1, i_1, δ_1 *be as in Step 2 and set* $C = C_{i_1}^{e_0}$. *Then* $\partial C \cup \Omega_{\lambda_0(e_1)}(e_1)$ *contains a subset* Γ *on which* u *is constant and whose projection on the hyperplane* $H_{\lambda_0(e_1)}(e_1)$ *contains on open subset of the hyperplane.*

Since $Du = 0$ on $\partial C \cup \Omega_{\lambda_0(e_1)}(e_1)$, Step 3 gives a contradiction with Proposition 5.1 and ends the proof of Theorem 5.1.

5.3. *Bounded domains: Results for all* $p > 1$

In this section we will describe some symmetry and monotonicity results which also hold when $p > 2$. They will require the nonlinearity f to be positive and will allow to get information about the set of the critical points of the solutions, proving, for some domains, that it reduces to only one point. However let us note that when the nonlinearity f changes sign there are counterexamples to the symmetry of C^1 solutions of (5.1), as shown in Section 1. These results have been proved in [27] and [28] as a consequence of some regularity theorems which allow to get new weak and strong comparison principles for solutions to (5.1). The key idea in [27] is to work in a weighted Sobolev space with weight $\rho = |Du|^{p-2}$. This new point of view in the study of p-Laplace equations was first introduced in [4] to define a Morse index for radial solutions of (5.1) in the ball. In this case the summability properties of $1/|Du|$ are quite easy to derive because the gradient of the solution vanishes only at the centre of the ball and the precise behaviour of Du near the origin can be obtained using L'Hospital's rule.

Instead, in general smooth bounded domains, the set of the critical points of a solution u, that we denote by Z, as in the previous section, may be very irregular that the study of the regularity of u, which depends on the behaviour of Du in the points where it vanishes, can be very complicated.

Nevertheless in [27] some summability properties of $1/|Du|$ are derived (see Theorem 1.1 in [27]). From this the authors deduce a weighted Poincaré-type inequality which in turn, allows to prove a weak comparison principle which is crucial to apply the moving-plane method as in the case $1 < p < 2$ considered in the previous section. To state the symmetry and monotonicity results, we use the same notations as in the previous section and in addition define, for any direction $e \in \mathbb{R}^N$, $|e| = 1$.

$$\Lambda_2(e) = \{\lambda > a(e) : \Omega'_\mu(e) \subseteq \Omega, \forall \mu \in (a(e), \lambda)\}$$

and $\lambda_2(e) = \sup \Lambda_2(e)$.

Note that since Ω is assumed smooth, neither $\Lambda_1(e)$ nor $\Lambda_2(e)$ are empty and $\Lambda_1(e) \subseteq \Lambda_2(e)$ so that $\lambda_1(e) \leq \lambda_2(e)$.

THEOREM 5.2. *Let Ω be a bounded smooth domain in $\mathbb{R}^N$, $N \geq 2$ and $u \in C^1(\overline{\Omega})$ a weak solution of* (5.1) *with* $1 < p < \infty$ *and* $f : [0, \infty) \to \mathbb{R}$ *a continuous function which is strictly positive and locally Lipschitz-continuous in* $(0, \infty)$. *For any direction a and for λ in the interval* $(a(e), \lambda_1(e)]$ *we have*

$$u(x) \leq u(x^e_\lambda) \quad \forall x \in \Omega_\lambda(e).$$

Moreover, for any λ with $a(e) < \lambda < \lambda_1(e)$ we have

$$u(x) < u(x^e_\lambda) \quad \forall x \in \Omega_\lambda(e) \setminus Z^e_\lambda.$$

Finally

$$\frac{\partial u}{\partial e}(x) > 0 \quad \forall x \in \Omega_{\lambda_1(e)}(e) \setminus Z.$$

If f is locally Lipschitz-continuous in the closed interval $[0, \infty)$ *then all results hold up to* $\lambda_2(e)$.

COROLLARY 5.2. *If f is locally Lipschitz-continuous in the closed interval* $[0, \infty)$ *and strictly positive in* $(0, \infty)$ *and the domain Ω is convex with respect to a direction e and symmetric with respect to the hyperplane* $H_0(e) = \{x \in \mathbb{R}^N : x \cdot e = 0\}$, *then u is symmetric and* $\frac{\partial u}{\partial e}(x) > 0$ *in* $\Omega_0(e) \setminus Z$. *In particular if Ω is a ball then u is radially symmetric and* $\frac{\partial u}{\partial r} < 0$, *where* $\frac{\partial u}{\partial r}$ *denotes the derivative in the radial direction.*

REMARK 5.1. The main interest of Theorem 5.2 and Corollary 5.2 is that they provide monotonicity and symmetry results in the case $p > 2$. However, in the case of positive nonlinearities, Theorem 5.2 improves slightly Theorem 5.1 because it allows to get the monotonicity for all $\lambda \in (a(e), \lambda_2(e))$.

The proof of Theorem 5.2 follows the same procedure sketched in the previous section to prove Theorem 5.1. However the assumption that f is positive together with the regularity obtained allows to simplify considerably the proof.

Let us note that in Theorems 5.1 and 5.2, as well as in Corollaries 5.1 and 5.2 the information about the derivative $\frac{\partial u}{\partial e}$ can be obtained only outside of the critical points of the solution and no results on the critical set Z is obtained, unless Ω is a ball. In the paper [28] the authors continue the study of the regularity of the solutions of (5.1) and are able to get some type of Harnack inequalities which allow to prove a strong maximum principle for solutions of the linearized equation as well as strong comparison principle for solutions of (5.1). Combining these results with those at Theorem 5.2 they are able to prove important properties of the critical set of a solution. More precisely they prove:

THEOREM 5.3. *Let Ω be a bounded smooth domain in $\mathbb{R}^N$, $N \geq 2$ and $u \in C^1(\overline{\Omega})$ be a weak solution of* (5.1) *where $f : [0, \infty) \to \mathbb{R}$ is a continuous function which is strictly positive and locally Lipschitz-continuous in* $(0, \infty)$ *and $p > \frac{2N+2}{N+2}$ or $p > 2$. Then for any direction e and for λ in the interval $(a(e), \lambda_1(e)]$ we have*

$$u(x) \leq u(x_\lambda^e) \quad \forall x \in \Omega_\lambda(e).$$

Moreover

$$\frac{\partial u}{\partial e}(x) > 0 \quad \forall x \in \Omega_{\lambda_1(e)}(e). \tag{5.2}$$

If f is locally Lipschitz-continuous in the closed interval $[0, \infty)$ then all results hold up to the value $\lambda_2(e)$.

Let us remark that in Theorem 5.2 the inequality (5.2) was proved only for $x \in \Omega_{\lambda_1(e)}(e) \setminus Z$.

Then we have

COROLLARY 5.3. *If the nonlinearity f is as in the previous theorem and the domain Ω is symmetric with respect to the hyperplane $H_0(e) = \{x \in \mathbb{R}^N : x \cdot e = 0\}$ and strictly convex in the e-direction, then u is symmetric and strictly increasing in the e-direction in $\Omega_0(e)$ with $\frac{\partial u}{\partial e} > 0$ in $\Omega_0(e)$. In particular the only points where Du vanishes belong to the hyperplane $H_0(e)$. Therefore if for N-orthogonal directions e_i, the domain Ω is symmetric with respect to any hyperplane $H_0(e_i)$ and $\lambda_1(e_i) = \lambda_1(-e_i) = 0$, then the critical set Z is just $\{0\}$, assuming that 0 is the centre of symmetry. Moreover if f is locally Lipschitz-continuous in the closed interval $[0, +\infty)$ then the same result holds assuming only that the domain Ω is convex and symmetric with respect to N-orthogonal directions.*

REMARK 5.2. As a consequence of the previous corollary we have that the solution u belongs to $C^2(\Omega \setminus \{0\})$ whenever Ω is convex and symmetric with respect to N-orthogonal directions. This follows easily from standard regularity results because the p-Laplace operator is uniformly elliptic in $\Omega \setminus \{0\}$ since it is only degenerate in the points where $|Du| = 0$.

We conclude this section by mentioning that a few more results about geometrical properties of the set of the critical points of a solution of (5.1) in the case of sign-changing nonlinearities can be found in [44].

5.4. *Symmetry results in* $\mathbb{R}^N$ *for* $1 < p < 2$

This section deals with symmetry and monotonicity results for ground state solutions of

$$\begin{cases} -\Delta_p u(x) = f(u(x)), & \text{in } \mathbb{R}^N \\ u > 0 & \text{in } \mathbb{R}^N \\ u(x) \to 0 & \text{as } |x| \to \infty \end{cases} \tag{5.3}$$

for $1 < p < 2$, with $\Delta_p u = \operatorname{div}(|Du|^{p-2} Du)$. The results described here following [25] and [26], concern the extensions of the results of Section 2.2 for the laplacian in $\mathbb{R}^N$, to the degenerate case of p-laplacian, for $1 < p < 2$. The first result is for the case of a nonlinearity f, nonpositive near the origin, as in Theorems 2.4 and 2.5.

THEOREM 5.4. *Under the assumptions:*
(H1) *f is locally Lipschitz-continuous in* $(0, \infty)$.
(H2) *there exists* $s_0 > 0$ *such that f is nonincreasing on* $(0, s_0)$.
If $u \in C^1(\mathbb{R}^N)$ *is a weak solution of* (5.3)*, for* $1 < p < 2$*, then u is radially symmetric about some point* $x_0 \in \mathbb{R}^N$*, i.e.* $u = u(r)$*, with* $r = |x - x_0|$*, and strictly radially decreasing, i.e.* $u'(r) < 0$ *for all* $r > 0$.

The second result applies to the case of a power nonlinearity, extending the result of Theorem 2.6:

THEOREM 5.5. *Under the assumptions:*
(H1) *f is locally Lipschitz-continuous in* $(0, \infty)$.
(H3) *there exists* $s_0 > 0$ *and* $\alpha > p - 2$ *such that for* $0 < u < v < s_0$,

$$\frac{f(v) - f(u)}{(v - u)} \le \begin{cases} C v^\alpha & \text{if } \alpha \ge 0 \\ C u^\alpha & \text{if } \alpha < 0. \end{cases}$$

If $u \in C^1(\mathbb{R}^N)$ *be a weak solution of* (5.3) *for* $1 < p < 2$ *satisfying, for some* $m > 0$,

$$\begin{aligned} & u(x) = 0\left(\frac{1}{|x|^m}\right) \quad \text{as } |x| \to \infty, \\ & \left(u(x) \ge \frac{C}{|x|^m} \quad \text{as } |x| \to \infty \quad \text{when } \alpha < 0\right), \\ & Du(x) = 0\left(\frac{1}{|x|^{m+1}}\right) \quad \text{as } |x| \to \infty \end{aligned} \tag{5.4}$$

and if

$$m(\alpha + 2 - p) > p, \tag{5.5}$$

then u is radially symmetric about some point $x_0 \in \mathbb{R}^N$ *and strictly radially decreasing.*

To prove these theorems, we introduce the half space,

$$\Sigma_\lambda(e) = \{x \in \mathbb{R}^N : x \cdot e < \lambda\}$$

for any direction e in $\mathbb{R}^N$ and $\lambda \in \mathbb{R}$. Following the notations of Section 2.1, we let

$$H_\lambda(e) = \{x \in \mathbb{R}^N : x \cdot e = \lambda\}.$$
$$x_\lambda^e = x + 2(\lambda - x \cdot e)e,$$

for any $x \in \mathbb{R}^N$ and

$$u_\lambda^e(x) = u(x_\lambda^e), \quad x \in \Sigma_\lambda(e)$$
$$Z_\lambda^e = Z_\lambda^e(u) = \{x \in \Sigma_\lambda(e) : Du(x) = Du_\lambda^e(x) = 0\}$$
$$\Lambda_0(e) = \{\lambda \in \mathbb{R}, : u \geq u_\mu^e \text{ in } \Sigma_\mu(e) \text{ for any } \mu > \lambda\}.$$

As before, in order to prove radial symmetry, we need to check that

(1) $\Lambda_0(e)$ is nonempty and bounded from below.

(2) If $\lambda_0(e) := \inf \Lambda_0(e)$, then $u \equiv u_{\lambda_0}$ in $\Sigma_{\lambda_0}(e)$.

We indicate here the proof of step (i) for both theorems. The proof of step (ii) will follow by arguments similar to those in Section 5.2.

Proof of step (i) of Theorem 5.4:

In all comparison theorems for p-Laplacian, the following inequality is often used:

$$(|\eta|^{p-2}\eta - |\eta'|^{p-2}\eta') \cdot (\eta - \eta') \geq c(|\eta| + |\eta'|)^{p-2}|\eta - \eta'|^2 \tag{5.6}$$

for any $\eta, \eta' \in \mathbb{R}^N$, where c is a constant depending on N and p. As in Section 2.2, we test the equations for u and v with $(v - u - \varepsilon)^+$ and subtract to get

$$\begin{aligned} c \int_{\Sigma_\lambda(e)\cap\{v\geq u+\varepsilon\}} (|Du| + |Dv|)^{p-2}|D(v-u)^+|^2\,dx \\ \leq \int_{\Sigma_\lambda(e)\cap\{v\geq u+\varepsilon\}} (f(v) - f(u))(v - u - \varepsilon)^+\,dx. \end{aligned} \tag{5.7}$$

Since $u \to 0$ as $|x| \to \infty$, there exists R such that for x outside B_R, $u(x) < s_0$. Then using (H2), we have for $\lambda > R$, the function $f(v) - f(u) \leq 0$ on the set $\Sigma_\lambda(e) \cap \{x : v \geq u + \epsilon\}$ since on this set $u < v < s_0$. Thus it follows from (5.7) that

$$c \int_{\Sigma_\lambda(e)\cap\{v\geq u+\varepsilon\}} (|Dv| + |Du|)^{p-2}|D(v-u)^+|^2\,dx \leq 0$$

and by monotone convergence theorem, as $\varepsilon \to 0$ we get

$$\int_{\Sigma_\lambda(e)} (|Dv| + |Du|)^{p-2}|D(v-u)^+|^2\,dx \leq 0. \tag{5.8}$$

Hence $D(v-u)^+ = 0$ a.e. on $\Sigma_\lambda(e)$ and since $(v-u)^+ = 0$ on $\partial\Sigma_\lambda(e)$, $(v-u)^+ = 0$ on $\Sigma_\lambda(e)$.

Proof of step (i) of Theorem 5.5: This is along the same lines as the proof above but uses the test function $[(v-u-\epsilon)^+]^t$ for a suitable t. The integral $\int (f(v) - f(u))[(v-u-e)^+]^t$ is then carefully estimated using weighted Hardy–Sobolev inequality (see Proposition 3.2 in [26], for details).

The proof of step (ii) for both theorems relies on a version of Proposition 5.2, adapted to unbounded domains and the arguments outlined in the three steps at the end of Section 5.2 in the proof of Theorem 5.1.

Acknowledgement

Research supported by M.I.U.R Project.

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CHAPTER 5

Stationary Boundary Value Problems for Compressible Navier–Stokes Equations

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Abstract

We give an overview of available results on the well-posedness of basic boundary value problem for equations of viscous compressible fluids.

Keywords: Navier–Stokes equations, Compressible fluids, Shape optimization, Bergman projection, Incompressible limit, Transport equations

AMS Subject Classifications: 35B50, 35B99, 35J60, 35J70

Contents

HANDBOOK OF DIFFERENTIAL EQUATIONS
Stationary Partial Differential Equations, volume 6
Edited by M. Chipot

1. Introduction

Compressible Navier–Stokes equations are the subject of current studies. We refer the reader to the books by Lions [34], Feireisl [16], Novotný and Straškraba [41] for the state of the art in the domain. The related references are listed at the end of the chapter.

Inhomogeneous boundary value problems for the equations are of some importance in applications, including optimal control theory, shape optimization and inverse problems. Such results, however, are not available in the literature on the subject in the full range of physical parameters. In our joint research, see e.g., [50–53], the specific problem studied is the minimization of the drag functional which is a representative shape optimization problem for the mathematical models in the form of compressible Navier–Stokes equations.

In this chapter, we present the results and the techniques which can be used to study not only the existence and uniqueness of weak solutions, but also the compactness of the set of solutions with respect to the boundary perturbations, and the differentiability of solutions with respect to the coefficients of differential operators. The class of mathematical models is of elliptic–hyperbolic types. We briefly describe the modelisation issue, define all the physical constants which are related to compressible Navier–Stokes equations. One of the constants, which is called the adiabatic constant $\gamma \geq 1$, which is present in the formula for the pressure in terms of the density, is quite important from the mathematical point of view. We present some results for the range of constants which includes the diatomic gases. The case of the ideal gas with $\gamma = 1$ is considered e.g. in [51] and leads to some singularities in three spatial dimensions, such a singularity is absent however in two spatial dimensions [50].

The content of the chapter can be described as follows.

In Section 2 the steady state equations of compressible fluid dynamics are introduced for the state variables including the density, the velocity field and the temperature. The boundary value problems considered are of elliptic–hyperbolic type.

In Section 3 the general properties of the boundary value problems are described. Local existence and uniqueness results for classical solutions known in the literature are recalled. The weak solutions to compressible Navier–Stokes equations are defined, and the existence of such solutions is discussed.

In Section 4 the mathematical tools including the interpolation theory, the Young measures and the Sobolev spaces are introduced.

In Section 5 the framework for the transport equations which constitute the hyperbolic component of the boundary value problems is established. The so-called emergent vector field conditions are given in order to assure the appropriate solvability of the transport equations.

In Section 6 important properties of the transport equations with discontinuous coefficients are analysed, among others the normalization procedure, the kinetic form of equations, the compactness of renormalized solutions, and the so-called oscillation defect measure. The strong convergence of solutions to the transport equation is shown in Theorem 6.4.

In Section 7 the existence of weak solutions for some range of adiabatic constant is shown, which is the original contribution.

In Sections 8–10, Appendices A and B all proofs of the technical results of the chapter are provided.

2. Equations of viscous compressible fluid dynamics

Let a compressible fluid occupy the domain Ω in Euclidian space $\mathbb{R}^3$. The steady state of a fluid at point $x \in \Omega$ is completely characterized by the macroscopic quantities: the *density* $\varrho(x)$, the *velocity* $\mathbf{u}(x)$, and the *temperature* $\vartheta(x)$. These quantities are called state variables in the sequel. The governing equations represent three basic principles of fluid mechanics: the mass balance

$$\operatorname{div}(\varrho\mathbf{u}) = 0 \quad \text{in } \Omega, \tag{2.1a}$$

the balance of momentum

$$\operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla p = \varrho\mathbf{f} + \mathbf{h} + \operatorname{div}\mathbb{S} \quad \text{in } \Omega, \tag{2.1b}$$

energy conservation law

$$\operatorname{div}((E+p)\mathbf{u}) = \operatorname{div}\mathbb{S}(\mathbf{u}) + \operatorname{div}(\kappa\nabla\vartheta) + (\varrho\mathbf{f}+\mathbf{h})\mathbf{u}. \tag{2.1c}$$

Here, given that the vector fields $\mathbf{f}$ and $\mathbf{h}$ denote the densities of external mass and volume forces, the *heat conduction coefficient* κ is a positive constant, the viscous stress tensor $\mathbb{S}$ has the form

$$\mathbb{S}(\mathbf{u}) = \nu_1\left(\nabla\mathbf{u} + \nabla\mathbf{u}^\top - \frac{2}{3}\operatorname{div}\mathbf{v}\,\mathbf{I}\right) + \nu_2\operatorname{div}\mathbf{v}\,\mathbf{I}, \tag{2.1d}$$

in which the viscous coefficients ν_i satisfy the inequality $\nu_1 4/3 + \nu_2 > 0$, the energy density E is given by

$$E = \frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e,$$

where e is the density of *internal energy*. The physical properties of a gas are reflected through constitutive equations relating the state variables to the pressure and the internal energy density. We restrict our considerations to the classic case of *perfect polytropic* gases with the pressure and the internal energy density defined by the formulae

$$p = R_m\varrho\vartheta \qquad e = c_v\vartheta. \tag{2.1e}$$

Here R_m is a positive constant inversely proportional to the molecular weight of the gas, that is

$$R_m = c_p - c_v, \quad \text{with } \gamma =: c_p/c_v > 1,$$

where c_v is the specific heat at constant volume and c_p is the specific heat at constant pressure, are positive constants. In this case the entropy density S takes the form

$$S = \log e - (\gamma - 1)\log\varrho. \tag{2.2}$$

The system of partial differential equations (2.1) is called *compressible Navier–Stokes–Fourier* equations.

It is useful to rewrite the governing equations in the dimensionless form, which is widely accepted in applications. To this end we denote by u_c, ϱ_c, and ϑ_c the characteristic values of the velocity, the density, and the temperature, and by l_c and $\Delta\vartheta_c$ the characteristic values of the length and the temperature oscillation. They form five dimensionless combinations: the Reynolds number, the Prandtl number, the Mach number, the viscosity ratio, and the relative temperature oscillation defined by the formulae, see [54],

$$\mathbb{R}\mathrm{e} = \frac{\varrho_c u_c l_c}{\nu_1}, \quad \mathbb{P}\mathrm{r} = \frac{\nu_1 c_p}{\kappa}, \quad \mathbb{M}\mathrm{a}^2 = \frac{u_c^2}{c_p \vartheta_c(\gamma - 1)},$$
$$\lambda = \frac{1}{3} + \frac{\nu_2}{\nu_1}, \quad b = \frac{\Delta\vartheta_c}{\vartheta_c}.$$

Note that the specific values of the constants γ, λ, and $\mathbb{P}\mathrm{r}$ depend only on physical properties of a fluid. For example, for the air under standard conditions, we have $\gamma = 7/5$, $\lambda = 1/3$, and $\mathrm{Pr} = 7/10$. The passage to the dimensionless variables is defined as follows

$$x \to l_c x, \quad \mathbf{u} \to u_c \mathbf{u}, \quad \varrho \to \varrho_c \varrho,$$
$$\vartheta \to \vartheta_c + \Delta\vartheta_c \vartheta, \quad \varrho\,\mathbf{f} \to \frac{\varrho_c l_c^2}{\nu_1 u_c}\varrho\,\mathbf{f}, \quad \mathbf{h} \to \frac{l_c^2}{\nu_1 u_c}\mathbf{h}$$

and (2.1) performed leading to the following system of differential equations for dimensionless quantities in the scaled domain $l_c^{-1}\Omega$, which we still denote by Ω,

$$\Delta\mathbf{u} + \lambda\nabla\,\mathrm{div}\,\mathbf{u} = k\varrho\mathbf{u}\cdot\nabla\mathbf{u} + \sigma\nabla(\varrho(1 + b\vartheta)) - \varrho\mathbf{f} - \mathbf{h} \quad \text{in } \Omega, \tag{2.3a}$$
$$\mathrm{div}(\varrho\mathbf{u}) = 0 \quad \text{in } \Omega, \tag{2.3b}$$
$$\Delta\vartheta = kb_1(\varrho\mathbf{u}\nabla\vartheta + (\gamma - 1)(b^{-1} + \vartheta)\varrho\,\mathrm{div}\,\mathbf{u}) - \frac{k}{\sigma}b_2((\nabla\mathbf{u} + \nabla\mathbf{u}^*)^2 + (\lambda - 1)\,\mathrm{div}\,\mathbf{u}^2), \tag{2.3c}$$

where

$$k = \mathbb{R}\mathrm{e}, \quad \sigma = \frac{\mathbb{R}\mathrm{e}}{\gamma\mathbb{M}\mathrm{a}^2}, \quad b_1 = \frac{\mathbb{P}\mathrm{r}}{\gamma}, \quad b_2 = \frac{\mathbb{P}\mathrm{r}}{2b}(\gamma - 1).$$

The asymptotic analysis of solutions is of mathematical and practical importance. In theoretical hydrodynamics the following cases are distinguished:

- the low compressible and hypersonic limits $\mathbb{M}\mathrm{a} \to 0, \infty$;
- the Stokes and the Euler limits $\mathbb{R}\mathrm{e} \to 0, \infty$;
- the elastic low compressible limit $\lambda \to \infty$.

From this point of view the quantities b, b_i do not play any important role in the theory, and further we shall assume that

$$b = b_1 = b_2 = 1.$$

2.1. *Barotropic flows*

The flow is *barotropic* if the pressure depends only on the density. The most important example of such flows are *isentropic flows*. In order to deduce the governing equations for isentropic flows we note that for perfect fluid with $\nu_i = \kappa = 0$, the entropy takes a constant value in each material point. Hence in this case the governing equations have a family of explicit solutions with the entropy $S = \text{const}$. By virtue of (2.1e) and (2.2) in this case we have

$$p(\varrho) = (\gamma - 1)\exp(S_c)\varrho^{\gamma},$$

where a positive constant S_c is a characteristic value of the entropy (without loss of generality we can take $(\gamma - 1)\exp(S_c) = 1$). Assuming that this relation holds for $\nu_i \neq 0$ we arrive at the system of *compressible Navier–Stokes equations* for isentropic flows of viscous compressible fluid

$$\Delta \mathbf{u} + \lambda \nabla \operatorname{div} \mathbf{u} = k\varrho \mathbf{u} \cdot \nabla \mathbf{u} + \sigma \nabla \varrho^{\gamma} - \varrho \mathbf{f} - \mathbf{h} \quad \text{in } \Omega, \tag{2.4a}$$

$$\operatorname{div}(\varrho \mathbf{u}) = 0 \quad \text{in } \Omega. \tag{2.4b}$$

Recall that the exponent γ depends on the physical properties of the fluid. In particular, $\gamma = 5/3$ for mono-atomic, $\gamma = 7/5$ for diatomic and $\gamma = 4/3$ for polyatomic gases, [14]. It is worthy of note that equations (2.4) are not compatible with (2.3), and that they are not *thermodynamically consistent*. Nevertheless, compressible Navier–Stokes equations play an important role in the theory as the only example of physically relevant equations of compressible fluid dynamics for which we have nonlocal existence result.

2.2. *Boundary conditions*

The governing equations should be supplemented with the boundary conditions. The typical boundary conditions for the velocity are: the first boundary condition (Dirichlet-type condition)

$$\mathbf{u} = \mathbf{U} \quad \text{on } \partial\Omega, \tag{2.5}$$

the second boundary condition (Neumann-type condition)

$$(\mathbb{S}(\mathbf{u}) - p\,\mathbf{I})\mathbf{n} = \mathbf{S}_n \quad \text{on } \partial\Omega, \tag{2.6}$$

where $\mathbf{n}$ is the outward normal vector to $\partial\Omega$, $\mathbf{U}$ and $\mathbf{S}_n$ are given vector fields. The important particular cases are the *no-slip boundary condition* with $\mathbf{U} = 0$, and zero normal stress condition with $\mathbf{S}_n = 0$. The third physically and mathematically reasonable condition is the *no-stick boundary condition*

$$\mathbf{u} \cdot \mathbf{n} = 0, \quad ((\mathbb{S}(\mathbf{u}) - p\,\mathbf{I})\mathbf{n}) \times \mathbf{n} = 0 \quad \text{on } \partial\Omega,$$

which corresponds to the case of frictionless boundary.

The typical boundary conditions for the temperature are the Dirichlet boundary condition

$$\vartheta = g \quad \text{on } \partial\Omega, \tag{2.7}$$

the Neumann boundary condition

$$\nabla\vartheta \cdot \mathbf{n} = g \quad \text{on } \partial\Omega,$$

and the third boundary condition

$$\nabla\vartheta \cdot \mathbf{n} + \mathbb{N}\mathrm{u}\,\vartheta = g \quad \text{on } \partial\Omega.$$

Here g is a given function, the dimensionless Nusselt number defined by the equality $\mathbb{N}\mathrm{u} = l_c\alpha/\kappa$, in which α is the positive heat transfer coefficient.

The formulation of boundary conditions for the density is a more delicate task. Assume that the velocity $\mathbf{u}$ satisfies the first boundary condition (2.5), and split the boundary of flow region into three disjoint sets called the inlet Σ_{in}, the outgoing set Σ_{out}, and the characteristic set Σ_0, and defined by the relations

$$\Sigma_{\text{in}} = \{x \in \partial\Omega : \mathbf{U}\cdot\mathbf{n} < 0\}, \quad \Sigma_{\text{out}} = \{x \in \partial\Omega : \mathbf{U}\cdot\mathbf{n} > 0\}, \quad \Sigma_0 = \{x \in \partial\Omega : \mathbf{U}\cdot\mathbf{n} = 0\}. \tag{2.8}$$

The density distribution must be given on the inlet

$$\varrho = \varrho_b \quad \text{on } \Sigma_{\text{in}}. \tag{2.9}$$

The boundary conditions for the density are not needed in the case $\Sigma_{\text{in}} = \emptyset$. In particular, there are no boundary conditions for the density if the velocity satisfies the no-slip and no-stick conditions when $\Sigma_{\text{in}} = \Sigma_{\text{out}} = \emptyset$. But in this case one extra scalar condition is required to fix the average density m of the fluid

$$\frac{1}{|\Omega|}\int_\Omega \varrho\,dx = m. \tag{2.10}$$

2.3. *Bibliographical comments*

The mathematical theory of compressible viscous flows is covered in the books by Lions [34], Feireisl [16], and Novotný and Straškraba [41]. The statement of basic principles of the fluid mechanics can be found in monographs by Landau and Lifschitz [35], and Serrin [55].

3. Mathematical aspects of the problem

In general the mathematical analysis of boundary value problems includes the following steps, with the mathematical proofs of the required facts,

- existence of solutions for the appropriate given data;
- uniqueness of solutions,
- stability of solutions with respect to perturbations of the given data, including the flow domain, and coefficients of the governing equations.

In spite of considerable progress having been made in the past two decades, the theory of compressible viscous flows is far from being complete. In this section we give a brief overview of available results. First note that governing equations (2.3) form a hyperbolic–elliptic system of differential equations, which include the Lame-type equation for the velocity, the transport equation for the density, and the Poisson-type equation for the temperature. There is a significant disparity between stationary and nonstationary problems:

- in contrast to the nonstationary case, for stationary problems the energy conservation law does not imply the boundedness of the total energy, and the derivation of the first energy estimate becomes nontrivial;
- there are no estimates for the total mass of the gas in the inflow and/or outflow problems, the absence of the mass control is the main difficulty because of which these problems remain essentially unsolved;
- the equation for the density is degenerate in points where $\mathbf{u} = 0$, and the mathematical treatment of the problem requires more extensive mathematical theory of transport equations;
- the governing equations do not guarantee automatically the nonnegativity of the density, and this question needs further consideration.

3.1. *Local existence and uniqueness results*

The local theory deals with *strong solutions* which are close, in an appropriate metric, to some given explicit or approximate solutions (e.g., the equilibrium rest state). By a strong solution is meant a solution which has locally integrable generalized derivatives satisfying the equations almost everywhere in the sense of the Lebesgue measure. The minimal smoothness properties which are usually required for strong solutions in the theory of compressible viscous flows are $\mathbf{u}, \vartheta \in C^1(\Omega)$, $\varrho \in C(\Omega)$. Since the governing equations involves the elliptic component, the scale of Sobolev spaces $H^{s,r}$ can be considered as the most suitable framework for the mathematical treatment of the problem.

In this frame the considerations are focused on the detailed analysis of the linearized problem. The main goal is in one way or another to eliminate the divergence of the velocity field, and to obtain the transport equation for the density distribution. Roughly speaking, there are three different approaches to this problem. The first is the simple algebraic scheme proposed in the pioneering paper by Padula [48] for the isothermal problem with no-slip boundary condition:

$$\Delta \mathbf{u} + \lambda \nabla \operatorname{div} \mathbf{u} = k\varrho \mathbf{u} \cdot \nabla \mathbf{u} + \sigma \nabla \varrho - \varrho \mathbf{f} \quad \text{in } \Omega, \tag{3.1a}$$

$$\operatorname{div}(\varrho \mathbf{u}) = 0 \quad \text{in } \Omega, \tag{3.1b}$$

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega, \quad \frac{1}{|\Omega|} \int_\Omega \varrho \, dx = m. \tag{3.1c}$$

The key observation employed in the scheme is that governing equations can be replaced by the Stokes-type equation for the velocity and for *effective viscous pressure* $q = \sigma\varrho - \lambda \operatorname{div} \mathbf{u}$:

$$\Delta \mathbf{u} - \nabla q = k\varrho \mathbf{u} \nabla \mathbf{u} - \varrho \mathbf{f} \quad \text{in } \Omega,$$

$$\operatorname{div}\mathbf{u} = \sigma_\lambda \varrho - \frac{1}{\lambda} q \quad \text{in } \Omega,$$
$$\mathbf{u} = 0 \quad \text{on } \partial\Omega, \qquad \sigma_\lambda = \sigma/\lambda,$$

and by the transport equation for the density:

$$\mathbf{u}\cdot\nabla\varrho + \sigma_\lambda\varrho^2 = \frac{q\varrho}{\lambda} \quad \text{in } \Omega, \qquad \frac{1}{|\Omega|}\int_\Omega \varrho = m.$$

This scheme is working properly for large σ_λ, λ and small k. In particular, the approach leads to the following existence and uniqueness result shown in [48], see also [49] and [20] for further details.

THEOREM 3.1. *Let Ω be a bounded domain with the smooth boundary, $\mathbf{f} \in C(\Omega)$ and $m > 0$. Then there exists $\sigma^* > 0$, depending only on Ω, m and $\mathbf{f}$, and positive λ^*, k^*, and R, depending only on Ω, $\mathbf{f}$, m and σ_λ such that for all $\sigma_\lambda > \sigma^*$, $\lambda > \lambda^*(\sigma_\lambda, m, \mathbf{f}, \Omega)$ and $0 < k < k^*(\sigma_\lambda, m, \mathbf{f}, \Omega)$, problem (3.1) has a unique solution in the ball*

$$\|\mathbf{u}\|_{H^{2,4}(\Omega)} + \|\varrho - m\|_{H^{1,4}(\Omega)} \le R.$$

The second approach is proposed in the papers [7] and [8] by Beirao da Veiga which are devoted to studying the local existence theory for the general boundary value problem

$$\Delta\mathbf{u} + \lambda\nabla\operatorname{div}\mathbf{u} = k\varrho\mathbf{u}\cdot\nabla\mathbf{u} + \sigma\nabla p(\varrho,\vartheta) - \varrho\mathbf{f} \quad \text{in } \Omega, \tag{3.2a}$$
$$\operatorname{div}(\varrho\mathbf{u}) = g \quad \text{in } \Omega, \tag{3.2b}$$
$$\Delta\vartheta = c_1 p'_\vartheta(\varrho,\vartheta)\operatorname{div}\mathbf{u} + c_2\varrho\mathbf{u}\nabla\vartheta \tag{3.2c}$$
$$\qquad - \psi(\nabla\mathbf{u},\nabla\mathbf{u}) + \varrho h, \tag{3.2d}$$
$$\mathbf{u} = 0, \quad \vartheta = 0 \quad \text{on } \partial\Omega, \qquad \frac{1}{|\Omega|}\int_\Omega \varrho\,dx = m, \tag{3.2e}$$

where c_i, $i = 1, 2$ are positive constants, $\psi(\cdot,\cdot)$ is a quadratic form, whose properties are not essential, and h is a given function. It is assumed the a function $p(\varphi,\vartheta)$ is sufficiently smooth and $p'_\varrho(m,0) > 0$. In this setting the principal part of linearized equations reads

$$\Delta\mathbf{u} + \lambda\nabla\operatorname{div}\mathbf{u} - \sigma p'(m,0)\varphi = \mathbf{F} \quad \text{in } \Omega,$$
$$\mathbf{u}\cdot\nabla\varphi + \varphi\operatorname{div}\mathbf{u} + m\operatorname{div}\mathbf{u} = g \quad \text{in } \Omega,$$
$$\Delta\vartheta - c_1 p'_\vartheta(m,0)\ \operatorname{div}\mathbf{u} = h,$$
$$\mathbf{u} = 0 \quad \text{on } \partial\Omega,$$

where $\varphi = \varrho - m$ is the density perturbation. The key observation in papers [7,8] is that this system can be reduced to a Poisson-type equation for $\mathbf{u}$ and the transport equation for the Laplacian of the density perturbation

$$\mathbf{u}\nabla(\Delta\varphi) + \frac{\sigma m}{1+\lambda}(\Delta\varphi) = F(\mathbf{f}, g, h),$$

in which F is some linear operator. Note that this method requires the existence and uniqueness results for a transport equation in negative Sobolev spaces, see [9]. Using this approach Beirao da Veiga proved the following existence result.

THEOREM 3.2. *Let $r \in (1, \infty)$ and $j \geq -1$. Assume that $\partial\Omega \in C^{3+j}$ and $p \in C^{3+j}(\mathbb{R}^2)$. Then there exist constants c_0', c_1' depending only on Ω, λ, σ, m and c_i, $i = 1, 2$ such that if* $(\mathbf{f}, g, h)$ *verifies the condition*

$$\|\mathbf{f}\|_{H^{1+j,r}(\Omega)} + \|g\|_{H^{2+j,r}(\Omega)} + \|h\|_{H^{1+j,r}(\Omega)} \leq c_0', \quad \langle, g, 1\rangle = 0,$$

then there exists a unique solution to problem (3.2) *in the ball*

$$\|\mathbf{u}\|_{H^{3+j,r}(\Omega)} + \|\varrho - m\|_{H^{2+j,r}(\Omega)} + \|\vartheta\|_{H^{3+j,r}(\Omega)} \leq c_1'.$$

It was also shown that in the case of barotropic flow a solution to the problem tends to a solution of incompressible Stokes equations as $\sigma \to \infty$.

If $\mathbf{h} = 0$ and the mass force is potential, i.e. $\mathbf{f} = \nabla\Phi(x)$, then equations (2.3) endowed with the boundary conditions

$$\mathbf{u} = 0, \quad \vartheta = 0 \quad \text{on } \Omega$$

has the only solution

$$\mathbf{u}_0 = 0, \quad \vartheta_0 = 0, \quad \varrho_0(x) = c(m)\exp(\sigma^{-1}\Phi(x)).$$

The equilibrium solutions of this type also exist for general constitutive law $p = p(\varrho, \vartheta)$ provided that the total mass $\mathcal{M}$ is sufficiently large. The existence of solutions to boundary value problems close to $(\mathbf{u}_0, \varrho_0, \vartheta_0)$ was established by the application of the decomposition method proposed in the paper by Novotny and Padula [42]. The main idea of the method is to split the velocity field $\mathbf{u}$ into an incompressible part $\mathbf{u}_{sol}$, satisfying the relation $\mathrm{div}(\varrho_0\mathbf{u}_{sol}) = 0$, and a compressible irrotational part $\mathbf{u}_{pot} = \nabla\phi$. In such a way the original mixed elliptic–hyperbolic system is split into several equations: a Stokes-type system for $\mathbf{u}_{sol}$, a Poisson-type equation for ϕ and the transport equation for the density ϱ.

In [43] and [44] the results were applied over the case of flow in exterior domains which has practical importance. In this case the problem can be formulated as follows.

Let $\Omega \subset \mathbb{R}^3$ be an exterior domain such that $\Omega_c = \mathbb{R}^3 \setminus \Omega$ is a compact which has a boundary of a class C^{k+2}, $k \geq 0$. It is necessary to find the velocity field $\mathbf{u}$ and the density distribution ϱ satisfying equations (2.4) with $\gamma = 1$ (the extension to the case $\gamma > 1$ is obvious) and the boundary conditions

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega, \quad \varrho(x) \to \varrho_\infty, \quad \mathbf{u}(x) \to \mathbf{u}_\infty \quad \text{as } |x| \to \infty, \tag{3.3}$$

where the given vector $\mathbf{u}_\infty = (u_\infty, 0, 0)$, $u_\infty \neq 0$. The following existence and local uniqueness result is due to Novotny and Padula.

Let $\mathcal{W}^{k,r}(\Omega)$ be a completion of $C_0^\infty(\Omega)$ in the norm $\|\nabla \cdot \|_{H^{k-1,r}(\Omega)}$. Introduce the Banach spaces $W^{k,q,r} = H^{1,q}(\Omega) \cap H^{k,r}(\Omega)$ and $\mathbb{D}^{k,q,r} = \mathcal{W}^{2,q} \cap \mathcal{W}^{k,r}$ endowed with the standard norm of intersections of Banach spaces. Finally introduce

$$\mathbb{Q}^{k+1,q,r} = W^{k,q,r} \times \left(\mathbb{D}^{k,q,r} \cap L^{\frac{4q}{4-q}}(\Omega)\right)^3.$$

Endowed with the norm

$$|||(\varphi, \mathbf{v})|||_{k+1,q,r,u_\infty} = \|\varphi\|_{W^{k,q,r}} + \|\mathbf{v}\|_{\mathbb{D}^{k+1,q,r}} + |u_\infty|^{1/4}\|\mathbf{v}\|_{L^{\frac{4q}{4-q}}(\Omega)}$$

$\mathbb{Q}^{k+1,q,r}$ becomes the Banach space. The set

$$\mathbb{M}^{k,q,r}_{u_\infty} = \{(\varphi, \mathbf{v}) \in \mathbb{Q}^{k+1,q,r} : \ \mathbf{v} = -\mathbf{u}_\infty \text{ on } \partial\Omega\}$$

is the convex closed subset of $\mathbb{Q}^{k+1,q,r}$.

THEOREM 3.3. *Let $r > 3$, $1 < s < 5/6$, integer $k \geq 0$, $\partial\Omega \in C^{3+k}$ and $\mathbf{f} \in L^s(\Omega) \cap H^{k,r}(\Omega)$. Let q be an arbitrary such that $3s/(3-s) \leq q \leq 2$. Then there exists a positive $k_0 < 1$ which depends only on $s, q, k, r, \partial\Omega$ such that for any u_∞ with*

$$0 < |u_\infty| \leq k_0$$

there exist positive constants γ_0, γ_1 (depending only on $s, q, k, r, \partial\Omega$ and $|u_\infty|$) with the property: If

$$\|\mathbf{f}\|_{L^s(\Omega)} + \|\mathbf{f}\|_{H^{k,r}(\Omega)} \leq \gamma_1,$$

then in the set

$$\{(\varrho, \mathbf{u}) : \ (\varrho - \varrho_\infty, \mathbf{u} - \mathbf{u}_\infty) \in \mathbb{M}^{k+2,q,r}_{u_\infty}, \ |||(\varrho - \varrho_\infty, \mathbf{u} - \mathbf{u}_\infty)|||_{k+2,q,r,u_\infty} \leq \gamma_0\}$$

there exists just one solution to problem (2.4), (3.3).

3.1.1. *Inhomogeneous boundary value problems*

It is worthy of note that there are only a few kinds of physically reasonable external forces: the gravitational force, the centrifugal and centripetal forces, the electromagnetic forces. Therefore the stationary boundary value problems for compressible Navier–Stokes equations in *bounded domains* with no-slip and no-stick boundary conditions are primarily of academic interest. The inflow–outflow boundary value problems for $2D$-compressible Navier–Stokes and compressible Navier–Stokes–Fourier equations were considered in papers [23,24] by Kellogg and Kweon. It was shown that the properties of solutions are very sensitive to the geometry of the flow domain Ω and behavior of the velocity vector field at the boundary of Ω. It turns out that in the case of a smooth domain Ω the density has a weak singularity on the characteristic manifold $\overline{\Sigma_{\text{in}}} \cap \overline{\Sigma_{\text{out}}}$. It was shown that if $\partial\Omega$ and $\mathbf{U}$ satisfy the *emergent field condition* (**H1**)–(**H2**) in Section 5.1, then the problem has a continuous solution close to the explicit solution $\mathbf{u}_0 = (u_0, 0)$, $\varrho = \text{const}$. The flow on polygon was investigated in [24]. We consider this problem in more detail in Section 9. The qualitative properties of solutions and the propagation of corner singularities were considered in the papers [22,25] by Kellogg and Kellogg and Kweon, and in the paper [56] by Xinfu Chen and Xie. It was shown that jump of discontinuity in the velocity on the boundary produces a discontinuous density across the streamline emanating from the point of discontinuity. Moreover, in polygons the interior singularities do occur even for continuous boundary data.

3.2. *Global existence of generalized solutions*

By now there are no results on the existence of strong solutions to compressible Navier–Stokes–Fourier equations for large data. The available results concern the existence of weak solutions to the boundary value problem for the barotropic compressible Navier–Stokes equations with no-slip conditions at the boundary:

$$\Delta \mathbf{u} + \lambda \nabla \operatorname{div} \mathbf{u} = k\varrho \mathbf{u} \otimes \mathbf{u} + \sigma \nabla p(\rho) - \varrho \mathbf{f} - \mathbf{h} \quad \text{in } \Omega, \tag{3.4a}$$

$$\operatorname{div}(\varrho \mathbf{u}) = 0 \quad \text{in } \Omega, \tag{3.4b}$$

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega, \tag{3.4c}$$

$$\int_\Omega \varrho \, dx = \mathcal{M}, \tag{3.4d}$$

where $p(\cdot)$ is a smooth strictly monotone function. Along with problem (3.4), in mathematical literature its different regularization is also considered. The most distributed is the *relaxed* boundary value problem, [34],

$$a\varrho \mathbf{u} + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\rho) = \varrho \mathbf{f} + \mathbf{h} + \operatorname{div} \mathbb{S}, \tag{3.5}$$

$$a\varrho + \operatorname{div}(\varrho \mathbf{u}) = h \quad \text{in } \Omega, \tag{3.6}$$

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega,$$

where $a(x)$ and $h(x)$ are given positive functions. This problem is of mathematical interest since for a = const., equations (3.5) can be regarded as the time discretization of the evolutionary problem.

The definition of a weak solution to compressible Navier–Stokes equations differs from a standard that is caused by the specific character of the transport equation, see [34] and [16].

DEFINITION 3.4. We say that $(\varrho, \mathbf{u})$ is a weak solution to equations (3.4a)–(3.4c) if $\mathbf{u} \in H_0^{1,2}(\Omega)$, $\varrho \in L^1(\Omega)$, nonnegative functions ϱ, $\varrho|\mathbf{u}|^2$, and $p(\varrho)$ are locally integrable and equations (3.4a)–(3.4b) are satisfied in the sense of distributions: for all vector fields $\varphi \in C_0^\infty(\Omega)$ and functions $G \in C^1(\mathbb{R})$, $\psi \in C^1(\Omega)$,

$$\int_\Omega \nabla\varphi : (\varrho \mathbf{u} \otimes \mathbf{u} + p\,\mathbf{I})\, dx = \int_\Omega (\nabla\varphi : \nabla\mathbf{u} + \lambda \operatorname{div}\varphi \operatorname{div}\mathbf{u} - (\varrho\mathbf{f} + \mathbf{h}) \cdot \varphi)\, dx \tag{3.7a}$$

$$\int_\Omega (G(\varrho)\mathbf{u} \cdot \nabla\psi + (G(\varrho) - G'(\varrho)\varrho)\psi \operatorname{div}\mathbf{u})\, dx = 0. \tag{3.7b}$$

Note that weak solutions whose definition involves arbitrary functions of unknown quantities are generally referred to as *renormalized* solutions.

The principal question in the theory concerns the compactness properties of the generalized solution. Suppose for a moment that we have a sequence of explicit or approximate solutions $(\mathbf{u}_\varepsilon, \varrho_\varepsilon)_{\varepsilon>0}$ to problem (3.4) and the energies of these solutions are uniformly bounded

$$\|\mathbf{u}_\varepsilon\|_{H^{1,2}(\Omega)} + \|\varrho_\varepsilon\|_{L^\gamma(\Omega)} + \|\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2\|_{L^1(\Omega)} < C < \infty.$$

After passing to a subsequence we can assume that

$$\mathbf{u}_\varepsilon \to \mathbf{u} \quad \text{weakly in } H_0^{1,2}(\Omega), \qquad \varrho_\varepsilon \to \varrho \quad \text{weakly in } L^\gamma(\Omega),$$

$$\varrho_\varepsilon \mathbf{v}_\varepsilon \otimes \mathbf{v}_\varepsilon \to \mathbb{M} \quad \text{star weakly in the space of Radon measures as } \varepsilon \to 0.$$

The question is under what conditions will a weak limit $(\mathbf{u}, \varrho)$ be a solution to the original equation. In order to answer this question we have to solve two problems:

First we have to resolve the *oscillation* problem, i.e. to prove the equality $w\text{-lim}\, p(\varrho_\varepsilon) = p(\varrho)$.

Next we have to prove that the *weak star defect measure* $\mathcal{S} =: \mathbb{M} - \varrho\mathbf{v}\otimes\mathbf{v} = 0$ and thereby to resolve the *concentration* problem.

The mathematical analysis of these problems in 2 and 3 dimensions originated in papers [32,33] by Lions, where the weak regularity of the *effective viscous flux* was established. This result allows to prove the compactness properties of the density and hence gives a way around the oscillation problem. The detailed mathematical treatment of the problems (3.4) and (3.7) was undertaken in the monograph [34] by Lions. In particular, there the following global result on solvability of problem (3.4) was proved.

THEOREM 3.5. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^1-boundary. Assume that the C^1-function p is strictly monotone and satisfies the following conditions: there exist $\gamma > 5/3$ and $c_0 > 0$ such that*

$$c_0^{-1} s^{\gamma-1} < p'(s) < c_0 s^{\gamma-1}, \quad c_0^{-1} s^{\gamma} < p(s) < c_0 s^{\gamma} \quad \textit{for all sufficiently large } s.$$

Then for each $\mathcal{M} > 0$, $\mathbf{f}, \mathbf{h} \in C(\Omega)$ boundary value problem (3.4) has a generalized solution satisfying the energy inequality

$$\int_\Omega (|\nabla\mathbf{u}|^2 + \lambda|\operatorname{div}\mathbf{u}|^2)\,dx \le \int_\Omega (\varrho\mathbf{f} + \mathbf{h})\cdot\mathbf{u}\,dx.$$

The restrictions on the adiabatic exponent were weakened by Novo and Novotny in [39] and Novotny and Straškraba in [41] in the case of *potential mass forces* $\mathbf{f}$. Using the Feireisl theory of the oscillation defect measure, [16,17], they proved that for potential mass forces the statement of the Lions Theorem remains true for all adiabatic constant $\gamma > 3/2$. The extension of these results on the case of the exterior boundary value problem and boundary value problems in domains with noncompact boundary was made by Novo and Novotny in [40].

It is easy to see that in the three-dimensional case the energy estimates guarantee the inclusion of $\varrho|\mathbf{v}|^2 \in L^r(\Omega)$ with $r > 1$ if and only if $\gamma > 3/2$. Hence, for $\gamma \le 3/2$ we have only an L^1 estimate for the density of the kinetic energy, and the concentrations problem becomes nontrivial. In this sense the critical value $\gamma = 3/2$ is the block for the existing nonlocal theory. Steps to overcome this threshold were taken by Frehse *et al.* in [19] and by Plotnikov and Sokolowski in [53], where it was proved that the boundedness of the total energy implies the compactness properties of solutions to problem (3.4) for all $\gamma > 1$. The proof of this result is based on pointwise estimates of the Newtonian potential of the pressure. Using this approach Plotnikov and Sokolowski have proved the existence of solutions to the relaxed problem (3.5) for all $\gamma > 1$. In Section 7 we show that this approach also leads to the existence theory for barotropic flows for all $\gamma > 4/3$.

4. Mathematical preliminaries

In this paragraph we assemble some technical results which are used throughout of the paper. Function spaces play a central role, and we recall some notations, fundamental definitions, and properties, which can be found in [1] and [10]. For the convenience of readers we also collect the basic facts from the theory of interpolation spaces and the Young measures. For our applications we need the results in three spatial dimensions, however the results are presented for the dimension $d \geq 2$.

4.1. *Interpolation theory*

In this paragraph we recall some results from the interpolation theory, see [10] for the proofs. Let A_0 and A_1 be Banach spaces. For simplicity assume that $A_1 \subset A_0$. For $t > 0$ introduce two nonnegative functions $K : A_0 \mapsto \mathbb{R}$ and $J : A_1 \mapsto \mathbb{R}$ defined by

$$K(t, u, A_0, A_1) = \inf_{\substack{u=u_0+u_1\\ u_i \in A_i}} \|u_0\|_{A_0} + t\|u_1\|_{A_1}$$

and

$$J(t, u, A_0, A_1) = \max\{\|u\|_{A_0}, t\|u\|_{A_1}\}.$$

For each $s \in (0, 1)$, $1 < r < \infty$, the K-interpolation space $[A_0, A_1]_{s,r,K}$ consists of all elements $u \in A_0$, having the finite norm

$$\|u\|_{[A_0,A_1]_{s,r,K}} = \left(\int_0^\infty t^{-1-sr} K(t, u, A_0, A_1)^r \, dt\right)^{1/r}. \tag{4.1}$$

On the other hand, J-interpolation space $[A_0, A_1]_{s,r,J}$ consists of all elements $u \in A_0+A_1$ which admit the representation

$$u = \int_0^\infty \frac{v(t)}{t} \, dt, \quad v(t) \in A_1 \quad \text{for } t \in (0, \infty) \tag{4.2}$$

and have the finite norm

$$\|u\|_{[A_0,A_1]_{s,r,J}} = \inf_{v(t)} \left(\int_0^\infty t^{-1-sr} J(t, v(t), A_0, A_1)^r \, dt\right)^{1/r} < \infty, \tag{4.3}$$

where the infimum is taken over the set of all $v(t)$ satisfying (4.2). The first main result of the interpolation theory reads: For all $s \in (0, 1)$ and $r \in (1, \infty)$ the spaces $[A_0, A_1]_{s,r,K}$ and $[A_0, A_1]_{s,r,J}$ are isomorphic, topologically and algebraically. Hence the introduced norms are equivalent, and we can omit indices J and K. The following simple properties of interpolation spaces directly follows from definitions.
(1) If $A_1 \subset A_0$ is dense in A_0, then $[A_0, A_1]_{s,r} \subset A_0$ is dense in A_0. To show this fix an arbitrary $u \in [A_0, A_1]_{s,r}$ and choose the v in representation (4.2) such that

$\|t^{-s}v\|_{L^r(0,\infty;dt/t)} < \infty$. It is easy to see that $u_n = \int_{n^{-1}}^n v(t)t^{-1}dt \in A_1$ and

$$\|u_n - u\|^r_{[A_0,A_1]_{s,r,J}} \le \left(\int_0^{n^{-1}} + \int_n^{\infty}\right) t^{-1-sr} J(t, v(t), A_0, A_1)^r \, dt \to 0 \quad \text{as } n \to \infty.$$

(2) If $\tilde{A}_i$, $i = 0, 1$, are closed subspaces of A_i, then $[\tilde{A}_0, \tilde{A}_1]_{s,r} \subset [A_0, A_1]_{s,r}$ and $\|u\|_{[A_0,A_1]_{s,r}} \le \|u\|_{[\tilde{A}_0,\tilde{A}_1]_{s,r}}$.

One of the important results of the interpolation theory is the following representation for the interpolation of dual spaces. Let A_i be Banach spaces such that $A_1 \cap A_0$ is dense in $A_0 + A_1$. Then the Banach spaces $[(A_0)', (A_1)']_{s,r'}$ and $([A_0, A_1]_{s,r})'$ are isomorphic topologically and algebraically. Hence the spaces can be identified with equivalent norms.

In particular, if $A_1 \subset A_0$ $A_0' \subset A_1'$ are dense in A_0 and A_1', then $([A_0, A_1]_{s,r})'$ is the completion of A_0' in $([A_0, A_1]_{s,r})'$-norm.

The following lemma is the central result of the interpolation theory.

LEMMA 4.1. *Let A_i, B_i, $i = 0, 1$, be Banach spaces and let $T : A_i \mapsto B_i$, be a bounded linear operator. Then for all $s \in (0, 1)$ and $r \in (1, \infty)$, the operator $T : [A_0, A_1]_{s,r} \mapsto [B_0, B_1]_{s,r}$ is bounded and*

$$\|T\|_{\mathcal{L}([A_0,A_1]_{s,r},[X_0,Y_1]_{s,r})} \le \|T\|^s_{\mathcal{L}(A_0,B_0)} \|T\|^{1-s}_{\mathcal{L}(A_1,B_1)}.$$

4.2. *Function spaces*

Let Ω be the whole space $\mathbb{R}^d$ or a bounded domain in $\mathbb{R}^d$ with the boundary $\partial\Omega$ of class C^1. For an integer $l \ge 0$ and for an exponent $r \in [1, \infty)$, we denote by $H^{l,r}(\Omega)$ the Sobolev space endowed with the norm $\|u\|_{H^{l,r}(\Omega)} = \sup_{|\alpha| \le l} \|\partial^\alpha u\|_{L^r(\Omega)}$.

For real $0 < s < 1$, the fractional Sobolev space $H^{s,r}(\Omega) = [H^{0,r}(\Omega), H^{1,r}(\Omega)]_{s,r}$ endowed with one of the equivalent norms (4.1) or (4.3) is obtained by the interpolation between $L^r(\Omega)$ and $H^{1,r}(\Omega)$. It consists of all measurable functions with the finite norm

$$\|u\|_{H^{s,r}(\Omega)} = \|u\|_{L^r(\Omega)} + |u|_{s,r,\Omega},$$

where

$$|u|^r_{s,r,\Omega} = \int_{\Omega\times\Omega} |x - y|^{-d-rs} |u(x) - u(y)|^r \, dxdy. \tag{4.4}$$

In the general case, the Sobolev space $H^{l+s,r}(\Omega)$ is defined as the space of measurable functions with the finite norm $\|u\|_{H^{l+s,r}(\Omega)} = \sup_{|\alpha| \le l} \|\partial^\alpha u\|_{H^{s,r}(\Omega)}$. For $0 < s < 1$, the Sobolev space $H^{s,r}(\Omega)$ is, in fact [10], the interpolation space $[L^r(\Omega), H^{1,r}(\Omega)]_{s,r}$.

Furthermore, the notation $H_0^{l,r}(\Omega)$, with an integer l, stands for the closed subspace of the space $H^{l,r}(\Omega)$ of all functions $u \in H^{l,r}(\Omega)$ which are being extended by zero outside Ω, and which belong to $H^{l,r}(\mathbb{R}^d)$.

Denote by $\mathcal{H}_0^{0,r}(\Omega)$ and $\mathcal{H}_0^{1,r}(\Omega)$ the subspaces of $L^r(\mathbb{R}^d)$ and $H^{1,r}(\mathbb{R}^d)$, respectively, of all functions vanishing outside of Ω. Obviously $\mathcal{H}_0^{1,r}(\Omega)$ and $H_0^{1,r}(\Omega)$ are isomorphic topologically and algebraically and we can identify them. However, we need the interpolation spaces $\mathcal{H}_0^{s,r}(\Omega)$ for nonintegers, in particular for $s = 1/r$.

DEFINITION 4.2. For all $0 < s \le 1$ and $1 < r < \infty$, we denote by $\mathcal{H}_0^{s,r}(\Omega)$ the interpolation space $[\mathcal{H}_0^{0,r}(\Omega), \mathcal{H}_0^{1,r}(\Omega)]_{s,r}$ endowed with one of the equivalent norms (4.1) or (4.3) defined by the interpolation method.

It follows from the property (2) of interpolation spaces that $\mathcal{H}_0^{s,r}(\Omega) \subset H^{s,r}(\mathbb{R}^d)$ and for all $u \in \mathcal{H}_0^{s,r}(\Omega)$,

$$\|u\|_{H^{s,r}(\mathbb{R}^d)} \le c(r,s)\|u\|_{\mathcal{H}_0^{s,r}(\Omega)}, \quad u = 0 \quad \text{outside } \Omega. \tag{4.5}$$

In other words, $\mathcal{H}_0^{s,r}(\Omega)$ consists of all elements $u \in H^{s,r}(\Omega)$ such that the extension $\overline{u}$ of u by 0 outside of Ω have the finite $[\mathcal{H}_0^{0,r}(\Omega), \mathcal{H}_0^{1,r}(\Omega)]_{s,r}$-norm. We identify u and $\overline{u}$ for the elements $u \in \mathcal{H}_0^{s,r}(\Omega)$. With this identification it follows that $H_0^{1,r}(\Omega) \subset \mathcal{H}_0^{s,r}(\Omega)$ and the space $C_0^\infty(\Omega)$ is dense in $\mathcal{H}_0^{s,r}(\Omega)$.

It is worthy of note that a function u belongs to the space $\mathcal{H}_0^{s,r}(\Omega)$, $0 \le s \le 1$, $1 < r < \infty$ if and only if

$$\operatorname{dist}(x, \partial\Omega)^{-s}|u| \in L^r(\Omega).$$

In particular $\mathcal{H}_0^{s,r}(\Omega) = H^{s,r}(\Omega)$ for $0 \le s < r^{-1}$. We also point out that the interpolation space $\mathcal{H}_0^{s,r}(\Omega)$ coincides with the Sobolev space $H_0^{s,r}(\Omega)$ for $s \ne 1/r$. Recall that the standard space $H_0^{s,r}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in the $H^{s,r}(\Omega)$-norm.

Embedding of Sobolev spaces. For $sr > d$ and $0 \le \alpha < s - r/d$, the embedding $H^{s,r}(\Omega) \hookrightarrow C^\alpha(\Omega)$ is continuous and compact. In particular, for $sr > d$, the Sobolev space $H^{s,r}(\Omega)$ is a commutative Banach algebra, i.e. for all $u, v \in H^{s,r}(\Omega)$,

$$\|uv\|_{H^{s,r}(\Omega)} \le c(r,s)\|u\|_{H^{s,r}(\Omega)}\|v\|_{H^{s,r}(\Omega)}. \tag{4.6}$$

If $sr < d$ and $t^{-1} = r^{-1} - d^{-1}s$, then the embedding $H^{s,r}(\Omega) \hookrightarrow L^t(\Omega)$ is continuous. In particular, for $\alpha \le s$, $(s-\alpha)r < d$ and $\beta^{-1} = r^{-1} - d^{-1}(s-\alpha)$,

$$\|u\|_{H^{\alpha,\beta}(\Omega)} \le c(r,s,\alpha,\beta,\Omega)\|u\|_{H^{s,r}(\Omega)}. \tag{4.7}$$

If $(s-\alpha)r \ge d$, then estimate (4.7) holds true for all $\beta \in (1,\infty)$. It follows from (4.5) that all the embedding inequalities remain true for the elements of the interpolation space $\mathcal{H}_0^{s,r}(\Omega)$.

Duality. We define

$$\langle u, v\rangle = \int_\Omega u\, v\, dx \tag{4.8}$$

for any function such that the right-hand side make sense. For $r \in (1, \infty)$, each element $v \in L^{r'}(\Omega)$, $r' = r/(r-1)$, determines the functional L_v of $(\mathcal{H}_0^{s,r}(\Omega))'$ by the identity $L_v(u) = \langle u, v\rangle$. We introduce the $(-s, r')$-norm of an element $v \in L^{r'}(\Omega)$ to be by definition the norm of the functional L_v, that is

$$\|v\|_{\mathcal{H}^{-s,r'}(\Omega)} = \sup_{\substack{u\in\mathcal{H}_0^{s,r}(\Omega)\\ \|u\|_{\mathcal{H}_0^{s,r}(\Omega)}=1}} |\langle u, v\rangle|. \tag{4.9}$$

Let $\mathcal{H}^{-s,r'}(\Omega)$ denote the completion of the space $L^{r'}(\Omega)$ with respect to $(-s, r')$-norm. By virtue of pairing (4.8), the space $L^{r'}(\Omega)$ can be identified with $(\mathcal{H}_0^{0,r})'$, which is dense in $\mathcal{H}^{-1,r}(\Omega) = (\mathcal{H}_0^{1,r}(\Omega))'$. Therefore, the space $(\mathcal{H}_0^{s,r})'$ is the completion of $L^{r'}(\Omega)$ in the norm of $(\mathcal{H}_0^{s,r}(\Omega))'$, which is exactly equal to the norm of $\mathcal{H}^{-s,r'}(\Omega)$. Hence $(\mathcal{H}_0^{s,r}(\Omega))' = \mathcal{H}^{-s,r'}(\Omega)$ which leads to the duality principle

$$\|u\|_{\mathcal{H}_0^{s,r}(\Omega)} = \sup_{\substack{v\in C_0^\infty(\Omega)\\ \|v\|_{\mathcal{H}^{-s,r'}(\Omega)}=1}} |\langle u, v\rangle|. \tag{4.10}$$

Moreover, we can identify $\mathcal{H}^{-s,r'}(\Omega)$ with the interpolation space $[L^{r'}(\Omega), H_0^{-1,r'}(\Omega)]_{s,r}$.

With applications to the theory of Navier–Stokes equations in mind, we introduce the smaller dual space defined as follows. We identify the function $v \in L^{r'}(\Omega)$ with the functional $L_v \in (H^{s,r}(\Omega))'$ and denote by $\mathbb{H}^{-s,r'}(\Omega)$ the completion of $L^{r'}(\Omega)$ in the norm

$$\|v\|_{\mathbb{H}^{-s,r'}(\Omega)} := \sup_{\substack{u\in H^{s,r}(\Omega)\\ \|u\|_{H^{s,r}(\Omega)}=1}} |\langle u, v\rangle|. \tag{4.11}$$

In the sense of this identification the space $C_0^\infty(\Omega)$ is dense in the interpolation space $\mathbb{H}^{-s,r}(\Omega)$. It follows immediately from the definition that

$$\mathbb{H}^{-s,r'}(\Omega) \subset (H^{s,r}(\Omega))' \subset \mathcal{H}^{-s,r'}(\Omega).$$

4.3. *Embedding $H_0^{1,2}(\Omega)$ into $L^2(\Omega, d\mu)$*

The question of integrability of elements of the Sobolev spaces with respect to general Borel measures μ is more difficult. The following theorem belonging to Adams [2] and Mazjia [37], see also [3] for the further discussion, gives necessary and sufficient conditions for the continuity of the embedding $H_0^{1,2}(\Omega) \hookrightarrow L^2(\Omega, d\mu)$

LEMMA 4.3. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^2-boundary. Then the embedding $H_0^{1,2}(\Omega) \hookrightarrow L^r(\Omega, d\mu)$ is continuous if and only if there is a constant C such that the inequality $\mu(K) \le C\,\mathrm{cap}\,K$ holds true for any compact $K \Subset \Omega$.*

Further we shall use the following consequence of this result.

COROLLARY 4.4. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^1 boundary and $\mathfrak{G}(x, y)$ is the Green of domain Ω. Assume that a Radon measure μ satisfies the inequality*

$$M =: ess \sup_{y\in\mathbb{R}^3} \int_\Omega \mathfrak{G}(x, y)\, d\mu < \infty. \tag{4.12}$$

Then there exists a constant c, depending only on Ω, such that inequality

$$\int_\Omega |v|^2\, d\mu \le cM\|v\|^2_{H^{1,2}(\Omega)} \tag{4.13}$$

holds true for all $v \in H^{1,2}_0(D)$.

PROOF. Recall that the capacity of the compact $K \subset D$ is defined by the equality

$$\text{cap } K = \inf\left\{\int_D |\nabla\varphi|^2\, dx :\ \varphi \in C_0^\infty(D), \varphi \ge 0\, \varphi \ge 1 \text{ on } K\right\}.$$

Choose an arbitrary admissible function φ. It is easily seen that

$$\int_K d\mu \le \int_K \varphi\, d\mu \le \|\Delta^{-1/2} 1_K \mu\|_{L^2(D)} \left(\int_D |\nabla\varphi|^2\, dx\right)^{1/2}.$$

On the other hand, we have

$$\|\Delta^{-1/2} 1_K \mu\|^2_{L^2(D)} = \int_D\int_D \mathfrak{G}(x, y) 1_K(x) 1_K(y)\, d\mu(x) d\mu(y) \le M \int_K d\mu(y)$$

which leads to

$$\int_K d\mu \le M^{1/2} \left(\int_K d\mu\right)^{1/2} \left(\int_D |\nabla\varphi|^2\, dx\right)^{1/2}.$$

Hence for any compact $K \subset D$,

$$\int_K d\mu \le M \inf_\varphi \int_D |\nabla\varphi|^2\, dx = M \text{ cap } K,$$

which along with Lemma 4.3 yields the desired estimate. □

4.4. *Di-Perna–Lions Lemma*

Let θ be a standard mollifying kernel in $\mathbb{R}$,

$$\theta \in \mathcal{D}_+(\mathbb{R}), \quad \int_\mathbb{R} \theta(t)\, dt = 1, \quad \text{spt } \theta \Subset \{|t| \le 1\}. \tag{4.14}$$

For any compactly-supported and locally-integrable function $f : \mathbb{R}^d \times \mathbb{R} \mapsto \mathbb{R}$ define the mollifiers

$$[f]_{,k}(x,\lambda) := k \int_{\mathbb{R}} \theta(k(\lambda - t)) f(x,t)dt, \tag{4.15}$$

$$[f]_{m,}(x,\lambda) := m^n \int_{\mathbb{R}^n} \Theta(m(x-y)) f(y,\lambda)\, dy \quad \Theta(z) = \prod_{i=1}^{n} \theta(z_i). \tag{4.16}$$

We shall write simply $[f]_{m,k}$ instead of $[[f]_{m,}]_{,k}$. The following Lemma is due to Di-Perna and Lions [29].

LEMMA 4.5. *Let $f \in L^2_{\text{loc}}(\mathbb{R}^n \times \mathbb{R})$ and $\mathbf{v} \in H^{1,2}_{\text{loc}}(\mathbb{R}^n)$. Then for any bounded measurable set $E \subset \mathbb{R}^n \times \mathbb{R}$,*

$$\operatorname{div}[f\mathbf{v}]_{m,} - \operatorname{div}([f]_{m,}\mathbf{v}) \to 0 \quad \textit{in } L^1(E) \textit{ as } m \to \infty. \tag{4.17}$$

The kinetic theory of transport equations operates with the extended velocity field

$$\mathbf{V}(x,\lambda) = (\mathbf{v}(x), -\lambda \operatorname{div} \mathbf{v}(x)),$$

which does not meet the requirements of the previous Lemma. Nevertheless, because of the specific structure of $\mathbf{V}$, the following assertion holds true, see [5].

LEMMA 4.6. *Assume that all the assumptions of Lemma 4.5 are satisfied, $f \in L^\infty(\mathbb{R}^n \times \mathbb{R})$ and*

$$I^{k,m}(x,\lambda) = \operatorname{div}[f\mathbf{V}]_{m,k} - \operatorname{div}([f]_{m,k}\mathbf{V}). \tag{4.18}$$

Then, for any bounded measurable set $E \subset \mathbb{R}^n \times \mathbb{R}$,

$$\lim_{k\to\infty} \{ \lim_{m\to\infty} I^{m,k} \} = 0 \quad \textit{in } L^1(E). \tag{4.19}$$

4.5. *Young measures*

Since the notion of weak limits plays a crucial role in our analysis, we begin with a short description of some basic facts concerning weak convergence and weak compactness. We refer the reader to [36] for the proofs of basic results.

Let A be an arbitrary bounded, measurable subset of $\mathbb{R}^3$ and $1 < r \le \infty$. Then for every bounded sequence $\{g_n\}_{n\ge 1} \subset L^r(A)$ there exists a subsequence, still denoted by $\{g_n\}$, and a function $g \in L^r(A)$ such that for $n \to \infty$,

$$\int_A g_n(x)h(x)dx \to \int_A g(x)h(x)dx \quad \text{for all } h \in L^{r/(r-1)}(A).$$

We say the sequence converges $g_n \to g$ weakly in $L^r(A)$ for $r < \infty$, and converges star-weakly in $L^\infty(A)$ in the limit case of $r = \infty$. In the very special case of $r = 1$ it is known that the sequence of g_n contains a weakly convergent subsequence in $L^1(A)$, if and only if there is a continuous function $\Phi : \mathbb{R} \to \mathbb{R}^+$ such that

$$\lim_{s\to\infty} \Phi(s)/s = \infty \quad \text{and} \quad \sup_{n\geq 1} \|\Phi(g_n)\|_{L^1(A)} < \infty.$$

If the sequence of g_n is only bounded in $L^1(A)$ and A is open, then after passing to a subsequence we can assume that g_n converges star-weakly to a bounded Radon measure μ_g i.e.,

$$\begin{aligned}\lim_{n\to\infty} &\int_A g_n(x)h(x)dx \\ &= \int_A h(x)d\mu_g(x) \quad \text{for all compactly supported } h \in C(A).\end{aligned}$$

In the sequel, the linear space of compactly supported functions on a set A is denoted by $C_0(A)$, and its dual by $C_0(A)^*$.

The Ball's version [6], see also [36], of the fundamental Tartar Theorem on Young measures gives a simple and effective representation of weak limits in the form of integrals over families of probabilities measures. The following lemma is a consequence of Ball's theorem.

LEMMA 4.7. *Suppose that a sequence $\{g_n\}_{n\geq 1}$ is bounded in $L^r(A)$, $1 < r \leq \infty$, where A is an open, bounded subset of $\mathbb{R}^3$. Then we have the following characterizations of weak limits.*

(i) *There exists a subsequence, still denoted by $\{g_n\}_{n\geq 1}$, and a family of probability measures $\sigma_x \in C_0(\mathbb{R})^*$, $x \in A$, with a measurable distribution function $f(x,\lambda) := \sigma_x(-\infty,\lambda]$ so that the function $\lambda \mapsto f(x,\lambda)$ is monotone and continuous from the right, and admits the limits $1, 0$ for $\lambda \to \pm\infty$. Furthermore, for any continuous function $G : A \times \mathbb{R} \mapsto \mathbb{R}$ such that*

$$\begin{aligned}&\lim_{|\lambda|\to\infty} \|G(\cdot,\lambda)\|_{C(A)}/|\lambda|^r = 0 \quad \textit{for } r < \infty \quad \textit{and} \\ &\quad \sup_{|\lambda|} \|G(\cdot,\lambda)\|_{C(A)} < \infty \quad \textit{for } r = \infty,\end{aligned}$$

the sequence of $G(\cdot, g_n)$ converges weakly in $L^1(A)$ to a function

$$\overline{G}(x) = \int_{\mathbb{R}} G(x,\lambda)d_\lambda f(x,\lambda). \tag{4.20}$$

Moreover, for $r < \infty$, the function

$$A \ni x \to \int_{\mathbb{R}} |\lambda|^r\, d_\lambda f(x,\lambda) \in \mathbb{R}$$

belongs to $L^1(A)$.

(ii) *If $G(x,\cdot)$ is convex and the sequence g_n converges weakly (star-weakly for $r = \infty$) to $g \in L^r(A)$, then $\overline{G}(x) \leq G(x, g(x))$. If the functions g_n satisfy the inequalities $g_n \leq M$ (resp. $g_n \geq m$), then $f(x,\lambda) = 1$ for $\lambda \geq M$ (resp. $f(x,\lambda) = 0$ for $\lambda < m$).*

(iii) *If* $f(1-f) = 0$ *a.e. in* A*, then the sequence* g_n *converges to* g *in measure, and hence in* $L^s(A)$ *for positive* $s < r$*. Moreover, in this case* $f(x,\lambda) = 0$ *for* $\lambda < g(x)$ *and* $f(x,\lambda) = 1$ *for* $\lambda \geq g(x)$.

5. Boundary value problems for transport equations

The first-order scalar differential equation

$$\mathbf{u} \cdot \nabla \varrho + c\varrho = f, \tag{5.1}$$

which is called the transport equation, is one of the basic equations of mathematical physics. It is widely used for mathematical modeling of the mass and heat transfer and plays an important role in the kinetic theory of such phenomena. In the frame of theory of the compressible Navier–Stokes equations the most important examples of transport equations are: the steady mass balance equation,

$$\operatorname{div}(\varrho \mathbf{u}) = 0, \tag{5.2}$$

and “relaxed” mass balance equation

$$\operatorname{div}(\varphi \mathbf{u}) + \alpha\varphi = h, \tag{5.3}$$

where a given vector field $\mathbf{u}$ belongs to the Sobolev space $H^{1,2}(\Omega)$ and satisfies the boundary condition

$$\mathbf{u} = \mathbf{U} \quad \text{on } \partial\Omega. \tag{5.4}$$

The typical boundary value problem for the transport equation can be formulated as follows. Split the boundary of Ω into three disjoint parts: inlet Σ_{in}, outgoing set Σ_{out}, and the characteristic set Σ^0 defined by inequalities (2.8). The boundary value problem is to find a solution to differential equation (5.1) which takes the prescribed value φ_b at the inlet Σ_{in}.

Suppose for a moment that the vector field $\mathbf{u}$ has continuous derivatives and does not vanish in Ω. If a C^1 curve $l : x = x(s)$, $0 \leq s \leq s^*$ is the integral line of $\mathbf{u}$, i.e., solution of ODE

$$\frac{dx}{ds} = \mathbf{u}(x),$$

then equation (5.1) can be rewritten as the ordinary differential equation

$$\frac{d\varphi(x(s))}{ds} + c(x(s))\varphi(x(s)) = h(x(s))$$

along the line l. If in addition for each point $x^* \in \Omega$, there is a unique integral line such that $x(0) \in \Sigma_{\text{in}}$ and $x(s^*) = x^*$, then any solution of the transport equation is completely defined by the boundary data. Therefore we can apply the classical method of characteristics for solving (5.1). Note that if inlet Σ^+ is not an isolated component of $\partial\Omega$ ($\operatorname{cl} \Sigma^+ \cap \operatorname{cl} (\partial\Omega \setminus \Sigma^+) \neq \emptyset$), then in the general case a solution to the boundary value problem for transport equation is not smooth. Moreover it is easy to construct an example of plane domain Ω such that a solution of simplest transport equation $\partial_{x_1}\varrho = 1$ with zero boundary data has a jump of discontinuity at $\mathrm{cl}\,\Sigma_{\mathrm{in}} \cap \mathrm{cl}\,\Sigma_{\mathrm{out}}$.

The method of characteristics does not work if the totality of integral lines has a complicated structure, for example if $\mathbf{u}$ has rest points within Ω, and when the velocity field is not smooth and therefore integral lines are not well defined.

To address the first issue note that the theory of linear transport equations is an integral part of the general theory of elliptic–parabolic equations known also as the theory of second-order equations with nonnegative quadratic form. Such a theory deals with the general second-order partial differential equation

$$-\sum_{i,j=1}^{n} a_{ij}\partial_{x_i}\partial_{x_j}\varphi + \mathbf{v}\cdot\nabla\varphi + c\varphi = h \tag{5.5}$$

under the assumption

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq 0 \quad \text{for all } x\in\Omega \quad \text{and} \quad \xi\in\mathbb{R}^n.$$

Boundary value problems for second-order PDE with nonnegative quadratic form were studied by many authors starting from the pioneering papers of Fichera [15], Kohn and Nirenberg [26], and Oleinik and Radkevich [47]. In this paragraph we give a short review of available results.

Weak solutions. The first result on the existence of weak solutions to the general equation (5.5) is due to Fichera. Note that the definition of weak solutions proposed by Fichera is not standard since the used set of test functions does not admit the algorithmic description. The complete theory of integrable weak solutions to elliptic–parabolic equations was developed by Oleinik and Radkevich. The following theorem on existence and uniqueness of weak solutions to boundary value problems

$$\varphi = 0 \quad \text{on } \partial\Omega \tag{5.6}$$

for equation (5.1) is a particular case of the Oleinik results, we refer to Theorems 1.5.1 and 1.6.2 in [47].

Recall that a function $\varphi\in L^1(\Omega)$ is a weak solution to problem (5.1), (5.6) if the integral identity

$$\int_\Omega (\varphi\mathcal{L}^*\zeta - f\zeta)\,dx = 0 \tag{5.7}$$

holds true for all test functions $\zeta\in C^1(\Omega)$ vanishing on Σ_{out}. Here the adjoint operator $\mathcal{L}^*$ are defined by

$$\mathcal{L}^*\varphi := -\operatorname{div}(\mathbf{u}\varphi) + c\varphi. \tag{5.8}$$

THEOREM 5.1. *Let Ω be a bounded domain with the boundary of a class C^2 and $1 < r \le \infty$. Assume that the vector field $\mathbf{u} \in C^1(\Omega)$ and a function $c \in C(\Omega)$ satisfy the condition*

$$\delta = \inf_{x\in\Omega} (c(x) - r^{-1} \operatorname{div} \mathbf{u}(x)) > 0.$$

Then problem (5.1), (5.6) *has a weak solution $\varphi \in L^r(\Omega)$ satisfying the inequality*

$$\|\varphi\|_{L^r(\Omega)} \le \delta^{-1} \|f\|_{L^r(\Omega)}.$$

If, in addition, $r > 3$ and the intersection $\Gamma = cl\,(\Sigma_{\text{out}} \cup \Sigma_0) \cap cl\,\Sigma_{\text{in}}$ is a smooth one-dimensional manifold, then a weak solution $\varphi \in L^r(\Omega)$ to problem (5.1), (5.6) *is unique.*

Moreover, in [47] it was shown that weak solutions are continuous at interior points of Σ_{in} and take the boundary value in a classic sense.

It is worthy of note that, under the assumptions of the theorem, the operator $\mathfrak{L}$ with the domain of definition $D(\mathfrak{L}) = \{\varphi \in L^r(\Omega) : \mathbf{u}\nabla\varphi \in L^r(\Omega),\ \varphi = 0 \text{ on } \Sigma_{\text{in}}\}$ is m-accretive.

Strong solutions. The question on the regularity of solutions to boundary value problems for transport equations is more difficult. All known results [26,47] related to the case of multi-connected domains with isolated inlet. We illustrate the theory by two theorems. The first is a consequence of the general result of Kohn and Nirenberg, see [26], on solvability boundary value problems for elliptic–parabolic equations.

THEOREM 5.2. *Let $\Omega \in \mathbb{R}^d$ be a bounded domain with a boundary $\partial\Omega \in C^\infty$, $\mathbf{u}, c \in C^\infty(\Omega)$, and $k \ge 1$ be an arbitrary integer. Furthermore assume that*

$$\text{cl}\ \Sigma_{\text{in}} \cap (\Sigma_{\text{out}} \cup \Sigma_0) = \emptyset, \tag{5.9}$$

and $c > c_0 > 0$, where c_0 is sufficiently large constant depending only on Ω $\|\mathbf{u}\|_{C^3(\Omega)}$, $\|c\|_{C^3(\Omega)}$, and k. Then for any $f \in H^{2,k}(\Omega)$ problem (5.1), (5.6) *has a unique solution satisfying the inequality*

$$\|\varphi\|_{H^{2,k}(\Omega)} \le C(k, \Omega, \mathbf{u}, c)\|f\|_{H^{2,k}(\Omega)}.$$

The second result is a consequence of Theorem 1.8.1 in the monograph [47].

THEOREM 5.3. *Assume that Ω is a bounded domain with the boundary of the class C^2 and $\mathbf{u}, c, f \in C^1(\mathbb{R}^d)$. Furthermore, let the following conditions hold.*

(1) *The vector field $\mathbf{U} = \mathbf{u}\big|_{\partial\Omega}$ and the manifold Ω satisfy condition* (5.9).

(2) *There is $\Omega' \Supset \Omega$ such that the inequality*

$$c(x) - \sup_{\Omega'} \left\{ |\operatorname{div} \mathbf{u}| - \frac{1}{2} \sup_i \sum_{j\neq i} \left|\frac{\partial u_i}{\partial x_j}\right| - \frac{1}{2} \sup_j \sum_{j\neq i} \left|\frac{\partial u_j}{\partial x_i}\right| \right\} > 0$$

is fulfilled. Then a weak solution to problem (5.1), (5.6) *satisfies the Lipschitz condition in* cl Ω.

If the vector field $\mathbf{u}$ satisfies the nonpermeability condition

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega,$$

then $\Sigma_{\text{in}} = \Sigma_{\text{out}} = \emptyset$ and we do not need the boundary conditions. This particular case was investigated in detail by Beirao Da Veiga in [9] and Novotny in [45], [46]. The following theorem is due to Berirao da Veiga

THEOREM 5.4. *Let $1 \le l \le k$ be arbitrary integers, $r \in (1, \infty)$, and $\Omega \subset \mathbb{R}^d$ be a bounded domain with the boundary $\partial\Omega \in C^k$. Let*

$$\mathbf{u}, c \in C^k(\Omega), \quad f \in H^{k,r}(\Omega) \cap H_0^{l,r}(\Omega), \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

Then there exists a constant c_Ω depending only on k,r, and Ω such that for all

$$\sigma > \sigma^* \equiv c_\Omega(\|\mathbf{u}\|_{C^k(\Omega)} + \|c\|_{C^k(\Omega)})$$

the equation

$$\mathcal{L}\varphi + \sigma\varphi \equiv \mathbf{u}\nabla\varphi + c(x)\varphi + \sigma\varphi = f$$

has a unique solution $\varphi \in H^{k,r}(\Omega)$ satisfying the inequality

$$\|\varphi\|_{H^{k,r}(\Omega)} \le (\sigma - \sigma^*)^{-1} \|f\|_{H^{k,r}(\Omega)}.$$

In [45] these results were extended to a broad class of domains including $\mathbb{R}^d$, $\mathbb{R}^d_+$, and exterior domains with compact complements. The case of noninteger k was covered in [46].

5.1. *Strong solutions. General case*

As was mentioned above, there are no results on regularity of solutions to transport equations in the general case of nonempty intersections of inlet and outgoing set. In this paragraph we formulate the theorem which partially fills this gap. With application to compressible Navier–Stokes equations in mind we restrict our considerations by the case when $\Omega \subset \mathbb{R}^3$ is a bounded domain and

$$\Gamma := \text{cl } \Sigma_{\text{in}} \cap \text{cl } \Sigma_{\text{out}} = \text{cl } \Sigma_{\text{in}} \cap \text{cl } \Sigma_0 \tag{5.10}$$

is a smooth one-dimensional manifold. For simplicity we shall assume that $\partial\Omega \in C^\infty$, $c(x) = \sigma = \text{const.}$, and consider the boundary value problem

$$\mathcal{L}\varphi := \mathbf{u}\nabla\varphi + \sigma\varphi = f \quad \text{in } \Omega, \quad \varphi = 0 \quad \text{on } \Sigma_{\text{in}}. \tag{5.11}$$

The main difficulty of the problem is that the smoothness properties of solutions are very sensitive to the behavior of the vector field $\mathbf{u}$ near the characteristic manifold Γ. Further we shall assume that a characteristic manifold and a vector field $\mathbf{U} = \mathbf{u}|_{\partial\Omega}$ satisfy the following conditions, referred to as the *emergent vector field conditions*.

(**H1**) The boundary of Ω belongs to class C^∞. For each point $P \in \Gamma$ there exists the local Cartesian coordinates (x_1, x_2, x_3) with the origin at P such that in the new coordinates $\mathbf{U}(P) = (U, 0, 0)$ with $U = |\mathbf{U}(P)|$, and $\mathbf{n}(P) = (0, 0, -1)$. Moreover, there is a neighborhood $\mathcal{O} = [-k, k]^2 \times [-t, t]$ of P such that the intersections $\Sigma \cap \mathcal{O}$ and $\Gamma \cap \mathcal{O}$ are defined by the equations

$$F_0(x) \equiv x_3 - F(x_1, x_2) = 0, \quad \nabla F_0(x) \cdot \mathbf{U}(x) = 0,$$

and $\Omega \cap \mathcal{O}$ is the epigraph $\{F_0 > 0\} \cap \mathcal{O}$. The function F satisfies the conditions

$$\|F\|_{C^2([-k,k]^2)} \le K, \quad F(0,0) = 0, \quad \nabla F(0,0) = 0, \tag{5.12}$$

where the constants $k, t < 1$ and $K > 1$ depend only on the curvature of Σ and are independent of the point P.

(**H2**) The manifold $\Gamma \cap \mathcal{O}$ admits the parameterization

$$x = \mathbf{x}^0(x_2) := (\Upsilon(x_2), x_2, F(\Upsilon(x_2), x_2)), \tag{5.13}$$

such that $\Upsilon(0) = 0$ and $\|\Upsilon\|_{C^2([-k,k])} \le C_\Gamma$, where the constant $C_\Gamma > 1$ depends only on Σ and $\mathbf{U}$.

(**H3**) There are positive constants $N^\pm$ independent of P such that for $x \in \Sigma \cap \mathcal{O}$ we have

$$\begin{aligned} N^-(x_1 - \Upsilon(x_2)) &\le -\nabla F_0(x) \cdot \mathbf{U}(x) \\ &\le N^+(x_1 - \Upsilon(x_2)) \quad \text{for } x_1 > \Upsilon(x_2), \\ -N^-(x_1 - \Upsilon(x_2)) &\le \nabla F_0(x) \cdot \mathbf{U}(x) \\ &\le -N^+(x_1 - \Upsilon(x_2)) \quad \text{for } x_1 < \Upsilon(x_2). \end{aligned} \tag{5.14}$$

These conditions are obviously fulfilled for all strictly convex domains and constant vector fields. They have simple geometric interpretation, that $\mathbf{U} \cdot \mathbf{n}$ only vanishes up to the first order at Γ, and for each point $P \in \Gamma$, the vector $\mathbf{U}(P)$ points to the part of Σ where $\mathbf{U}$ is an exterior vector field. Note that *emergent vector field* condition plays an important role in the theory of the classical oblique derivative problem, see [21]. It seems that this condition is necessary for continuity of solutions to problem (5.11).

THEOREM 5.5. *Assume that Σ and $\mathbf{U}$ satisfy conditions* (**H1**)–(**H3**), *the vector field* $\mathbf{u}$ *belongs to the class $C^1(\Omega)$, and satisfies the boundary condition*

$$\mathbf{u} = \mathbf{U} \quad \text{on } \Sigma, \quad \mathbf{u} = 0 \quad \text{on } \partial S. \tag{5.15}$$

Furthermore, let s, r and α be the constants satisfying

$$\begin{aligned} 0 < s \le 1, \quad 1 < r < \infty, \quad 2s - 3/r < 1, \\ \max\{s, 2s - 3/r\} < \alpha < \min\{2s, 1\} \quad \textit{for } 0 < s < 1, \\ \alpha = 1 - 1/r \quad \textit{for } s = 1 \quad \textit{and} \quad 1 < r < 2, \\ 2 - 3/r < \alpha < 1 \quad \textit{for } s = 1 \quad \textit{and} \quad 2 \le r < 3r < 2. \end{aligned} \tag{5.16}$$

Then there are positive constants $\sigma^* > 1$, and C depending only on Σ, $\mathbf{U}$, s, r, α, and $\|\mathbf{u}\|_{C^1(\Omega)}$ such that: for any $\sigma > \sigma^*$ and $f \in H^{s,r}(\Omega) \cap L^\infty(\Omega)$ problem (5.11) *has a unique solution $\varphi \in H^{s,r}(\Omega) \cap L^\infty(\Omega)$, which admits the estimates*

$$\|\varphi\|_{H^{s,r}(\Omega)} \le C\sigma^{-1}\|f\|_{H^{s,r}(\Omega)} + \sigma^{-1+\alpha}C\|f\|_{L^\infty(\Omega)}. \tag{5.17}$$

PROOF. The proof is given in Section 10. □

Since for $sr > 3$ the Sobolev space $H^{s,r}(\Omega)$ is the Banach algebra, Theorem 5.5 along with the contraction mapping principle implies the following result on solvability of the adjoint problem

$$\mathcal{L}^*\varphi := -\operatorname{div}(\varphi\mathbf{u}) + \sigma\varphi = f \quad \text{in } \Omega, \quad \varphi = 0 \quad \text{on } \Sigma_{\text{out}}. \tag{5.18}$$

THEOREM 5.6. *Let Σ and $\mathbf{U}$ comply with hypotheses* (**H1**)–(**H3**), *the vector field $\mathbf{u} \in C^1(\Omega)$ satisfies boundary condition* (5.15), *exponents s, r, α satisfy conditions* (5.16). *Moreover, assume that $sr > 3$ and $\operatorname{div}\mathbf{u} \in H^{s,r}(\Omega)$. Then there are positive constants $\sigma^* > 1, C$ depending only on Σ, $\mathbf{U}$, s, r, α, $\|\mathbf{u}\|_{C^1(\Omega)}$, and $\|\operatorname{div}\mathbf{u}\|_{H^{s,r}(\Omega)}$ such that: for any $\sigma > \sigma^*$ and $f \in H^{s,r}(\Omega) \cap L^\infty(\Omega)$ problem* (5.18) *has a unique solution $\varphi \in H^{s,r}(\Omega) \cap L^\infty(\Omega)$, which admits the estimates*

$$\|\varphi\|_{H^{s,r}(\Omega)} \le C\sigma^{-1}\|f\|_{H^{s,r}(\Omega)} + \sigma^{-1+\alpha}C\|f\|_{L^\infty(\Omega)}. \tag{5.19}$$

Solution to problem (5.11) is continuous if $sr > 3$, which along with (5.16) yields inequality $s < 1$. Therefore in this case φ is only a weak solution. Under the assumptions of Theorem 5.5, a solution to problem (5.11) is strong, if $s = 1$. But in this case inequalities (5.16) yield $r < 3$, which does not guarantee the continuity property. In order to obtain strong continuous solutions we introduce the Banach space $X^{s,r} = H^{s,r}(\Omega) \cap H^{1,r_0}(\Omega)$ equipped with the norm

$$\|u\|_{X^{s,r}} = \|u\|_{H^{s,r}(\Omega)} + \|u\|_{H^{1,r_0}(\Omega)}.$$

We shall assume that exponents s, r, r_0, and α satisfy conditions (5.16) and the inequalities

$$rs > 3, \quad 0 < \alpha = 1 - 1/r_0 < 1, \quad r_0 < 3. \tag{5.20}$$

It follows from the embedding theory that in this case $X^{s,r}$ is a Banach algebra. Theorems 5.5 and 5.6 imply the following result.

THEOREM 5.7. *Let Σ and $\mathbf{U}$ comply with hypotheses* (**H1**)–(**H3**), *a vector field $\mathbf{u}$ satisfies boundary condition* (5.15), *exponents s, r, r_0, α satisfy conditions* (5.16) *and* (5.20). *Moreover, assume that $\|\mathbf{u}\|_{H^{1+s,r}(\Omega)} < R < \infty$. Then there are positive constants $\sigma^* > 1$ and C depending only on Σ, $\mathbf{U}$, s, r, r_0, α, and R such that: for any $\sigma > \sigma^*$ and $f \in X^{s,r}$, each of the problems* (5.11), (5.18) *has a unique solution $\varphi \in X^{s,r}$ which admits the estimate*

$$\|\varphi\|_{X^{s,r}} \le C\sigma^{-1}\|f\|_{X^{s,r}} + \sigma^{-1+\alpha}C\|f\|_{L^\infty(\Omega)}, \tag{5.21}$$

$$\|\varphi\|_{C(\Omega)} \le (\sigma - cR)^{-1}\|f\|_{C(\Omega)}. \tag{5.22}$$

6. Transport equations with discontinuous coefficient

The theory of linear transport equations with discontinuous coefficients originated in the pioneering paper [29] by Di-Perna and Lions. The key point of the theory is the concept of *renormalized* solutions as a new class of generalized solutions to linear problems which play a similar role as the Kruzhkov entropy solutions in the theory of scalar nonlinear conservation laws. The main result obtained in [29] is the existence and uniqueness of renormalized solutions to the Cauchy problem for transport equations generated by vector fields with integrable derivatives and bounded divergence. Note that the Cauchy problem for nonstationary transport equations is a particular case of boundary value problem (5.1), (5.6) with the domain $\Omega = \mathbb{R}^{d+1}$ the vector field $\mathbf{u} = (1, u_1, \dots, u_d)$). Nowadays we have the developed theory of the Cauchy problem for transport equations with discontinuous coefficients, see [5] and [28] for the overview. But all available results related to transport equations generated by vector fields $\mathbf{u}$ with $\operatorname{div}\mathbf{u}$ bounded from below, while the theory of compressible Navier–Stokes equations operates with vector fields whose divergence only is integrable with square. In this section we prove the compactness of a totality renormalized solution's relaxed mass balance equation

$$\operatorname{div}(\mathbf{u}\varrho) = h \in L^r(\Omega) \quad \text{in } \Omega \tag{6.1}$$

with a vector field $\mathbf{u}$ satisfying the conditions

$$\mathbf{u} \in H_0^{1,2}(\Omega). \tag{6.2}$$

Before formulation of the results we recall some basic facts and definitions.

6.1. *Renormalized solutions*

We begin with the definition of renormalized solution to equation (6.1).

DEFINITION 6.1. For a given $\mathbf{u} \in H^{1,2}(\Omega)$ satisfying conditions (6.2) and $h \in L^1(\Omega)$, a renormalized solution to equation (6.1), is a function $\varrho \in L^1(\Omega)$, with $\varrho\mathbf{v} \in L^1(\Omega)$ such that the integral identity

$$\int_\Omega (G(\varrho)\mathbf{u} \cdot \nabla\psi + (G(\varrho) - G'(\varrho)\rho)\psi \operatorname{div}\mathbf{u})\, dx + \int_\Omega G'(\varrho)h(x)\psi\, dx = 0 \tag{6.3}$$

holds for any functions $\psi \in C^1(\Omega)$, and any function $G \in C^1_{\mathrm{loc}}[0,\infty)$ with the properties

$$\limsup_{|s|\to\infty} |G'(s)| < \infty,$$
$$[0,\infty) \ni s \mapsto G(s) - G'(s)s \in \mathbb{R} \quad \text{is continuous and bounded.}$$

The generic property of equation (6.1) is extendability of renormalized solutions through $\partial\Omega$ onto $\mathbb{R}^3$. Define the extensions of the vector field $\mathbf{u}(x)$ and the function h onto $\mathbb{R}^3$ by the equalities,

$$\mathbf{u}(x) = 0, \quad h(x) = 0 \quad \text{for } x \in \mathbb{R}^3 \setminus \Omega. \tag{6.4}$$

LEMMA 6.2. *Let ϱ be a renormalized solution to* (6.1). *Then the extended functions serves as renormalized solution to equation* (6.1) *in* $\mathbb{R}^3$.

Proof obviously follows from Definition 6.1. □

It is clear that each renormalized solution to transport equation is a weak solution. The inverse is also true if, for instance $\varrho \in L^2(\Omega)$. By virtue of the extension principle it is sufficient to prove this fact in the case when $\Omega = \mathbb{R}^3$ and $\mathbf{u}$, h and ϱ are compactly supported in $\mathbb{R}^3$. Fix an arbitrary point $x_0 \in \mathbb{R}^3$. Denote by Θ the mollification kernel in $\mathbb{R}^3$ and recall formula (4.16). Substituting $\psi(x) = m^n \Theta(m(x_0 - x))$ into integral identity (6.3) with $G(\varrho) = \varrho$ leads to the equality

$$\operatorname{div}[\varrho\mathbf{u}]_{m,}(x_0) = [h]_{m,}(x_0),$$

which holds true for all $x_0 \in \mathbb{R}^3$. Now fix an arbitrary function G satisfying (6.4). Multiplying both the sides of the last identity by $G'([\varrho]_{m,})$ and noting that the function $[\varrho]_{m,}$ has continuous derivatives of all orders we arrive at the equality

$$\begin{aligned}&\operatorname{div}(G([\varrho]_{m,})\mathbf{u}) + (G'([\varrho]_{m,})[\varrho]_{m,} - G([\varrho]_{m,})\operatorname{div}\mathbf{u}\\&\quad + G'([\varrho]_{m,})(\mathfrak{r}^m - [h]_{m,}) = 0,\end{aligned}$$

where

$$\mathfrak{r}^m = \operatorname{div}[\varrho\mathbf{u}]_{m,} - \operatorname{div}([\varrho]_{m,}\mathbf{u}).$$

Multiplying both the sides by a function $\psi \in C_0^\infty(\mathbb{R}^3)$ and integrating the result over $\mathbb{R}^3$ we conclude that

$$\begin{aligned}&\int_{\mathbb{R}^3} (G([\varrho]_{m,})\mathbf{u}\cdot\nabla\psi - (G'([\varrho]_{m,})[\varrho]_{m,} - G([\varrho]_{m,})\operatorname{div}\mathbf{v}\psi)\,dx\\&\quad + \int_{\mathbb{R}^3} G'([\varrho]_{m,})([h]_{m,} - \mathfrak{r}^m)\psi\,dx = 0.\end{aligned}$$

Since ψ is compactly supported, Lemma 4.5 yields

$$\lim_{m\to\infty}\int_{\mathbb{R}^3} \mathfrak{r}^m\psi\,dx = 0.$$

Letting $m \to \infty$ and noting that $[\varrho]_{m,}$ tends to ϱ in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$ we conclude that ϱ satisfies integral identity (6.3). Therefore, ϱ is the renormalized solution to equation (6.1) which is the desired conclusion.

6.2. *Kinetic formulation of a transport equation*

If for a differential equation, linear or nonlinear, the renormalization procedure can be performed, then the equation can be equivalently rewritten as a linear differential equation for the so-called *distribution function* in the extended space of (x, λ), where λ is the extra *kinetic variable*. This fact underlies the *kinetic equation method*, which is one of the most powerful methods in the modern PDE theory. The method of kinetic equations has been created and applied recently to study a wide range of problems, for example, to study the equations of isentropic gas dynamics and p-systems and the first- and second-order quasilinear conservation laws [11,30,31]. In this paragraph we explain how the kinetic formulation of transport equations can be obtained from the definition of renormalized solution.

Assume that a renormalized solution to (6.1) meets all requirements of Definition 6.1. For each $(x, \lambda) \in \Omega \times \mathbb{R}$, define the distribution function $f(x, \lambda)$ by the equalities

$$f(x, \lambda) = 0 \quad \text{for } \lambda < \varrho(x), \qquad f(x, \lambda) = 1 \quad \text{otherwise.} \tag{6.5}$$

Obviously, for a.e. $x \in \Omega$, the distribution function $f(x, \cdot)$ is monotone and continuous from the right with respect to λ, and the associated Stieltjes measure is given by

$$d_\lambda f(x, \cdot) = \partial_\lambda f(x, \lambda) = \delta(\lambda - \varrho(x)),$$

where $\delta(\lambda - \varrho(x))$ is the Dirac measure concentrated at $\varrho(x)$. In particular, for each continuous function $G : \mathbb{R} \mapsto \mathbb{R}$,

$$G(\varrho(x)) = \int_{\mathbb{R}} G(\lambda) d_\lambda f(x, \lambda) \quad \text{a.e. in } \Omega.$$

LEMMA 6.3. *Under the above assumptions the function $f(x, \lambda)$ satisfies the extended transport equation* (6.6)

$$\Omega : \ \operatorname{div}(f(x, \lambda)\mathbf{V}(x, \lambda)) + h(x) f(x, \lambda) = 0, \tag{6.6}$$

which is understood in the sense of distributions. Here, the velocity field $\mathbf{V} : \Omega \times \mathbb{R} \mapsto \mathbb{R}^4$ is defined by

$$\mathbf{V}(x, \lambda) = (\mathbf{u}(x), -\lambda \operatorname{div} \mathbf{u}(x)). \tag{6.7}$$

PROOF. Choose an arbitrary $\psi \in C^\infty(\Omega)$ vanishing near $\partial D \setminus \Sigma^+$ and a function $\eta \in C^\infty(\mathbb{R})$. Set

$$G(\lambda) = \int_\lambda^\infty \eta(s)\, ds.$$

Note that

$$\begin{aligned} G(\varrho(x)) &= \int_{\mathbb{R}} \left(\int_s^\infty \eta(\lambda) d\lambda \right) d_s f(x, s) \\ &= \int_{\mathbb{R}} \eta(\lambda) \left(\int_{(-\infty, \lambda]} d_s f(x, s) \right) d\lambda \\ &= \int_{\mathbb{R}} \eta(\lambda) f(x, \lambda)\, d\lambda. \end{aligned}$$

Repeating these arguments gives

$$\begin{aligned} G(\varrho) - G'(\varrho)\varrho &= \int_{\mathbb{R}} \eta(\lambda) f(x, \lambda)\, d\lambda + \int_{\mathbb{R}} \lambda \eta(\lambda) d_\lambda f(x, \lambda) \\ &= -\int_{\mathbb{R}} \lambda f(x, \lambda) \eta'(\lambda)\, d\lambda, \end{aligned}$$

$$G(\varrho)a(x,\varrho) = \int_{\mathbb{R}} \eta(\lambda)a(x,\lambda)f(x,\lambda)\,d\lambda - \int_{\mathbb{R}} \eta(\lambda)\left(\int_{-\infty}^{\lambda} a(x,s)f(x,s)\,ds\right)d\lambda.$$

Substituting these inequalities into integral identity (6.3) we arrive at

$$\int_{\Omega\times\mathbb{R}} f(x,\lambda)(\mathbf{u}(x)\cdot\nabla_x\psi\eta - \lambda\,\mathrm{div}\,\mathbf{u}(x)\psi\partial_\lambda\eta + \alpha(h(x)-a(x,\lambda))\psi\eta)\,d\lambda\,dx + \int_{\Omega\times\mathbb{R}} \psi\eta(\lambda)\left(\int_{-\infty}^{\lambda} \partial_s a(x,s)f(x,s)\,ds\right)d\lambda\,dx + \int_{\Sigma^+\times\mathbb{R}} f_\infty(\lambda)\psi\eta(\lambda)\mathbf{U}_\infty\cdot\mathbf{n}\,d\lambda\,d\Sigma.$$

Noting that an arbitrary function $\zeta\in C^\infty(\Omega\times\mathbb{R})$ vanishing for all sufficiently large λ can be approximate in the norm $C^1(\Omega\times\mathbb{R})$ by linear combinations of functions $\psi(x)\eta(\lambda)$ we arrive at the integral identity

$$\int_{\Omega\times\mathbb{R}} f(x,\lambda)(\mathbf{u}(x)\cdot\nabla_x\zeta - \lambda\,\mathrm{div}\,\mathbf{u}(x)\partial_\lambda\zeta + \alpha(h(x)-a(x,\lambda))\zeta)\,d\lambda\,dx + \int_{\Omega\times\mathbb{R}} \zeta(x,\lambda)\left(\int_{-\infty}^{\lambda} \partial_s a(x,s)f(x,s)\,ds\right)d\lambda\,dx + \int_{\Sigma^+\times\mathbb{R}} f_\infty(\lambda)\zeta(x,\lambda)\mathbf{U}_\infty\cdot\mathbf{n}\,d\lambda\,d\Sigma,$$

which yields (6.6). □

The preferences of the kinetic formulation of the equation are:

- kinetic equation (6.6) is a linear equation in any case;
- the unknown function is a priori uniformly bounded and monotone with respect to kinetic variable;
- the velocity field $\mathbf{V}$ is divergence-free i.e., $\mathrm{div}\,\mathbf{V} = 0$ in any case and the kinetic transport equation is a *Liouville*-type equation.

6.3. *Compactness of solutions to mass conservation equation*

In this paragraph we prove the main theorem on compactness of renormalized solutions to equation (6.1). Assume that sequences of a vector fields $\mathbf{u}_n\in H^{1,2}(\mathbb{R}^3)$ and nonnegative functions $\varrho_n:\mathbb{R}^3\mapsto\mathbb{R}^+$ satisfy the following conditions.

(H4) Vector fields $\mathbf{u}_n$ and functions ϱ_n vanish outside of bounded domain $\Omega\Subset\mathbb{R}^3$. There exist positive κ and c such that for all $n\geq 1$,

$$\|\mathbf{u}_n\|_{H^{1,2}(\mathbb{R}^3)}\leq c, \qquad \int_{\mathbb{R}^3} p_n\,dx + \int_{\mathbb{R}^3} |\varrho_n\mathbf{u}_n|^{1+\kappa}\,dx\leq c.$$

Here the functions $p_n = p(\varrho_n)$ are defined by the equalities

$$p_n(\varrho) = \varrho_n^\gamma + \varepsilon_n \varrho_n^k, \tag{6.8}$$

in which $1 < \gamma < k$ are given constants and a sequence of positive numbers ε_n tends to zero as $n \to \infty$. Moreover, for each compact $\Omega' \Subset \Omega$,

$$\int_{\Omega'} p_n^{1+\kappa}\, dx + \int_{\Omega'} (\varrho_n |\mathbf{u}_n|^2)^{1+\kappa}\, dx \le c(\Omega'),$$

where $c(\Omega')$ does not depend on n.

(**H5**) The vector fields $\mathbf{u}_n$ converges weakly in $H^{1,2}(\mathbb{R}^3)$ to a vector field $\mathbf{u}$. For each compact $E \subset \mathbb{R}^3$ and an arbitrary function $G : \mathbb{R} \to \mathbb{R}$ satisfying the condition $\lim_{\varrho\to\infty} \varrho^{-\gamma} G(\varrho) = 0$, the functions $G(\varrho_n))$ converges weakly in $L^1(E)$ to a function $\overline{G} \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$. Moreover, if G satisfies the more weak condition $\limsup_{\varrho\to\infty} \varrho^{-\gamma}|G(\varrho)| < \infty$, then the sequence $G(\varrho_n))$ converges weakly in $L^1(\Omega')$ the function $\overline{G}$ in any subdomain $\Omega' \Subset \Omega$. In particular,

$$\varrho_n^\gamma \to \overline{p} \in L^1(\Omega) \cap L^{1+\kappa}_{\mathrm{loc}}(\Omega) \quad \text{weakly in } L^1(\Omega') \quad \text{for all } \Omega' \Subset \Omega.$$

(**H6**) There exists a function $h \in C_0(\mathbb{R}^3)$ such that the limiting relation

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} (G(\varrho_n)\mathbf{u}_n \cdot \nabla\psi + (G(\varrho_n) \\ - G'(\varrho_n)\varrho_n)\psi \operatorname{div} \mathbf{u}_n + G'(\varrho_n) h\psi)\, dx = 0 \tag{6.9}$$

holds true for any function $G \in C^2(\mathbb{R})$ with $\lim_{\varrho\to\infty} G''(\varrho) \to 0$, and any nonnegative function $\psi \in C_0^\infty(\mathbb{R}^3)$.

(**H7**) There exists a constant $\mu \ne 0$ such that for all functions $\psi \in C_0(\Omega)$ and $g \in C(\mathbb{R})$,

$$\liminf_{n\to\infty} \int_\Omega \psi g(\varrho_n)(p(\varrho_n) - \mu \operatorname{div} \mathbf{v}_n)\, dx = \int_\Omega \psi \overline{g}(\overline{p} - \mu \operatorname{div} \mathbf{v})\, dx.$$

THEOREM 6.4. *If sequences $\mathbf{u}_n$, ϱ_n satisfy conditions* (**H4**)–(**H7**), *then $\varrho_n \to \varrho$ strongly in $L^r_{\mathrm{loc}}(\mathbb{R}^3)$ for any $r \in [1, 1+\kappa)$.*

The rest of this section is devoted to the proof of Theorem 6.4. We begin with the following lemma which describes the properties of the Young measure associated with the sequence of ϱ_n.

LEMMA 6.5. *Under the assumptions of Theorem 6.4 there exists a distribution function $f : \mathbb{R}^n \times \mathbb{R} \mapsto [0, 1]$ such that:*

(i)

$$f(x,\lambda) = 0 \quad \text{for } \lambda < 0 \quad \text{a.e. in } \mathbb{R}^3,$$

$$f(x,\lambda) = 1 \quad \text{for } \lambda \ge 0 \quad \text{a.e. in } \mathbb{R}^3 \setminus \Omega.$$

(ii) *f meets all requirements of Lemma 4.7. For any bounded set $E \subset \mathbb{R}^3$ and a continuous function $G : \mathbb{R}^3 \times \mathbb{R}$ with $\lim_{\rho\to\infty} \rho^{-\gamma}\|G(\cdot,\rho)\|_{C(E)} = 0$, the sequence $G(\cdot, \varrho_n)$ converges weakly in $L^1(A)$ to the function*

$$\overline{G}(x) = \int_{[0,\infty)} G(x,\lambda)\, d_\lambda f(x,\lambda) \quad \text{a.e. in } \mathbb{R}^n. \tag{6.10}$$

In particular, the weak limit of the sequence ϱ_n admits the representation

$$\rho(x) = \int_{[0,\infty)} \lambda\, d_\lambda f(x,\lambda) \equiv \int_{[0,\infty)} (1 - f(x,\lambda))\, d\lambda \quad \text{a.e. in } \mathbb{R}^3. \tag{6.11}$$

(iii) *There is a function* $\overline{p} \in L^1(\Omega) \cap L^{1+\kappa}_{\text{loc}}(\Omega)$, *such that for any compact* $\Omega' \Subset \Omega$, *the sequence* ϱ^γ *converges weakly to* $\overline{p}$ *in* $L^{1+\kappa}_{\text{loc}}(\Omega')$. *The function* $\overline{p}$ *vanishes outside* Ω *and admits the representation*

$$\overline{p}(x) = \int_{[0,\infty)} \lambda^\gamma \, d_\lambda f(x,\lambda) \equiv \gamma \int_{[0,\infty)} \lambda^{\gamma-1}(1 - f(x,\lambda))\, d\lambda. \tag{6.12}$$

PROOF. Assertions (i)–(ii) are the obvious consequence of Lemma 4.7 and conditions (**H4**)–(**H5**). Recall that the sequence $\{p(\rho_n)\}$ is bounded in $L^{1+\kappa}(E)$ for any compact $E \Subset \mathbb{R}^3 \setminus \partial\Omega$. Hence the function $\overline{p}$ is well defined and belongs to the class $L^{1+\kappa}_{\text{loc}}(\mathbb{R}^3 \setminus \partial\Omega)$. Since $\|\overline{p}\|_{L^1(K)} \le \liminf_{n\to\infty} \|p(\rho_n)\|_{L^1(K)} \le c$, the function $\overline{p}$ is integrable in $\mathbb{R}^3$. □

6.4. *Kinetic equation*

In this paragraph we deduce the kinetic equation for the distribution function $f(x,\lambda)$ pointed out in Lemma 6.5.

LEMMA 6.6. *Under the assumption 6.4 the distribution function* $f(x,\lambda)$ *satisfies the equation*

$$\operatorname{div}(f\mathbf{V}) + \partial_\lambda(\lambda\mathcal{M}) + h\partial_\lambda f = 0 \quad \text{in } D'(\mathbb{R}^4). \tag{6.13}$$

Here bilinear operator $\mathcal{M}$ *and the divergence-free vector field* $\mathbf{V} : \mathbb{R}^4 \mapsto \mathbb{R}^4$ *defined by the equalities*

$$\begin{aligned}\mathcal{M}(x,\lambda) &= -\frac{1}{\mu}\int_{(-\infty,\lambda)} (s^\gamma - \overline{p})\, d_s f(x,s)\\ &= \frac{1}{\mu}\int_{[\lambda,\infty)} (s^\gamma - \overline{p})\, d_s f(x,s),\\ \mathbf{V}(x,\lambda) &= (\mathbf{u}(x), -\lambda \operatorname{div}\mathbf{u}(x)).\end{aligned} \tag{6.14}$$

PROOF. Choose arbitrary functions $\psi \in C_0^\infty(\mathbb{R}^3)$, $\eta \in C_0^\infty(\mathbb{R})$ and set $G(\lambda) = \int_\lambda^\infty \eta(s)\, ds$. Substituting G and ψ into (6.9) we obtain

$$\begin{aligned}&\int_{\mathbb{R}^3} (\overline{G}\mathbf{v}\cdot\nabla\psi + \overline{G'}h\psi)\, dx\\ &\qquad + \lim_{n\to\infty}\int_{\mathbb{R}^3} (G(\varrho_n) - G'(\varrho_n)\varrho_n)\psi \operatorname{div}\mathbf{v}_n\, dx = 0.\end{aligned} \tag{6.15}$$

Since the function G is continuous and vanishes near ∞, it follows from Lemma 6.5 that

$$\begin{aligned}\overline{G}(x) &= \int_{\mathbb{R}} \left(\int_s^\infty \eta(\lambda) d\lambda\right) d_s f(x,s) = \int_{\mathbb{R}} \eta(\lambda) f(x,\lambda)\, d\lambda,\\ \overline{G'}(x) &= -\int_{\mathbb{R}} \eta(\lambda) d_\lambda f(x,\lambda).\end{aligned}$$

Substituting these relations into (7.22) we arrive at the integral identity

$$\int_{\mathbb{R}^4} f(x,\lambda)\eta(\lambda)\mathbf{v}\cdot\nabla\psi\,dxd\lambda - \int_{\mathbb{R}^4} h(x))\eta\psi\,d_\lambda f(x,\lambda)\,dx$$
$$+\lim_{n\to\infty}\int_{\mathbb{R}^3}(G(\varrho_n)-G'(\varrho_n)\varrho_n)\psi\,\operatorname{div}\mathbf{u}_n\,dx = 0. \tag{6.16}$$

Denote by O_δ δ-neighborhood of $\partial\Omega$. Since the sequence $(G-G'\varrho_n)\operatorname{div}\mathbf{u}_n$ is bounded $L^2(\mathbb{R}^3)$, we can apply Condition (**H7**) to obtain

$$\begin{aligned}
&\lim_{n\to\infty}\int_{\mathbb{R}^3}(G(\varrho_n)-G'(\varrho_n)\varrho_n)\psi\,\operatorname{div}\mathbf{u}_n\,dx\\
&\quad=\lim_{\delta\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^3\setminus O_\delta}(G(\varrho_n)-G'(\varrho_n)\varrho_n)\psi\,\operatorname{div}\mathbf{u}_n\,dx\\
&\quad=\lim_{\delta\to 0}\int_{\mathbb{R}_3\setminus O_\delta}\overline{(G-G'\varrho)}\,\operatorname{div}\mathbf{u}\psi\,dx\\
&\qquad+\frac{1}{\mu}\lim_{\delta\to 0}\int_{\mathbb{R}_3\setminus O_\delta}(\overline{(G-G'\varrho)p}-\overline{(G-G'\varrho)}\overline{p})\,dx\\
&\quad=\int_{\mathbb{R}_3}\overline{(G-G'\varrho)}\,\operatorname{div}\mathbf{u}\psi\,dx+\frac{1}{\mu}\int_{\mathbb{R}_3}(\overline{(G-G'\varrho)p}-\overline{(G-G'\varrho)}\overline{p}).
\end{aligned}\tag{6.17}$$

Convergence of the integrals in this relation follows from integrability of the function λ^γ with respect to the measure $d_\lambda f\,dx$. Thus we get

$$\int_{\mathbb{R}^3}\psi(\overline{(G-G'\varrho)p}-\overline{(G-G'\varrho)}\overline{p})dx=\int_{\mathbb{R}^4}\psi\eta\lambda(\lambda^\gamma-\overline{p})d_\lambda f(x,\lambda)dx$$
$$+\int_{\mathbb{R}^4}\psi\left\{\int_\lambda^\infty\eta(s)ds\right\}(\lambda^\gamma-\overline{p})d_\lambda f(x,\lambda)dx=-\mu\int_{\mathbb{R}^4}\psi\eta\,d_\lambda(\lambda\mathcal{M}(x,\lambda))dx,$$

which leads to the limiting relation

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}(G(\varrho_n)-G'(\varrho_n)\varrho_n)\operatorname{div}\mathbf{u}_n\psi\,dx$$
$$=-\int_{\mathbb{R}^3}\operatorname{div}\mathbf{v}f(x,\lambda)\lambda\eta'(\lambda)\psi\,dx+\int_{\mathbb{R}^4}\psi\eta'(\lambda)(\lambda\mathcal{M}(x,\lambda))\,dxd\lambda.$$

Substituting this result into (6.16) gives the integral identity

$$\int_{\mathbb{R}^4} f\mathbf{V}\cdot\nabla_{x,\lambda}(\eta\psi)\,dxd\lambda-\int_{\mathbb{R}^4}h\eta\psi\,d_\lambda f\,dx+\int_{\mathbb{R}^4}(\psi\eta')(\lambda\mathcal{M})\,dxd\lambda=0,$$

which is equivalent to equation (6.13). □

Our next task is to justify the renormalization procedure for kinetic equation (6.13). To this end introduce the concave function

$$\Psi(f):=f(1-f), \tag{6.18}$$

and note that Theorem 6.4 will be proved if we show that $\Psi(f)=0$. The following lemma gives the kinetic equation for the function $\Psi(f)$.

THEOREM 6.7. *Under the assumptions of Theorem 6.4 the function* $\Psi(f)$ *satisfies the equation*

$$\operatorname*{div}_{x,\lambda}(\Psi\mathbf{V}) + 2\lambda\mathfrak{M}\partial_\lambda f - \partial_\lambda(\Psi'(f)\lambda\mathfrak{M}) + \partial_\lambda h\Psi(f) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^4), \tag{6.19}$$

in which the function $\mathfrak{M}$ *is defined by*

$$\mathfrak{M}(x, .\lambda) = \frac{1}{2}\lim_{s\to\lambda+0}\mathcal{M}(s) + \frac{1}{2}\lim_{s\to\lambda-0}\mathcal{M}(s). \tag{6.20}$$

PROOF. The technical difficulty which arises in the proof of the theorem concerns the integration of discontinuous function with respect to a Stieltjes measure. The following lemma alleviates the problem.

LEMMA 6.8. *If a function* $g : \mathbb{R} \mapsto \mathbb{R}$ *has the bounded variation, then for any* $\chi \in C_0(\mathbb{R})$,

$$\lim_{k\to\infty}[\chi[g]_{,k}]_{,k} = \lim_{k\to\infty}[\chi g]_{,k} = \chi\tilde{g},$$
$$\text{where } \tilde{g}(\lambda) = \frac{1}{2}\lim_{s\to\lambda+0} g(s) + \frac{1}{2}\lim_{s\to\lambda-0} g(s).$$

Moreover, for any $F \in C^1(\mathbb{R})$ *the function* $F'(\tilde{g})$ *is integrable with respect to the Stieltjes measure* $d_\lambda g(\lambda)$ *and*

$$\int_{\mathbb{R}}\chi(\lambda)F'(\tilde{g}(\lambda))d_\lambda g = \int_{\mathbb{R}}\chi(\lambda)d_\lambda F(g(\lambda)).$$

In particular, the function $\mathfrak{M}$ *defined by formula* (6.20) *is integrable over* $\mathbb{R}^3$ *with respect to the measure* $d_\lambda f(x,\lambda)dx$. *Recall that mollifier* $[]_{,k}$ *is defined by formula* (4.15).

PROOF. The first assertion is obvious, and the second follows from the first. Next note that for almost each $x \in \mathbb{R}^3$, the function $\mathfrak{M}(x,\cdot)$ is the pointwise limit of the sequence of the continuous functions $[\mathcal{M}(x,\cdot)]_k$ and hence is measurable with respect to any finite Borel measure. It remain to note that the nonnegative function $\mathcal{M}$ has the integrable majorant $\overline{p}$. □

Let us turn to the proof of Theorem 6.7. Applying mollifiers (4.15), (4.16) to both the sides of equation (6.13) we arrive at the equality

$$\operatorname{div}([f]_{m,k}\mathbf{V}) + \partial_\lambda[\lambda\mathcal{M}]_{m,k} + [h\partial_\lambda f]_{m,k} = I^{m,k},$$

where $I^{m,k}$ denotes the commutator $\operatorname{div}([f]_{m,k}\mathbf{V}) - \operatorname{div}[f\mathbf{V}]_{m,k}$. Multiplying both the sides by $\Psi'([f]_{m,k})$ and noting that $\operatorname{div}\mathbf{V} = 0$ we arrive at

$$\operatorname{div}(\Psi([f]_{m,k})\mathbf{V}) + \Psi'([f]_{m,k})\partial_\lambda[\lambda\mathcal{M}]_{m,k}$$
$$+ \Psi'([f]_{m,k})[h\partial_\lambda f]_{m,k} = \Psi'([f]_{m,k})I^{k,m}.$$

Next multiplying both the sides of this equality by a test function $\zeta \in C_0^\infty(\mathbb{R}^3)$ and integrating the result over $\mathbb{R}^4$ we obtain the integral identity

$$\begin{aligned}
&-\int_{\mathbb{R}^4} \Psi([f]_{m,k})\mathbf{V}\cdot\nabla_{x,\lambda}\zeta\,dx\,d\lambda + 2\int_{\mathbb{R}^4}\zeta[\lambda\mathcal{M}]_{m,k}\partial_\lambda[f]_{m,k}\,dx\,d\lambda\\
&\quad -\int_{\mathbb{R}^4}\partial_\lambda\zeta\Psi'([f]_{m,k})[\lambda\mathcal{M}]_{m,k}\,dx\,d\lambda + \int_{\mathbb{R}^4}\zeta\Psi'([f]_{m,k})[h\partial_\lambda f]_{m,k}\,dx\,d\lambda\\
&\quad = \int_{\mathbb{R}^4}\zeta\Psi'([f]_{m,k})I^{m,k}\,dx\,d\lambda.
\end{aligned}$$

Now our task is to pass to the limit as $n, k \to \infty$. To this end note that the nonnegative function $\mathcal{M}$ vanishes outside of the cylinder $\Omega\times\mathbb{R}^+$ in which it has the integrable majorant $\mathcal{M} \le \overline{p}$. Therefore, the sequence $[\lambda\mathcal{M}]_{m,k}$ converges to the function $\lambda\mathcal{M}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^4)$ as $m, k \to \infty$. From this and Lemma 4.6 we conclude that

$$\begin{aligned}
&-\int_{\mathbb{R}^4}\Psi(f)\mathbf{V}\cdot\nabla_{x,\lambda}\zeta\,dx\,d\lambda - \int_{\mathbb{R}^4}\partial_\lambda\zeta\Psi'(f)\lambda\mathcal{M}\,dx\,d\lambda\\
&\quad + \lim_{k\to\infty}\lim_{m\to\infty}\left\{\int_{\mathbb{R}^4}\zeta[\lambda\mathcal{M}]_{m,k}\partial_\lambda[f]_{m,k}\,dx\,d\lambda\right.\\
&\quad \left. + \int_{\mathbb{R}^4}\zeta\Psi'([f]_{m,k})[h\partial_\lambda f]_{m,k}\,dx\,d\lambda\right\} = 0. \qquad (6.21)
\end{aligned}$$

Note that for any positive k, the function $\partial_\lambda[f]_{,k}$ is a smooth function of the variable λ, the function $\mathcal{M} \le \overline{p}(x)$ belongs to the class $L^1(\mathbb{R}^4)$ and

$$[\lambda\mathcal{M}]_{m,k} \to [\lambda\mathcal{M}]_{,k} \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^4) \quad \text{as } m \to \infty.$$

From this we conclude that

$$\lim_{m\to\infty}\int_{\mathbb{R}^4}\zeta[\lambda\mathcal{M}]_{m,k}\partial_\lambda[f]_{m,k}\,dx\,d\lambda = \int_{\mathbb{R}^4}\left(\int_{\mathbb{R}}\zeta[\lambda\mathcal{M}]_{,k}\partial_\lambda[f]_{,k}\,d\lambda\right)dx.$$

On the other hand, for each fixed x, the function $\zeta[\lambda\mathcal{M}]_{,k}(x,\cdot)$ is compactly supported and smooth in λ which leads to the identity

$$\int_{\mathbb{R}}\zeta[\lambda\mathcal{M}]_{,k}\partial_\lambda[f]_{,k}\,d\lambda = \int_{\mathbb{R}}[\zeta[\lambda\mathcal{M}]_{,k}]_{,k}\,d_\lambda f(x,\lambda).$$

It is easy to see that the functions $[\zeta[\lambda\mathcal{M}]_{,k}]_{,k}(x,\cdot)$ have the common integrable majorant

$$|[\zeta[\lambda\mathcal{M}]_{,k}]_{,k}(x,\cdot)| \le \overline{p}(x)\sup_{\lambda}\zeta(\lambda)\sup_{\mathrm{spt}\,\eta}\{\lambda\},$$

and for each fixed x, they converge to the function $\zeta\lambda\mathfrak{M}(x,\cdot)$ everywhere in $\mathbb{R}$. From this and the Lebesgue dominant convergence theorem we conclude that for a.e. fixed $x \in \mathbb{R}^3$,

$$\lim_{k\to\infty}\int_{\mathbb{R}}\zeta[[\lambda\mathcal{M}]_{,k}]_{,k}(x,\lambda)\,d_\lambda f(x,\lambda) = \int_{\mathbb{R}}\zeta\lambda\mathfrak{M}\,d_\lambda f(x,\lambda).$$

Applying once again the Lebesgue dominant convergence theorem we finally obtain

$$\lim_{k\to\infty}\lim_{m\to\infty}\int_{\mathbb{R}^4}\zeta[\lambda\mathcal{M}]_{m,k}\partial_\lambda[f]_{m,k}\,dx\,d\lambda=\int_{\mathbb{R}^4}\zeta\lambda\mathfrak{M}\,d_\lambda f(x,\lambda)\,dx. \tag{6.22}$$

Next note that for a fixed positive k, the functions $\zeta[h\partial_\lambda f]_{,k}$ are continuous with respect to λ and uniformly bounded in $L^2(\mathbb{R}^4)$ which along with the identity $\Psi'([f]_{,k})=[\Psi'(f)]_{,k}$ yields the relation

$$\lim_{m\to\infty}\int_{\mathbb{R}^4}\zeta\Psi'([f]_{m,k})[h\partial_\lambda f]_{m,k}\,dx\,d\lambda=\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}}h[\zeta\Psi'(f)]_{,k}\,d_\lambda f\right)dx.$$

On the other hand, for a.e. x, the functions $h[\zeta[\Psi'(f)]_{,k}]_{,k}(x,\cdot)$ have the common integrable majorant $c|h|\sup_{\,\mathrm{spt}\zeta}|\lambda|$, and converge on the number axis to the function $h\zeta\Psi'(\tilde{f})(x,\cdot)$. Applying the Lebesgue dominant convergence theorem and invoking Lemma 6.8 we finally obtain

$$\lim_{k\to\infty}\lim_{m\to\infty}\int_{\mathbb{R}^4}\zeta\Psi'([f]_{m,k})[h\partial_\lambda f]_{m,k}\,dx\,d\lambda=\int_{\mathbb{R}^4}\zeta\,h\,d_\lambda\Psi(f)\,dx. \tag{6.23}$$

Finally, substituting (6.22), (6.23) into (6.21) we obtain the integral identity

$$\begin{aligned}&2\int_{\mathbb{R}^4}\zeta\lambda\mathfrak{M}\,d_\lambda f(x,\lambda)\,dx\\&\quad=-\int_{\mathbb{R}^4}\partial_\lambda\zeta\Psi'(f)\lambda\mathcal{M}\,dx\,d\lambda+\int_{\mathbb{R}^4}\partial_\lambda\zeta\,h\,\Psi(f)\,dx\,d\lambda\\&\qquad+\int_{\mathbb{R}^4}\Psi(f)\mathbf{V}\cdot\nabla_{x,\lambda}\zeta\,dx\,d\lambda\end{aligned}\tag{6.24}$$

which is equivalent to the renormalized equation (6.19). □

6.5. *The oscillation defect measure*

The notion of oscillation defect measure was introduced in [17] in order to justify the existence theory for isentropic flows with *small* values of the adiabatic constant γ. Following [17,16] the r-oscillation defect measure associated with the sequence $\{\varrho_n\}_{n\le 1}$ is defined as follows

$$\mathbf{osc}_r[\varrho_n\to\varrho](K):=\sup_{k\ge 1}\limsup_{n\to\infty}\|T_k(\varrho_n)-T_k(\varrho)\|^r_{L^r(K)},$$

where $T_k(z)=kT(z/k)$, $T(z)$ is a smooth concave function, which is equal to z for $z\le 1$ and is a constant for $z\ge 3$. The smoothness properties of T_k are not important and we can take the simplest form $T_k(z)=\min\{z,k\}$. The unexpected result was obtained by Feireisl et al. in papers [17,18], where it was shown that $(1+\gamma)$-oscillation defect measure associated with the sequence $\{\varrho_n\}$ of solutions to compressible Navier–Stokes equations is uniformly bounded on all compact subsets of Ω.

Note that in the assumptions of Theorem 6.4 we cannot replace the compact subsets $K \Subset \Omega$ by the domain Ω itself, since the oscillation defect measure is not any regular set additive function on the family of compact subsets of Ω, i.e., it is not any measure in the sense of measure theory. In order to bypass this difficulty we observe that the finiteness of the oscillation defect measure on compacts gives some additional information on the properties of the distribution function Γ. Our task is to extract this information and then to use in the proof of Theorem 6.4. In order to formulate the appropriate auxiliary result we define the function $\mathcal{T}_\theta(x)$ by the equality

$$\mathcal{T}_\theta(x) = \overline{\min\{\varrho, \theta\}}(x) - \min\{\varrho(x), \theta(x)\} \quad \text{for each } \theta \in C(\Omega).$$

LEMMA 6.9. *Under the assumptions of Theorem 6.4, there is a constant c independent of θ and K such that the inequalities*

$$\|\mathcal{T}_\theta\|^{1+\gamma}_{L^{1+\gamma}(K)} \le \lim_{n\to\infty} \int_\Omega |\min\{\varrho_n(x), \theta(x)\} - \min\{\varrho(x), \theta(x)\}|^{1+\gamma}\, dx \le c \tag{6.25}$$

hold for all $\theta \in C(\Omega)$ and $K \Subset \Omega$. We point out that the limit in (6.25) does exist by the choice of the sequence ϱ_n.

PROOF. The proof imitates the proof of Lemma 4.3 from [18]. It can be easily seen that

$$\|\mathcal{T}_\theta\|^{1+\gamma}_{L^{1+\gamma}(K)} \le \limsup_{n\to\infty} \int_K |\min\{\varrho_n(x), \theta(x)\} - \min\{\varrho(x), \theta(x)\}|^{1+\gamma}\, dx. \tag{6.26}$$

Hence, it suffices to show that the right-hand side of this inequality admits a bound independent of θ. From the properties of $\min\{\cdot, \cdot\}$ it follows that

$$\begin{aligned} &|\min\{s', \theta\} - \min\{s'', \theta\}|^{1+\gamma} \\ &\quad \le (\min\{s', \theta\} - \min\{s'', \theta\})(s'^\gamma - s''^\gamma) \quad \text{for all } s', s'' \in \mathbb{R}^+, \end{aligned}$$

furthermore, for the weak limits we have the inequalities $\overline{\varrho^\gamma} \ge \varrho^\gamma$, and $\overline{\min\{\varrho, \theta\}} \le \min\{\varrho, \theta\}$, therefore, for any compactly supported, nonnegative function $h \in C(\Omega)$, we get

$$\begin{aligned} &\lim_{n\to\infty} \int_\Omega h|\min\{\varrho_n, \theta\} - \min\{\varrho, \theta\}|^{1+\gamma}\, dx \\ &\le \lim_{n\to\infty} \int_\Omega h(\min\{\varrho_n, \theta\} - \min\{\varrho, \theta\})(\varrho_n^\gamma - \varrho^\gamma)\, dx \\ &\quad \le \lim_{n\to\infty} \int_\Omega h(\min\{\varrho_n, \theta\} - \min\{\varrho, \theta\})(\varrho_n^\gamma - \varrho^\gamma)\, dx \\ &\qquad + \int_\Omega (\overline{\varrho^\gamma} - \varrho^\gamma)(\min\{\varrho, \theta\} - \overline{\min\{\varrho, \theta\}})\, dx \\ &\quad = \lim_{n\to\infty} \int_\Omega h(\varrho_n^\gamma \min\{\varrho_n, \theta\} - \overline{\varrho^\gamma}\,\overline{\min\{\varrho, \theta\}})\, dx \\ &\quad = \lim_{n\to\infty} \int_\Omega h(p(\varrho_n) \min\{\varrho_n, \theta\} - \overline{p}\,\overline{\min\{\varrho, \theta\}})\, dx. \end{aligned} \tag{6.27}$$

By Condition (**H7**), the right-hand side of (6.27), divided by μ, is equal to

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega h(\min\{\varrho_n,\theta\}\operatorname{div}\mathbf{u}_n-\overline{\min\{\varrho,\theta\}}\operatorname{div}\mathbf{u})\,dx\\
&\quad=\lim_{n\to\infty}\int_\Omega h(\min\{\varrho_n,\theta\}-\min\{\varrho,\theta\})\operatorname{div}\mathbf{u}_n dx\\
&\qquad-\lim_{n\to\infty}\int_\Omega h(\min\{\varrho_n,\theta\}-\min\{\varrho,\theta\})\operatorname{div}\mathbf{u}\,dx\\
&\quad\le\delta\limsup_{n\to\infty}\int_\Omega h|\min\{\varrho_n,\theta\}-\min\{\varrho,\theta\}|^{1+\gamma}\,dx\\
&\qquad+\delta^{-\gamma}\lim_{n\to\infty}\int_\Omega h(|\operatorname{div}\mathbf{u}_n|+|\operatorname{div}\mathbf{u}|)^{(1+\gamma)/\gamma}\\
&\quad\le\delta\lim_{n\to\infty}\int_\Omega h|\min\{\varrho_n,\theta\}-\min\{\varrho,\theta\}|^{1+\gamma}\,dx+c\delta^{-\gamma}\|h\|_{C(\Omega)}.
\end{aligned}\tag{6.28}$$

Combining (6.28) and (6.27), choosing $h=1$ on K, and $\delta>0$, δ sufficiently small we obtain (6.25). □

We reformulate this result in terms of the distribution function f. Recall that the functions $\min\{\varrho_n,\lambda\}$ are uniformly bounded in $\mathbb{R}^3$ and $\min\{\varrho_n,\lambda\}\operatorname{div}\mathbf{u}_n$ converges weakly in $L^2(D)$ for all nonnegative λ. Introduce the functions

$$\mathcal{V}_\lambda=\overline{(\min\{\varrho,\lambda\}\operatorname{div}\mathbf{u})}-\overline{\min\{\varrho,\lambda\}}\operatorname{div}\mathbf{u}\in L^2(D),\tag{6.29}$$

$$\mathfrak{H}(x)=\int_{[0,\infty)}f(x,s)(1-f(x,s))\,ds,\qquad \mathfrak{H}\in L^\gamma(D).$$

LEMMA 6.10. *There is a constant c independent of λ such that*

$$\|\mathfrak{H}\|_{L^{1+\gamma}(D\setminus S)}+\sup_\lambda\|\mathcal{V}_\lambda\|_{L^1(D\setminus S)}\le c.\tag{6.30}$$

PROOF. Recall that $\mathfrak{H}=\mathcal{V}_\lambda=0$ on $\mathbb{R}^3\setminus\Omega$. Hence, it is sufficient to prove that for all compacts $K\Subset\Omega$, we have

$$\|\mathfrak{H}\|_{L^{1+\gamma}(K)}+\sup_\lambda\|\mathcal{V}_\lambda\|_{L^1(K)}\le c\tag{6.31}$$

with the constant c independent of K. We begin with the observation that by Lemma 6.5,

$$\mathcal{T}_\theta(x)=\int_{[0,\infty)}\min\{\lambda,\theta(x)\}\,d_\lambda f(x,\lambda)-\min\left\{\int_{[0,\infty)}\lambda\,d_\lambda f(x,\lambda),\theta(x)\right\}\tag{6.32}$$

for all functions $\theta\in C(D)$. From this and the identity $\varrho(x)=\int_{[0,\infty)}(1-f(x,\lambda))\,d\lambda$ we conclude that

$$\mathcal{T}_\theta(x)=\int_0^{\theta(x)}f(x,s)\,ds\quad\text{for }\theta(x)\ge\varrho(x)$$

$$\text{and}\quad\mathcal{T}_\theta(x)=\int_{\theta(x)}^\infty(1-f(x,s))\,ds\quad\text{otherwise.}\tag{6.33}$$

Next, choose a sequence of continuous nonnegative functions $\{\theta_k\}_{k\geq 1}$ which converges for $k \to \infty$ to ϱ a.e. in RR^3. By Lemma 6.9 the norms in $L^{1+\gamma}(K)$ of functions $\mathcal{T}_{\theta_k}$ are uniformly bounded by a constant independent of k and K. Moreover, $\mathcal{T}_{\theta_k}$ converges a.e. in K to the function

$$\mathcal{T}_\varrho(x) = \int_0^{\varrho(x)} f(x,s)\,ds = \int_{\varrho(x)}^{\infty} (1-f)\,ds,$$

which yields the estimates $\|\mathcal{T}_\varrho\|_{L^{1+\gamma}(K)} \leq c$ with the constant c independent of K. It remains to note that estimate (6.31) for $\mathfrak{H}$ obviously follows from the inequality $\mathfrak{H} \leq 2\mathcal{T}_\varrho$.

In order to estimate $\mathcal{V}_\lambda$ note that

$$\begin{aligned}\mathcal{V}_\lambda = {}& \text{w-}\lim_{n\to\infty} ((\min\{\varrho_n,\lambda\} - \min\{\varrho,\lambda\})\operatorname{div}\mathbf{u}_n) \\ & - \Big(\text{w-}\lim_{n\to\infty} \min\{\varrho_n,\lambda\} - \min\{\varrho,\lambda\}\Big)\operatorname{div}\mathbf{u},\end{aligned}$$

where by w-lim is denoted the weak limit in $L^1(\mathbb{R}^3)$. From this and the boundedness of norms $\|\operatorname{div}\mathbf{u}_n\|_{L^2(\mathbb{R}^3)}$ hence, we obtain

$$\begin{aligned}\|\mathcal{V}_\lambda\|_{L^1(K)} \leq {}& \limsup_{n\to\infty}(\|\operatorname{div}\mathbf{u}_n\|_{L^2(K)} + \|\operatorname{div}\mathbf{u}\|_{L^2(K)}) \\ & \times \|\min\{\varrho_n(x),\lambda\} - \min\{\varrho(x),\lambda\}\|_{L^2(K)},\end{aligned}$$

which along with (6.25) implies (6.31) and the proof of Lemma 6.10 is completed. □

At the end of this paragraph we describe the basic properties of the function $\mathfrak{M}(x,\lambda)$, which are important for the further analysis.

LEMMA 6.11. *For a.e. $x \in \Omega$,* (i) *$\mathcal{M}(x,\cdot)$ is nonnegative and vanishes on $\mathbb{R}^-$. Moreover, if the Borel function $\mathfrak{M}(x,.)$ given by* (6.20) *vanishes $d_\lambda f(x,\cdot)$-almost everywhere on the interval (ω,∞) with $\omega = \overline{p}(x)^{1/\gamma}$, then $d_\lambda f(x,\cdot)$ is a Dirac measure and*

$$f(x,\lambda) = 0 \quad \textit{for } \lambda < \overline{p}(x)^{1/\gamma}, \qquad f(x,\lambda) = 1 \quad \textit{for } \lambda \geq \overline{p}(x)^{1/\gamma}.$$

(ii) *For all $g \in C_0^\infty(0,\infty)$,*

$$\int_{\mathbb{R}} g(\lambda)\mathcal{M}(x,\lambda)d\lambda = -\int_{[0,\infty)} g'(\lambda)\mathcal{V}_\lambda(x)d\lambda, \tag{6.34}$$

where $\mathcal{V}_\lambda$ is defined by (6.29).

PROOF. By abuse of notations we will write simply f_k instead of $[f]_{,k}$. The mollified distribution function $f_k(x,\cdot)$ belongs to the class $C^\infty(\mathbb{R})$ and generates the absolutely continuous Stieltjes measure σ_{kx} of the form $d\sigma_{kx} = \partial_\lambda f_k d\lambda$. It is easy to see that for $k \to \infty$ the sequence of measures σ_{kx} converges star-weakly to the measure $\sigma_x = d_\lambda f$ in the space of Radon's measures on $\mathbb{R}$. In particular, for all λ with $d_\lambda f(x,\cdot)\{\lambda\} := \lim_{s\to\lambda+0} f(x,s) - \lim_{s\to\lambda-0} f(x,s) = 0$, we can pass to the limit, to obtain

$$\int_{[0,\lambda)} (t^\gamma - \overline{p})\partial_t f_k(x,t)\,dt \to \int_{[0,\lambda)} (t^\gamma - \overline{p})\,d_t f(x,t) \quad \text{for } k \to \infty. \tag{6.35}$$

In other words, relation (6.35) holds true for all λ, possibly except for some countable set. Since $\partial_\lambda f_k \geq 0$, the function on the left-hand side of (6.35) increases on $(-\infty,\omega)$ and decreases on (ω, ∞). From this and (6.35) we conclude that $\mathcal{M}(x,\cdot)$ does not decrease for $\lambda < \omega$ and does not increase for $\lambda > \omega$, which along with the obvious relations $\lim_{\lambda\to\pm\infty}\mathcal{M}(x,\lambda) = 0$ yields the nonnegativity of $\mathcal{M}$.

In order to prove the second part of *(i)* note that $\mathfrak{M}(x,\lambda) = \lim_{k\to\infty}\mathbf{S}_k\mathcal{M}(x,\lambda)$ belongs to the first Baire class, and hence is measurable in σ_x. It follows from the monotonicity of $\mathfrak{M}(x,\cdot)$ on the interval (ω,∞) that if $\mathfrak{M}(x,\alpha) = 0$ for some $\alpha > \omega$, then $\mathfrak{M}(x,\lambda) = 0$ and $f(x,\lambda) = 1$ on (α,∞). Assume that $\mathfrak{M}(x,\cdot)$ vanishes $d_\lambda f(x,\cdot)$-almost everywhere on (ω,∞), and consider the set

$$\mathcal{O} = \left\{\alpha > \omega : \sigma_x(\omega,\alpha) \equiv \lim_{s\to\alpha-0} f(x,s) - \lim_{s\to\omega+0} f(x,s) = 0\right\}.$$

Let us prove that $\mathcal{O} = (\omega,\infty)$. If the set $\mathcal{O}$ is empty, then there is a sequence of points $\lambda_k \searrow \omega$ with $\mathfrak{M}(x,\lambda_k) = 0$, which yields $f(x,\cdot) = 1$ on (ω,∞) thus $\mathcal{O} = (\omega,\infty)$. Hence $\mathcal{O} \neq \emptyset$. If $m = \sup\mathcal{O} < \infty$, then there is a sequence $\lambda_k \searrow m$ with $\mathfrak{M}(x,\lambda_k) = 0$, which yields $f(x,\cdot) = 1$ on (m,∞). By construction, $f(x,\lambda) = c = \text{constant}$ on (ω,m). In other words, restriction of the Stieltjes measure $d_\lambda f(x,\cdot)$ to (ω,∞) is the mono-atomic measure $(1-c)\delta(\cdot - m)$. Hence $\mathfrak{M}(x,m) = 2^{-1}(1-c)(m^\gamma - \omega^\gamma) = 0$ which yields $c = 1$. From this we can conclude that $f(x,\cdot) = 1$ on (ω,∞), and $d_\lambda f(x,\cdot)$ is a probability measure concentrated on $[0,\omega]$. Recalling that $\omega^\gamma = \overline{p}(x)$ we obtain

$$\mu\mathcal{M}(x,0) = \int_{[0,\omega]} (\lambda^\gamma - \omega^\gamma)\, d_\lambda f(x,\lambda) \geq 0.$$

Hence $d_\lambda f(x,\lambda)$ is the Dirac measure concentrated at ω, which implies (i).

The proof of (ii) is straightforward. It is easily seen that

$$\begin{aligned}
&-\mu\int_{\mathbb{R}} g(\lambda)\mathcal{M}(x,\lambda)\,d\lambda\\
&= \int_{[0,\infty)}\left(\int_{[\lambda,\infty)} g'(s)\,ds\right)\left(\int_{[\lambda,\infty)} (t^\gamma - \overline{p})d_t f(x,t)\right)d\lambda\\
&= \int_{[0,\infty)} g'(s)\left(\int_{[0,s)} d\lambda \int_{[\lambda,\infty)} (t^\gamma - \overline{p})\,d_t f(x,t)\right)ds\\
&= \int_{[0,\infty)} g'(s)\left(\int_{[0,\infty)} \min\{t,s\}(t^\gamma - \overline{p})\,d_t\Gamma(x,t)\right)ds\\
&= \int_{[0,\infty)} g'(s)(\overline{\min\{\varrho,s\}p} - \overline{\min\{\varrho,s\}}\overline{p}))\,ds.
\end{aligned}$$

On the other hand, Condition (**H7**) of Theorem 6.4 yields $\overline{\min\{\varrho,\lambda\}p} - \overline{\min\{\varrho,\lambda\}}\overline{p} = \mu\mathcal{V}_\lambda(x)$, and the proof of Lemma 6.11 is completed. □

6.6. *Proof of Theorem 6.4*

We are now in a position to complete the proof of Theorem 6.4. We begin with the observation that all terms in equation (6.19) vanish outside of the slab $\Omega\times\mathbb{R}^+$, and integral identity (6.24) holds true after substituting any smooth function $\zeta = \eta(\lambda)$ vanishing near $+\infty$. Set $\eta(\lambda) = \int_\lambda^\infty \theta(s-t)\,ds$, where θ is defined by relations (4.14) and t is an arbitrary number from the interval $(2, \infty)$. Note that η' vanishes outside of the segment $[t-1, t+1]$, and $\eta(\lambda) = 1$ on the interval $(-\infty, t-1)$. Substituting η into integral identity (6.24) and noting that $\eta' \leq 0$ we arrive at the integral inequality

$$2\int_{\mathbb{R}^3\times(-\infty,t]} \eta\lambda\mathfrak{M}\,d_\lambda f(x,\lambda)\,dx \leq -(t+1)\int_{\mathbb{R}^4} \eta'\mathcal{M}\,dx\,d\lambda$$
$$-(t+1)\int_{\mathbb{R}^4} \eta'\Psi(f)(|h| + |\operatorname{div}\mathbf{v}|)\,dx\,d\lambda.$$

Identity (6.34) gives the representation for the first integral on the right-hand side of this inequality

$$-\int_{\mathbb{R}^4} \eta'(\lambda)\mathcal{M}(x,\lambda)d\lambda = \int_{[0,\infty)} \eta''(\lambda)\wp_1(\lambda)\,d\lambda, \quad \text{where } \wp_1(\lambda) = \int_{\mathbb{R}^3} \mathfrak{V}_\lambda(x).$$

The second admits the similar representation

$$-\int_{\mathbb{R}^4} \eta'\Psi(f)(|h| + |\operatorname{div}\mathbf{v}|)\,dx = \int_{[0,\infty)} \eta''(\lambda)\wp_2(\lambda)\,d\lambda,$$
$$\wp_2(\lambda) = \int_{\mathbb{R}^3} (|h| + |\operatorname{div}\mathbf{v}|)\left\{\int_0^\lambda \Psi(f(x,s))\,ds\right\}dx.$$

Using these relations we can rewrite (6.6) in the form

$$2\int_{(D\setminus S)\times(-\infty,t]} \eta\lambda\mathfrak{M}\,d_\lambda f(x,\lambda)\,dx \leq (1+t)\int_{[1,\infty)} \eta''(\lambda)\wp(\lambda)d\lambda, \tag{6.36}$$

where $\wp = \wp_1 + \wp_2$. Let us prove that the function $\wp$ is bounded on the positive semi-axis. By virtue of Lemma 6.10 we have $|\wp_1(\lambda)| \leq \|\mathfrak{V}_\lambda\|_{L^1(\mathbb{R}^3)} \leq c$. On the other hand, the obvious inequality $\int_0^\lambda \Psi(f(x,s))\,ds \leq \mathfrak{H}(x)$ along with Lemma 6.10 implies the estimate

$$|\wp_2(\lambda)| \leq c\||\operatorname{div}\mathbf{v}| + a + h\|_{L^2(\mathbb{R}^3)}\|\mathfrak{H}\|_{L^2(\mathbb{R}^3)} \leq c, \tag{6.37}$$

which yields the boundedness of $\wp$.

Recalling the expression for $\eta(\lambda)$ we can rewrite inequality (6.36) in the form

$$2\int_{\mathbb{R}^3}\left(\int_{[0,t-1)} \lambda\mathfrak{M}d_\lambda f(x,\lambda)\right)dx \leq (1+t)\frac{d}{dt}(\theta * \wp)(t). \tag{6.38}$$

Since the smooth function $(\theta * \wp)(t)$ is uniformly bounded on the positive semi-axis, there exist a sequence $t_k \to \infty$ such that $\lim_{k\to\infty}(t_k+1)(\theta*\wp)'(t_k) \leq 0$. Substituting $t = t_k$ into (6.38) and letting $k \to \infty$, we finally obtain

$$\int_{\mathbb{R}^3}\left\{\int_{[0,\infty)} \lambda\mathfrak{M}\,d_\lambda f(x,\lambda)\right\}dx = 0.$$

Therefore, for almost every $x \in \mathbb{R}^3$, the functions $\mathcal{M}(x,\cdot)$ is equal to zero $d_\lambda f(x,\lambda)$-a.e. on $(0,\infty)$. From this and Lemma 6.11 we conclude $f(1-f) = 0$ a.e. in $\mathbb{R}^4$, which implies the strong convergence of the sequence ϱ_n. □

7. Isentropic flows with adiabatic constants $\gamma \in (4/3, 5/3]$

In this section we prove the existence of a solution to the boundary value problem for compressible Navier–Stokes equations

$$-\Delta \mathbf{u} - \lambda \operatorname{div} \mathbf{u} + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\rho) = \varrho \mathbf{f} + \mathbf{h} \quad \text{in } \Omega, \tag{7.1a}$$

$$\operatorname{div}(\varrho \mathbf{u}) = 0 \quad \text{in } \Omega, \tag{7.1b}$$

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega. \tag{7.1c}$$

Here $p = \rho^\gamma$, where $\gamma > 1$- is the adiabatic constant, and $\mathbf{f}$, $\mathbf{h}$ are given continuous vector fields. Furthermore, we shall assume that the density satisfies the weighted-mass condition

$$\int_\Omega d(x)^{-s} \varrho \, dx = \mathbf{M}, \tag{7.1d}$$

in which $\mathbf{M}$ is a given constant, $d(x)$ is the distance between a point $x \in \Omega$ and the boundary of Ω, exponent $s \in (0, 1/2)$, depending only on γ will be specified below. The flow is characterized by the *internal energy* $\mathbf{E}$ and the *energy dissipation rate* $\mathbf{D}$, given by the formulae

$$\mathbf{E} = \int_\Omega P(\varrho(x)) \, dx \quad \mathbf{D} = \int_\Omega (|\nabla \mathbf{u}|^2 + \lambda \operatorname{div} \mathbf{u}^2) \, dx, \tag{7.2}$$

where a nonnegative function P is defined with an accuracy to the inessential linear function from the equation $sP'(s) - P(s) = p(s)$. The typical form of P is

$$P(\varrho) = \varrho \int_{\varrho_e}^{\varrho} s^{-2} p(s) \, ds \equiv \varrho e(\varrho),$$

where e is the specific internal energy and ϱ_e is some equilibrium value of the density. In the case of the isentropic flow with $p = \varrho^\gamma$ we can take $P(\varrho) = (\gamma - 1)^{-1} \varrho^\gamma$. Introduce also the weighted kinetic energy

$$\mathbf{K} = \int_\Omega d(x)^s \varrho |\mathbf{u}|^2 \, dx. \tag{7.3}$$

Taking formally the product of the moment equation with $\mathbf{u}$ and integrating the result by parts we obtain the integral identity

$$\int_\Omega (|\operatorname{rot} \mathbf{u}|^2 + (1 + \lambda)|\operatorname{div} \mathbf{u}|^2) dx = \int_\Omega (\varrho \mathbf{f} + \mathbf{h}) \cdot \mathbf{u} dx, \tag{7.4}$$

which express the energy balance law. The energy identity (7.4) implies the energy inequality

$$\int_\Omega |\nabla \mathbf{u}|^2 \, dx \le c \int_\Omega \varrho |\mathbf{f}||\mathbf{u}| \, dx + c(\Omega) \|\mathbf{h}\|^2_{C(\Omega)},$$

which, in its turn, yields the estimate

$$\mathbf{D} \le c \|\mathbf{f}\|_{C(\Omega)} \sqrt{\mathbf{MK}} + c(\Omega) \|\mathbf{h}\|^2_{C(\Omega)}$$

of the energy dissipation rate via the weighted mass and the kinetic energy. It is important to note that, in contrast with the nonstationary problems, the energy identity do not imply the boundedness of the total energy. In what follows, we shall give an outline of the nonlocal existence theory in the class of weak solutions having a finite weighted energy and, in particular, shall prove the following theorem on solvability of problem (7.1).

THEOREM 7.1. *Let the adiabatic constant γ and the exponent s satisfy the inequalities*

$$\gamma > 4/3, \quad s \in ((5\gamma - 4)^{-1}, 2^{-1}). \tag{7.5}$$

Furthermore assume that the Ω is a bounded region with the boundary $\partial\Omega \in C^3$. Then for any $\mathbf{f}, \mathbf{h} \in C(\Omega)$, problem (7.1) has a generalized solution which meets all requirements of Definition 3.4 and satisfies the inequalities

$$\|\mathbf{u}\|_{H_0^{1,2}(\Omega)} + \int_\Omega d^{-s}(p(\varrho) + \varrho|\mathbf{u}\cdot\nabla d|^2)\,dx + \int_\Omega d\varrho|\mathbf{u}|^2\,dx \le c, \tag{7.6}$$

where c depends only on γ, s, Ω and $\mathbf{M}$, $\|\mathbf{f}\|_{C(\Omega)}$, $\|\mathbf{h}\|_{C(\Omega)}$.

Inequalities (7.6) show that the internal energy and the normal component of the kinetic energy tensor $\varrho\mathbf{u}\otimes\mathbf{u}$ take moderate values in the vicinity of the boundary, while the tangential components of the kinetic energy tensor may concentrate near the wall. Does this phenomena really take place or is it a peculiarity of our method is the question which we cannot decide with certainty.

The rest of the section is devoted to the proof of this theorem. Recall that the scheme for solving nonlinear problems consists of the following steps:

- First of all, we have to choose an approximate scheme and to construct a family of approximate solutions.
- Next we find *a priori estimates* which guarantee the uniform boundedness of the family of approximate solutions in the suitable norms.
- Finally we have to prove that any limiting points of the set of approximation solutions in suitable topology is a solution to the original problem.

Hence our first task is to construct approximate solutions to problem (7.1).

7.1. *Approximate solutions*

There are several possibilities on how to construct the family of approximate solutions. Here, we use a two-level approximate scheme based on solving the momentum equation with artificial pressure $p = \varrho^\gamma + \epsilon\varrho^3$ and continuity equation with vanishing diffusion and homogeneous boundary conditions

$$\Delta\mathbf{u} + \lambda\nabla\operatorname{div}\mathbf{u} = \operatorname{div}((t\varrho\mathbf{u} - \varepsilon\nabla\varrho)\otimes\mathbf{u}) + \nabla p(\varrho) - t\varrho\mathbf{f} - t\mathbf{h} \quad \text{in } \Omega, \tag{7.7a}$$

$$-\varepsilon\Delta\varrho + t\operatorname{div}(\varrho\mathbf{u}) = 0 \quad \text{in } \Omega, \tag{7.7b}$$

$$\mathbf{u} = 0, \quad \partial_n\varrho = 0 \quad \text{on } \partial\Omega, \tag{7.7c}$$

$$\int_\Omega d^{-s}\varrho\,dx = \mathbf{M}. \tag{7.7d}$$

Here $t \in [0, 1]$ is an artificial parameter needed for application of the Leray–Schauder fixed-point theorem. The following theorem establishes the existence of a solution to this problem for $p \sim \varrho^3$.

THEOREM 7.2. *Let Ω be a bounded domain with the boundary of the class C^3 and $\gamma > 1$, $0 < \beta < \gamma - 1$. Furthermore assume that a function $p \in C^\gamma(\mathbb{R}^1)$ satisfies the conditions*

$$\begin{aligned} &c_0^{-1}\varrho^3 \le p(\varrho) \le c_0\varrho^3, \quad c_0^{-1}\varrho^2 \le p'(\varrho) \le c_0\varrho^2 \quad \textit{for } \varrho \ge 1,\\ &c_0^{-1}\varrho^\gamma \le p(\varrho) \le c_0\varrho^\gamma, \quad c_0^{-1}\varrho^{\gamma-1} \le p'(\varrho) \le c_0\varrho^{\gamma-1} \quad \textit{for } \varrho \in [0,1]. \end{aligned} \tag{7.8}$$

Then for any $\mathbf{f}, \mathbf{h} \in C^\infty(\Omega)$, problem (7.1) has a solution $(\mathbf{u}, \varrho) \in C^{2+\beta}(\Omega)^3 \times C^{2+\beta}(\Omega)$, $\varrho \ge 0$ satisfying the inequalities

$$\mathbf{D} \le c_{ab}(1 + \mathbf{M}^2 + \sqrt{\mathbf{MK}}), \tag{7.9a}$$

$$\|\mathbf{u}\|_{H^{1,2}(\Omega)} + \|\varrho\|_{L^{9/2}(\Omega)} + \sqrt{\varepsilon}\|\varrho\|_{H^{1,2}(\Omega)} \le C(1 + \mathbf{M}), \tag{7.9b}$$

$$\|\mathbf{u}\|_{C^{2+\beta}(\Omega)} + \|\varrho\|_{C^{2+\beta}(\Omega)} \le C_\varepsilon, \tag{7.9c}$$

where a constant c_{ab} depends only on Ω, $\|\mathbf{f}, \mathbf{h}\|_{C(\Omega)}$, a constant C depends only on Ω, $\mathbf{f}$, $\mathbf{h}$, c_0, and a constant C_ε does not depend on t, The energy dissipation rate and the weighted kinetic energy are defined by the equalities (7.2) and (7.3).

PROOF. We begin with proving a priori estimates. Assume that $(\mathbf{u}, \varrho)$ be a C^2-solution to the problem with a nonnegative density ϱ. Introduce a convex differentiable function $g(\varrho) = \varrho \int_0^\varrho s^{-2} p(s)\,ds$ satisfying the equation $sg'(s) - g(s) = p(s)$. It is clear that

$$0 < c^{-1} \le g''(\varrho) \le c(1 + \varrho).$$

Multiplying both the sides of equation (7.7a) by $\mathbf{u}$ and integrating by parts we obtain the integral identity

$$\int_\Omega (|\nabla \mathbf{u}|^2 + \lambda \operatorname{div} \mathbf{u}^2 + \varepsilon g''(\varrho)|\nabla\varrho|^2)\,dx = t\int_\Omega (\varrho\mathbf{f} + \mathbf{h}) \cdot \mathbf{u}\,dx,$$

which along with the Poincare inequality leads to the estimate

$$\|\mathbf{u}\|_{H^{1,2}(\Omega)} + \sqrt{\varepsilon}\|\varrho\|_{H^{1,2}(\Omega)} \le C(1 + \sqrt{\varepsilon}\mathbf{M}) + C\|\varrho\mathbf{u}\|_{L^1(\Omega)}{}^{1/2}. \tag{7.10}$$

Moreover, since

$$\int_\Omega \varrho\mathbf{f}\mathbf{u}\,dx \le C\int_\Omega \varrho|\mathbf{u}|\,dx \le 2\sqrt{\mathbf{MK}},$$

the energy identity obviously yields the energy estimate (7.9a). □

Our next task is to obtain the estimate for ϱ. Recall that by virtue of the Bogovskii Lemma for any $\psi \in C^\infty(\Omega)$ with

$$\psi_{av} = \frac{1}{|\Omega|}\int_\Omega \psi\, dx = 0,$$

there exists a vector field $\mathbf{q} \in H_0^{1,3}(\Omega)$ satisfying the conditions

$$\operatorname{div}\mathbf{q} = \psi \quad \text{in } \Omega, \qquad \|\mathbf{q}\|_{H^{1,3}(\Omega)} \le c\|\psi\|_{H^{1,3}(\Omega)}.$$

Multiplying both the sides of equation (7.7a) by $\mathbf{q}$ and integrating by parts we obtain

$$\begin{aligned}\int_\Omega p(\varrho)\psi\, dx = \int_\Omega (\nabla\mathbf{u} : \nabla\mathbf{q} + \lambda \operatorname{div}\mathbf{u}\psi)\, dx \\ - \int_\Omega (t\varrho\mathbf{u}\otimes\mathbf{u} : \nabla\mathbf{q} + \varepsilon\nabla\varrho\otimes\mathbf{u} : \nabla\mathbf{q})\, dx - \int_\Omega (\varrho\mathbf{f} + \mathbf{h})\mathbf{q} dx,\end{aligned}$$

which along with the standard duality arguments leads to the estimate

$$\begin{aligned}\|p - p_{av}\|_{L^{3/2}(\Omega)} \le C(\|\mathbf{u}\|_{H^{1,2}(\Omega)} + \|\varrho|\mathbf{u}|^2\|_{L^{3/2}(\Omega)} \\ + \varepsilon\||\mathbf{u}|\nabla\varrho\|_{L^{3/2}(\Omega)} + \|\varrho\|_{L^{3/2}(\Omega)} + 1).\end{aligned} \tag{7.11}$$

Since by the embedding theorem $\|\mathbf{u}\|_{L^6(\Omega)} \le c\|\mathbf{u}\|_{H^{1,2}(\Omega)}$, we have

$$\begin{aligned}\|\varrho|\mathbf{u}|^2\|_{L^{3/2}(\Omega)} \le c\|\varrho\|_{L^3(\Omega)}\|\mathbf{u}\|^2_{H^{1,2}(\Omega)}, \\ \||\mathbf{u}|\nabla\varrho\|_{L^{3/2}(\Omega)} \le c\|\nabla\varrho\|_{L^2(\Omega)}\|\mathbf{u}\|_{H^{1,2}(\Omega)}.\end{aligned}$$

Combining this result with (7.11) and using the obvious inequalities

$$p_{av} \le c\|\varrho\|^3_{L^3(\Omega)}, \qquad \|\varrho\|^3_{L^{9/2}(\Omega)} \le c\|p\|_{L^{3/2}(\Omega)}$$

we obtain the estimate

$$\begin{aligned}\|\varrho\|^3_{L^{9/2}(\Omega)} \le C(\|\mathbf{u}\|_{H^{1,2}(\Omega)} + \|\varrho\|_{L^3(\Omega)}\|\mathbf{u}\|^2_{H^{1,2}(\Omega)} \\ + \varepsilon\|\nabla\varrho\|_{L^2(\Omega)}\|\mathbf{u}\|_{H^{1,2}(\Omega)} + \|\varrho\|^3_{L^3(\Omega)} + 1).\end{aligned} \tag{7.12}$$

Next note that the Holder inequality implies the estimate

$$\|\varrho\|_{L^3(\Omega)} \le \left(\int_\Omega \varrho\, dx\right)^{1/7}\left(\int_\Omega \varrho^{9/2}\right)^{4/21} \le C\mathbf{M}^{1/7}\|\varrho\|^{6/7}_{L^{9/2}(\Omega)},$$

which along with the Young inequality yields the estimates

$$\begin{aligned}\|\varrho\|_{L^3(\Omega)}\|\mathbf{u}\|^2_{H^{1,2}(\Omega)} \le \delta\|\varrho\|^3_{L^{9/2}(\Omega)} + C(\delta)(\mathbf{M}^3 + \|\mathbf{u}\|^3_{H^{1,2}(\Omega)}), \\ \|\varrho\|^3_{L^3(\Omega)} \le \delta\|\varrho\|^3_{L^{9/2}(\Omega)} + C(\delta)(\mathbf{M}^3 + \|\mathbf{u}\|^3_{H^{1,2}(\Omega)}),\end{aligned}$$

where δ is an arbitrary positive number. Substituting these inequalities into the right-hand side of (7.12) we arrive at the estimate

$$\|\varrho\|_{L^{9/2}(\Omega)} \le C(\|\mathbf{u}\|_{H^{1,2}(\Omega)} + \varepsilon\|\nabla\varrho\|_{L^2(\Omega)} + \mathbf{M} + 1).$$

Combining this estimate with the energy inequality (7.10) and choosing δ sufficiently small we obtain the estimate

$$\|\varrho\|_{L^{9/2}(\Omega)} + \|\mathbf{u}\|_{H^{1,2}(\Omega)} + \sqrt{\varepsilon}\|\varrho\|_{H^{1,2}(\Omega)} \le C(1 + \mathbf{M} + \|\varrho\mathbf{u}\|_{L^1(\Omega)}{}^{1/2}). \quad (7.13)$$

Applying the Holder inequality we can obtain the following estimate for the last term on the right-hand side of

$$\|\varrho\mathbf{u}\|_{L^1(\Omega)} \le \|\mathbf{u}\|_{L^6(\Omega)} \|\varrho\|_{L^1(\Omega)}^{11/14} \|\varrho\|_{L^{9/2}(\Omega)}^{3/14}$$

which together with the Young inequality leads to the estimate

$$\|\varrho\mathbf{u}\|_{L^1(\Omega)}{}^{1/2} \le \delta(\|\varrho\|_{L^{9/2}(\Omega)} + \|\mathbf{u}\|_{H^{1,2}(\Omega)}) + C(\delta)\mathbf{M}.$$

Substituting this result in (7.13) we obtain estimate (7.9b), which along with the standard bootstrap arguments, see for example [34], leads to the estimate

$$\|\varrho\|_{L^\infty(\Omega)} + \|\mathbf{u}\|_{L^\infty(\Omega)} \le C(\varepsilon, \mathbf{M}, \Omega, p, \|\mathbf{f}, \mathbf{h}\|_{L^\infty(\Omega)}).$$

From this and the results from the theory of weakly nonlinear elliptic equations, see Theorem 13.1 in [4], we conclude that the inequality

$$\|(\varrho, \mathbf{v})\|_{C^{2+\beta}(\Omega)} < C(\varepsilon, \Omega, \|(\mathbf{f}, \mathbf{h})\|_{C^\beta(\Omega)}, \mathbf{M}, p) \quad (7.14)$$

holds for every solution $(\varrho, \mathbf{u}) \in C^{1+\beta}(\Omega)$, $\varrho > 0$, to problem (7.1).

To tackle the existence question we need to reformulate problem (7.1) as a nonlinear operator equation in the form $(\varrho, \mathbf{v}) = \Phi_t(\varrho, \mathbf{u})$. Introduce the mapping $\Phi_t : (\hat{\varrho}, \hat{\mathbf{u}}) \mapsto (\varrho, \mathbf{u})$ defined as a solution to the boundary value problems

$$\Delta\mathbf{u} + \lambda\nabla \operatorname{div} \mathbf{u} = \operatorname{div}((t\varrho\hat{\mathbf{u}} - \varepsilon\nabla\hat{\varrho}) \otimes \hat{\mathbf{u}}) + \nabla p(\hat{\varrho}) - t\hat{\varrho}\,\mathbf{f} - t\mathbf{h} \quad \text{in } \Omega, \quad (7.15a)$$

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega, \quad (7.15b)$$

$$-\varepsilon\Delta\varrho + t\,\operatorname{div}(\varrho\,\hat{\mathbf{u}}) = 0 \quad \text{in } \Omega, \quad (7.15c)$$

$$\partial_n\varrho = 0 \quad \text{on } \partial\Omega, \quad (7.15d)$$

$$\int_\Omega d^{-s}\varrho\,dx = \mathbf{M}. \quad (7.15e)$$

The solvability of boundary value problem (7.15a)–(7.15b) is a well-known fact from the theory of Lame equation. The question on existence of *positive* nontrivial solution to equations (7.15c) is less trivial. This fact is a consequence of the general theory of positive solutions to linear operator equations and results from the following lemma.

LEMMA 7.3. *Let Ω be a bounded region in $\mathbb{R}^d$ with the boundary $\partial\Omega \in C^{2+\beta}$, $0 < \beta < 1$, and a vector field $\mathbf{u} \in C^{1+\beta}(\Omega)$ vanishing at $\partial\Omega$. Then for any positive $\mathbf{M}$ and $s < 1$, there exists a unique positive solution to the problem* (7.7b)–(7.7d) *which satisfies*

$$\int_\Omega d^{-s}\varrho(x)dx = \mathbf{M}.$$

We refer to [50] for the proof. Note that only the existence of nontrivial solutions follows from the Fredholm theory since 0 is the simple eigenvalue of the adjoint problem

$$-\varepsilon\Delta\varrho - t\hat{\mathbf{u}}\nabla\varrho = 0 \quad \text{in } \Omega, \qquad \partial_n\varrho = 0 \quad \text{on } \Omega.$$

The positivity of ϱ follows from the positivity of the first eigenfunction of a second-order elliptic operator.

Hence the mapping $(\hat{\varrho}, \hat{\mathbf{u}}, t) \mapsto (\varrho, \mathbf{v})$ denoted by $\Phi : C^{1+\beta}(\Omega)^4 \times [0,1] \mapsto C^{2+\beta}(\Omega)^4$ is well defined and continuous. Denote by $\mathcal{J}$ a subset of $C^{1+\beta}(\Omega)^4$ defined by the inequalities $\{(\varrho, \mathbf{v}) : \varrho \geq 0, \|(\varrho, \mathbf{v})\|_{C^{1+\beta}(\Omega)} \leq C_\varepsilon\}$, where C_ε is a constant from estimate (7.9c). It follows from Lemma 7.3 that every fixed point $(\varrho, \mathbf{u})$ of Φ_t satisfies inequality (7.9c). Moreover, ϱ is strictly positive. Hence there are no fixed points of Φ_t at $\partial\mathcal{J}$ for all $t \in [0,1]$. On the other hand, the mapping Φ_0 has the unique fixed point inside $\mathcal{J}$. By the Leray–Schauder fixed-point theorem, problem (7.7) has a solution $(\varrho, \mathbf{u}) \in \operatorname{int}\mathcal{J}$ and the proof of Theorem 7.2 is completed. □

7.2. *Weak convergence results*

Theorem 7.2 guarantees the existence of smooth vector fields $\mathbf{u}_\varepsilon$ and positive functions ϱ_ε satisfying conditions (7.7b)–(7.7d) and the equations

$$\Delta\mathbf{u}_\varepsilon + \lambda\nabla \operatorname{div}\mathbf{u}_\varepsilon - \operatorname{div}(\varrho_\varepsilon\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) - \nabla p(\varrho_\varepsilon) + \varrho_\varepsilon\mathbf{f} + \mathbf{h} = \mathbf{O}_\varepsilon \tag{7.16a}$$

$$\operatorname{div}(\varrho\mathbf{u}) = \mathbf{o}_\varepsilon \quad \text{in } \Omega, \tag{7.16b}$$

where

$$\mathbf{O}_\varepsilon = \varepsilon \operatorname{div}(\nabla\varrho_\varepsilon \otimes \mathbf{u}_\varepsilon), \qquad \mathbf{o}_\varepsilon = \varepsilon\Delta\varrho_\varepsilon.$$

It follows from a priori estimates that there exist a subsequence, still denoted by $\mathbf{u}_\varepsilon$, ϱ_ε, and functions $\mathbf{u} \in H_0^{1,2}(\Omega)$, $\varrho \in L^{9/2}(\Omega)$ $\overline{p} \in L^{3/2}(\Omega)$, such that

$$\begin{aligned} \mathbf{u}_\varepsilon &\to \mathbf{u} \quad \text{weakly in } H^{1,2}(\Omega), \\ \varrho_\varepsilon &\to \varrho \quad \text{weakly in } L^{9/2}(\Omega), \\ \varrho_\varepsilon\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon &\to \varrho\mathbf{u}\otimes\mathbf{u} \quad \text{weakly in } L^{3/2}(\Omega), \\ p(\varrho_\varepsilon) &\to \overline{p} \quad \text{weakly in } L^{3/2}(\Omega). \end{aligned} \tag{7.17}$$

Moreover, estimate (7.9b) for $\nabla\varrho_\varepsilon$ implies the limiting relations

$$\mathbf{O}_\varepsilon \to 0 \quad \text{in } H^{-1,3/2}(\Omega), \qquad \mathbf{o}_\varepsilon \to 0 \quad \text{in } H^{-1,2}(\Omega). \tag{7.18}$$

Letting $\varepsilon \to 0$ in equations (7.16) we obtain

$$\Delta\mathbf{u} + \lambda\nabla \operatorname{div}\mathbf{u} - \operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) - \nabla\overline{p} + \varrho\mathbf{f} + \mathbf{h} = 0 \tag{7.19a}$$

$$\operatorname{div}(\varrho\mathbf{u}) = 0 \quad \text{in } \mathcal{D}'(\Omega). \tag{7.19b}$$

Applying the renormalization procedure, see Section 6.1, to (7.19b) we conclude that ϱ serves as the renormalized solution to the mass balance equation. Hence in order to show

that $(\mathbf{u}, \varrho)$ is a weak solution to the original problem in the sense of Definition 3.4 we have to establish the equality $\overline{p} = p(\varrho)$, which, in fact is equivalent to strong convergence of the sequence ϱ_ε.
The proof of this fact is the heart of the theory. It is based on the detailed consideration of the properties of the so-called *effective viscous flux*.

7.3. *Effective viscous flux*

Following [34] we defined the effective viscous flux by the equalities

$$V(\varrho, \mathbf{u}) =: p - \nabla\Delta^{-1}\nabla : \mathbb{S} = p - (1+\lambda)\,\mathrm{div}\,\mathbf{u}.$$

As was shown in [34,17,18] the effective viscous flux enjoys many remarkable properties. The most important is the multiplicative relation

$$\overline{bV} = \overline{b}\,\overline{V}, \quad \text{where } \overline{b} = \text{w-}\lim_{\varepsilon\to 0} b(\varrho_\varepsilon), \quad \overline{bV} = \text{w-}\lim_{\varepsilon\to 0} b(\varrho_\varepsilon)V(\varrho_\varepsilon, \mathbf{u}_\varepsilon), \tag{7.20}$$

which was discovered in [34]. The simple and effective proof of this relation, based on the new version of *compensated compactness principle*, was given in papers [17,18]. The following result is due to Feireisl [16].

LEMMA 7.4. *Let the quantities $\mathbf{u}_\varepsilon$, ϱ_ε satisfy equations (7.16) and for some $q > 3/2$, $r > 1$,*

$$\mathbf{u}_\varepsilon \to \mathbf{u} \quad \textit{weakly in } H^{1,2}(\Omega), \qquad \varrho_\varepsilon \to \varrho \quad \textit{weakly in } L^q(\Omega),$$
$$\mathbf{O}_\varepsilon \to 0, \quad \mathbf{o}_\varepsilon \to 0 \quad \textit{in } H^{-1,r'}(\Omega).$$

Then equality (7.20) holds true for any continuously differentiable function $b : \mathbb{R} \mapsto \mathbb{R}$ with $|b'| \le c$.

This result can be easy generalized if we take into account that:

- the assertion of the lemma is local and the behavior of the quantities near the boundary does not play an important role,
- we need the restriction $q > 3/2$ only to guarantee the integrability of the energy density with exponent greater than one, and the lemma remains true if we require the boundedness of $\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2$ in $L^r_{\mathrm{loc}}(\Omega)$ with $r > 1$.

Thus we come to the following version of Lemma 7.4, [51,52].

LEMMA 7.5. *Let the quantities $\mathbf{u}_\varepsilon$, ϱ_ε satisfy equations (7.16) with $\mathbf{O}_\varepsilon = 0$, $\mathbf{o}_\varepsilon = 0$. Furthermore assume that they satisfy the following conditions:*
(1) *There exist positive κ and c such that for all $n \ge 1$,*

$$\int_\Omega p_\varepsilon\,dx + \int_\Omega |\varrho_\varepsilon \mathbf{u}_\varepsilon|^{1+\kappa}\,dx \le c.$$

Moreover, for each compact $\Omega' \Subset \Omega$,

$$\int_{\Omega'} p_\varepsilon^{1+\kappa}\,dx + \int_{\Omega'} (\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2)^{1+\kappa}\,dx \le c(\Omega'),$$

where $c(\Omega')$ does not depend on n.
(2) *For each compact $E \subset \mathbb{R}^3$ and an arbitrary function $G : \mathbb{R} \to \mathbb{R}$ satisfying the condition $\lim_{\varrho\to\infty} \varrho^{-\gamma} G(\varrho) = 0$, the function $G(\varrho_\varepsilon))$ converges weakly in $L^1(E)$ to a function $\overline{G} \in L^1_{\text{loc}}(\Omega)$. Moreover, if G satisfies more weak condition* $\limsup_{\varrho\to\infty} \varrho^{-\gamma}|G(\varrho)| < \infty$*, then the sequence $G(\varrho_\varepsilon))$ converges weakly in $L^1(\Omega')$ to the function $\overline{G}$ in any subdomain $\Omega' \Subset \Omega$.*
(3) *For some $r > 1$,*

$$\mathbf{u}_\varepsilon \to \mathbf{u} \quad \textit{weakly in } H^{1,2}(\Omega),$$
$$\varrho_\varepsilon \to \varrho \quad \textit{weakly in } L^r(\Omega).$$

Then the limiting relation

$$\int_\Omega \overline{\Phi(\cdot,\varrho)V(\varrho,\mathbf{v})}dx = \int_\Omega \overline{\Phi}\;\overline{V}dx, \quad \textit{where } \overline{V} = \overline{p} - (2+\nu)\operatorname{div}\mathbf{v} \tag{7.21}$$

holds true for any function $\Phi \in C(\Omega \times \mathbb{R})$ satisfying the conditions

$$\Phi(\cdot,\lambda) \in C_0(\Omega) \quad \textit{for all } \lambda \in \mathbb{R}^+,$$
$$\Phi(\cdot,\lambda) = \Phi_\infty(\cdot) \in C_0(\Omega) \quad \textit{for all } \lambda > N > 0.$$

7.4. *Proof of Theorem 7.1*

We begin with the observation that for any $\gamma > 1$ and $\epsilon > 0$, the artificial pressure $p_\epsilon(\varrho) = \varrho^\gamma + \epsilon\varrho^3$ meets all requirements of Theorem 7.2. Therefore, the corresponding boundary value problem (7.7), with $t = 1$ and $p = p_\epsilon$ has a family of strong solutions $(\mathbf{u}_\varepsilon, \varrho_\varepsilon)$ which admit estimates (7.9). After passing to a subsequence we can assume that this sequence satisfies limiting relations (7.17) and its weak limit $(\mathbf{u}, \varrho)$ serves as a generalized solution to equations (7.19). In particular for any function G with $G' \in C^1(\Omega)$ and any function $\psi \in C^1(\Omega)$ we have

$$\begin{aligned}\int_\Omega &(G(\varrho_\varepsilon)\mathbf{u}_\varepsilon \cdot \nabla\psi + (G(\varrho_\varepsilon) - G'(\varrho_\varepsilon)\varrho_\varepsilon)\psi \operatorname{div}\mathbf{u}_\varepsilon)\,dx \\ &= \varepsilon\int_\Omega (G'(\varrho_\varepsilon)\nabla\varrho_\varepsilon\nabla\psi + G'(\varrho_\varepsilon)|\nabla\varrho_\varepsilon|^2\psi)\,dx \to 0 \quad \text{as } \varepsilon \to 0.\end{aligned} \tag{7.22}$$

Moreover, estimate (7.9b) implies that $(\mathbf{u}_\varepsilon, \varrho_\varepsilon)$ satisfy all assumptions of Lemma 7.4. It follows from this, inequalities (7.9b), and limiting relations (7.22) that being extended by zero outside of Ω the functions $(\mathbf{u}_\varepsilon, \varrho_\varepsilon)$ satisfy conditions (**H3**)–(**H7**) of Theorem 6.4. Applying this theorem we conclude that ϱ_ε converge to ϱ strongly in any space $L^r(\Omega)$ with $r < 9/2$. Therefore the limiting functions $(\mathbf{u}, \varrho)$ serve as a weak solution to problem 7.1 with the artificial pressure function $p = p_\epsilon(\varrho)$.

Our next task is to pass to the limit as $\epsilon \to 0$. Note that estimate (7.9b), which plays the key role in the previous considerations, depends on ϵ and cannot be used in the analysis of the second-level approximation. The only estimate which does not depend on ϵ is the weak energy inequality (7.17), but it does not guarantee the boundedness of $\|\mathbf{u}\|_{H^{1,2}(\Omega)}$ and $\|p\|_{L^1(\Omega)}$. The following theorem whose proof is given in the next section allows us to obviate the difficulty.

Let us consider the general moment relation

$$\operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u})+\nabla p=\mathbf{F}+\operatorname{div}\mathbb{S}\quad\text{in }\mathcal{D}'(\Omega),\tag{7.23}$$

which links nonnegative functions ϱ, p, vector fields $\mathbf{u}$, $\mathbf{F}$, and a tensor field $\mathbb{S}$. We do not suppose the existence of any other relations between these quantities. The unexpected result is that this relation along with the weak energy inequality implies the effective estimates for p and $\mathbf{u}$.

THEOREM 7.6. *Let nonnegative functions $\varrho\in L^\gamma(\Omega)$, $p\in L^1(\Omega)$, vector fields $\mathbf{u}\in H_0^{1,2}(\Omega)$, $\mathbf{F}\in L^1(\Omega)$, and a tensor field $\mathbb{S}\in L^2(\Omega)$ satisfy relation (7.23) and the inequalities*

$$\|\mathbb{S}\|^2_{L^2(\Omega)}\le c_e\mathbf{D}\le c_e(\sqrt{\mathbf{K}}+1),\qquad p\ge\varrho^\gamma,\tag{7.24}$$

where the energy dissipation rate $\mathbf{D}$ and the weighted kinetic energy $\mathbf{K}$ are defined by the formulae (7.2) and (7.3). Furthermore assume that

$$\frac{1}{5\gamma-4}<s<\frac{1}{2},\qquad \gamma>\frac{4}{3},\qquad \textit{and}\quad t>0.$$

Then there exist constants c and $\sigma>1$, depending only on Ω, $\|\mathbf{F}\|_{L^1(\Omega)}$, s, c_e, γ, a constant $c(t)$ depending only on Ω, $\|\mathbf{F}\|_{L^1(\Omega)}$, s,γ c_e, t

$$\mathbf{D}+\|d^s\varrho|\mathbf{v}|^2\|_{L^\sigma(\Omega)}+\|p\|_{L^1(\Omega)}\le c,\qquad \|p\|_{L^\sigma(\Omega_t)}\le c(t).\tag{7.25}$$

Denote by $(\mathbf{u}_\epsilon,\varrho_\epsilon)$ a sequence of generalized solutions to problem (7.1) with p replaced by p_ϵ. It follows from (7.1a) that the functions ϱ_ϵ, p_ϵ, and the vector fields $\mathbf{u}_\epsilon$ satisfy relation (7.23) with

$$\mathbf{F}=\mathbf{f}\varrho_\epsilon+\mathbf{h},\qquad \mathbb{S}=-(\nabla\mathbf{u}_\epsilon+\nabla\mathbf{u}_\epsilon{}^*)+(\lambda-1)\operatorname{div}\mathbf{u}_\epsilon\mathbf{I}.$$

Obviously

$$\|\mathbf{F}\|_{L^1(\Omega)}\le C(\mathbf{M}+1),\qquad \|\mathbb{S}\|^2_{L^2(\Omega)}\le c_e\mathbf{D}_\epsilon,$$

where $\mathbf{D}_\epsilon$ is given by formula (7.2) with $\mathbf{u}$ replaced by $\mathbf{u}_\epsilon$. On the other hand, by virtue of (7.17), ϱ_ϵ and $\mathbf{u}_\epsilon$ satisfy the weak energy inequality (7.24) with the constant c_e depending only on $\mathbf{M}$ and $\|\mathbf{f},\mathbf{h}\|_{C(\Omega)}$. Hence ϱ_ϵ, p_ϵ, and $\mathbf{u}_\epsilon$ meet all requirements of Theorem 7.6

and satisfy inequalities (7.25) with a constant c, $c(t)$ independent on ϵ. After passing to a subsequence we can assume that

$$\begin{aligned} \mathbf{u}_\epsilon \to \mathbf{u} \quad &\text{weakly in } H^{1,2}(\Omega), \\ \mathbf{u}_\epsilon \to \mathbf{u} \quad &\text{strongly in } L(\Omega), \quad r < 6, \\ \varrho_\epsilon \to \varrho \quad &\text{weakly in } L^\gamma(\Omega). \end{aligned} \tag{7.26}$$

Moreover, the limiting relation

$$G(\varrho_\epsilon) \to \overline{G} \quad \text{weakly in } L^1(\Omega) \tag{7.27}$$

holds true for any continuous function G satisfying the condition $\lim_{\varrho\to\infty} |G(\varrho)|/\varrho = 0$. Since $\gamma > 4/3$ the sequence $\varrho_\epsilon \mathbf{u}_\epsilon$ is bounded in $L^{12/11}(\Omega)$ we also have

$$\varrho_\epsilon \mathbf{u}_\epsilon \to \varrho \mathbf{u} \quad \text{weakly in } L^{12/11}(\Omega). \tag{7.28}$$

Next, it follows from inequalities (7.25) that for any subdomain $\Omega' \Subset \Omega$,

$$\begin{aligned} \varrho_\epsilon \mathbf{u}_\epsilon \otimes \mathbf{u}_\epsilon \to \varrho \mathbf{u} \otimes \mathbf{u} \quad &\text{weakly in } L^\sigma(\Omega'), \\ p(\varrho_\epsilon) \to \overline{p} \quad &\text{weakly in } L^\sigma(\Omega'). \end{aligned} \tag{7.29}$$

Let us show that for any positive $\kappa < \sigma - 1$,

$$v_\epsilon \equiv \epsilon \varrho_\epsilon^3 \to 0 \quad \text{in } L^{1+\kappa}(\Omega') \quad \text{as } \epsilon \to 0. \tag{7.30}$$

To this end note that the sequence v_ϵ is bounded in $L^\sigma(\Omega')$, and hence the inequality

$$\int_E v_\epsilon^{1+\kappa}\, dx \le \left(\int_{\Omega'} v_\epsilon^\sigma\, dx \right)^{\frac{1+\kappa}{\sigma}} \left(\int_E dx \right)^{1-\frac{1+\kappa}{\sigma}} \le c(\Omega')(\text{meas } E)^{1-\frac{1+\kappa}{\sigma}}$$

holds true for any measurable set $E \subset \Omega$. On the other hand, by virtue of the Chebyshev inequality we have

$$\text{meas}\, \{\varrho_\epsilon \ge N\} \le \mathbf{M} N^{-1} \quad \text{and} \quad \int_{\{\varrho_\epsilon \le N\}} v_\epsilon^{1+\kappa}\, dx \le \epsilon^{1+\kappa} N^{1+\kappa}.$$

Combining these results we obtain

$$\begin{aligned} &\limsup_{\epsilon\to 0} \int_{\Omega'} v_\epsilon^{1+\kappa}\, dx \le \limsup_{\epsilon\to 0} (\epsilon^{1+\kappa} N^{1+\kappa} + (\mathbf{M} N^{-1})^{1-\frac{1+\kappa}{\sigma}}) \le \\ &c(\mathbf{M}, \Omega') N^{-1+\frac{1+\kappa}{\sigma}}) \to 0 \quad \text{as } N \to \infty, \end{aligned}$$

which gives (7.30). It follows from (7.29) and (7.30) that for any $\Omega' \Subset \Omega$,

$$\varrho_\epsilon^\gamma \to \overline{p} \quad \text{weakly in } L^\sigma(\Omega'). \tag{7.31}$$

Finally note that functions ϱ_ϵ serve as renormalized solutions to mass balance equations associated with vector fields $\mathbf{u}_\epsilon$. It follows from this that ϱ_ϵ, $p(\varrho_\epsilon)$, and $\mathbf{u}_\epsilon$ meet all requirements of Lemma 7.5 and hence satisfy all assumptions of Theorem 6.4. Applying this theorem we conclude that the sequence ϱ_ϵ converges to ϱ a.e. in Ω and $p(\varrho_\epsilon) \to \varrho^\gamma$ in $L^1_{\text{loc}}(\Omega)$. Therefore the functions $(\mathbf{u}, \varrho)$ serve as the generalized solution to problem (7.1), which completes the proof of Theorem 7.1. □

8. Estimate of a Green potential. Proof of Theorem 7.6

This section is devoted to the proof of Theorem 7.6. Before the presentation of the formal proof we outline the basic ideas of our method. The key observation is that in compressible fluids the energy tensor $\varrho\mathbf{u}\otimes\mathbf{u}+p\,\mathbf{I}$ is a nonnegative symmetric matrix. Using this property we can obtain a priori estimates by means of the choice of a test vector field φ in the integral identity (3.7a) determining a weak solution to the moment balance equation. Calculations show that in order to get the optimal result one should take $\varphi(x) = \Phi(x)$ with a convex "potential" $\Phi(x) = |x-y|$ depending of the current point $y \in \Omega$. As was shown in [19] and [51] these simple arguments lead to new internal a priori estimates for solutions to compressible Navier–Stokes equations.

The question on existence of a priori estimates near the boundary is more difficult since there are no nontrivial convex functions with gradients vanishing at the boundary. Nevertheless, we show that the choice of a potential Φ in the form $\Phi(x) = |x-y| + \Phi_0(x,y)$, where y is an arbitrary point of Ω and Φ_0 is some regular function, leads to pointwise estimates for Newtonian potential of p, which, together with the Corollary 4.4, give the efficient estimates for the density of the kinetic energy. The following theorem gives the explicit formulation of this result

THEOREM 8.1. *Assume that nonnegative functions $\varrho \in L^\gamma(\Omega)$, $p \in L^1(\Omega)$, vector fields $\mathbf{u} \in H_0^{1,2}(\Omega)$, $\mathbf{F} \in L^1(\Omega)$, and a tensor field $\mathbb{S} \in L^2(\Omega)$ satisfy the equation*

$$\operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla p = \mathbf{F} + \operatorname{div}\mathbb{S} \quad \text{in } \mathcal{D}'(\Omega), \tag{8.1}$$

and inequality $p \geq \varrho^\gamma$. Then for any $\iota \in [0, 1/2)$, there exist positive constants r and c depending only on ι, γ, Ω, and $\|\mathbb{S}\|_{L^2(\Omega)}$ such that for all $y \in \Omega$,

$$\int_\Omega d(x)\mathfrak{G}(x,y)p(x)\,dx \leq c\left(\int_{\Omega_r} p\,dx + \int_\Omega d^\iota\varrho|\mathbf{u}|^2dx + \sqrt{\mathbf{D}} + 1 + \|\mathbf{F}\|_{L^1(\Omega)}\right). \tag{8.2}$$

We emphasize that ϱ, p, $\mathbf{u}$, and $\mathbb{S}$ are independent and connected only by equation (8.1).

PROOF. In order to avoid technicalities we give the complete proof for the case when a domain Ω is a ball in $\mathbb{R}^3$. The extension to the case of bounded domains with smooth boundary is obvious. The proof is based on the following construction which reduces the original problem to an auxiliary problem in the half-space.

Further we shall assume that all coordinates are contravariant. Note that in the original Cartesian coordinate system x_i covariant and contravariant components of any object are coincident but after passing to the general curvilinear coordinates they become different. For any $t > 0$, denote the subdomain by Ω_t

$$\Omega_t = \{x \in \Omega : \ d(x) =: \operatorname{dist}(x, \partial\Omega) \geq t\}.$$

Since Ω is conformally equivalent to a half-space, for each point of $\partial\Omega$ there exists a standard neighborhood $\mathcal{U}_0$ and a conformal mapping $\mathcal{O} : y = y(x)$, which takes $\Omega \cap \mathcal{U}_0$ onto the cylinder

$$\mathcal{V} = \mathcal{D} \times [0, 2R], \quad \text{where } \mathcal{D} = \{(y^1, y^2) : |y^1|^2 + |y^2|^2 < R\}$$

such that $c^{-1}d(x) < y^3(x) < cd(x)$. A finite collection of standard neighborhoods $\mathcal{U}_k$ covers the set $\Omega \setminus \Omega_{2R}$. There exists also a finite collection of smooth cut-off functions $\chi_k : \mathcal{V} \to [0, 1]$ such that χ_k vanishes in a vicinity of the set $\partial\mathcal{V} \setminus \{y^3 = 0\}$. Moreover, the functions $\chi_k \circ \mathcal{O}$, being extended by zero onto the set $\Omega \setminus \mathcal{U}_k$, belong to the class $C^\infty(\Omega)$ and satisfy the equality

$$\sum_k \chi_k \circ \mathcal{O}_k = 1 \quad \text{in } \Omega \setminus \Omega_R.$$

Fix an arbitrary standard neighborhood and write $\mathcal{U}$ and χ instead of $\mathcal{U}_k$ and χ_k. We shall consider the velocity vector field $\mathbf{u}$ and the force vector field $\mathbf{F}$ as contravariant vector fields, and a test vector field φ as a covariant vector field. In the conformal coordinate system y^i their components are defined by the equalities

$$\bar{\mathbf{u}} = (\bar{u}^1, \bar{u}^2, \bar{u}^3), \quad \bar{\varphi} = (\bar{\varphi}_1, \bar{\varphi}_2, \bar{\varphi}_3),$$
$$\bar{u}^i(y) = \frac{\partial y^i}{\partial x^j}(x(y))u^j(x(y)), \quad \bar{\varphi}_i(y) = \frac{\partial x^j}{\partial y^i}(y)\varphi_j(x(y)).$$

If $\mathbf{u}$ and ϱ, p satisfy equation (8.1) and $\operatorname{spt}\varphi \subset \mathcal{U}$, $\varphi|_{\partial\Omega} = 0$, then $\bar{\mathbf{u}}$, ϱ and $\bar{\varphi}$ satisfy the integral identity

$$\int_{\mathcal{V}} \varrho \bar{u}^i \bar{u}^j \nabla_j \bar{\varphi}_i g^{1/2}\, dy + \int_{\mathcal{V}} p \operatorname{div}(g^{1/6}\bar{\varphi})\, dy - \int_{\mathcal{V}} g^{1/2} \nabla_i \bar{\varphi}_j\, \mathbb{S}^{ij}\, dy$$
$$+ \int_{\mathcal{V}} g^{1/2} \overline{\mathbf{F}} \bar{\varphi}\, dy = 0. \tag{8.3}$$

Here $\sqrt{g(y)} = \det x'(y)$, the covariant derivatives ∇_i are defined by

$$\nabla_i \bar{\varphi}_j = \frac{\partial \bar{\varphi}_j}{\partial y^i} - \Gamma^k_{ij} \bar{\varphi}_k, \quad \Gamma^k_{ij} = \frac{1}{6g}\left(\frac{\partial g}{\partial y^i}\delta_{jk} + \frac{\partial g}{\partial y^j}\delta_{ik} - \frac{\partial g}{\partial y^k}\delta_{ij}\right).$$

Let us turn to the proof of Theorem 8.1. It naturally falls into three steps. □

The first step. First we show that the function p does not oscillate near the boundary of Ω.

LEMMA 8.2. *Under the assumptions of Theorem 8.1 for any $\iota < 1/2$ there exist constants c and r_0, depending only on Ω and s such that for all $r \in (0, r_0]$,*

$$\int_{\{0<d<r\}} d(x)^{-\iota}(\varrho|\mathbf{u}_n|^2 + p)\, dx \le c \int_\Omega d^{1/2}(\varrho|\mathbf{u}|^2 + p)\, dx$$
$$+ c(\|\mathbb{S}\|_{L^2(\Omega)} + \|\mathbf{F}\|_{L^1(\Omega)}), \tag{8.4}$$

where $\mathbf{u}_n = \nabla d \cdot \mathbf{u}$.

PROOF. Introduce the family of functions depending on a parameter $0 < \delta < t$ and given by the formulae

$$\eta(y) = 1 \quad \text{for } y^3 < t, \quad \eta(y) = \frac{t + \delta - y^3}{\delta}$$
$$\text{for } t \le y^3 \le t + \delta, \quad \eta(y) = 0 \quad \text{for } t > 0.$$

Next set $\bar{\varphi}_1 = \bar{\varphi}_2 = 0, \quad \bar{\varphi}_3 = \eta(y)\chi(y)y^3$. Substituting $\bar{\varphi}$ into (8.3) leads to the equality

$$-\frac{1}{\delta}\int_{\mathcal{D}\times[t,t+\delta]} y^3\Psi\,dy + \int_{\mathcal{D}\times[0,t+\delta]} \eta(y)\Psi dy + \int_{\mathcal{D}\times[0,t+\delta]} y^3\eta(y)\Xi\,dy = 0, \tag{8.5}$$

where

$$\Psi = \chi(g^{1/2}\varrho(\bar{u}^3)^2 + pg^{1/6} - \overline{\mathbb{S}}^{33}g^{1/2}),$$
$$\Xi = \chi\Gamma^3_{ij}(\varrho\bar{u}^i\bar{u}^j - \overline{\mathbb{S}^{ij}})g^{1/2} + p\frac{\partial}{\partial y^3}(g^{1/6}\chi) + \chi\bar{F}^3g^{1/2} + \frac{\chi}{\partial y^i}(\overline{\mathbb{S}}^{3i} - \varrho\bar{u}^3\bar{u}^i)g^{1/2}.$$

It is easy to see that

$$c\chi(p + \varrho(\bar{u}^3)^2 - |\overline{\mathbb{S}}|) \le \Psi \le C\chi(p + \varrho(\bar{u}^3)^2 + |\overline{\mathbb{S}}|), \tag{8.6}$$
$$|\Xi| \le c(p + |\overline{\mathbf{F}}| + \varrho + |\overline{\mathbb{S}}| + \varrho|\bar{u}|^2), \tag{8.7}$$

where c, C depends only on Ω. Letting $\delta \to 0$ in integral identity (8.5), multiplying both sides of the obtained relation by t^{-2}, and integrating the result with respect to t over the interval $(r, 2R)$ we arrive at the identity

$$\frac{1}{r}\int_{\mathcal{D}\times[0,r]} \Psi\,dy = \frac{1}{2R}\int_{\mathcal{D}\times[0,2R]} \Psi\,dy - \int_r^{2R} \frac{1}{t^2}\left(\int_{\mathcal{D}\times[0,t]} y^3\Xi\,dy\right) dt.$$

Since

$$\int_r^{2R} \frac{1}{t^2}\left(\int_{\mathcal{D}\times[0,t]} y^3\Xi\,dy\right) dt = \int_{\mathcal{D}\times[0,r]} y^3\left(\frac{1}{r} - \frac{1}{2R}\right)\Xi\,dy + \int_{\mathcal{D}\times[r,2R]} \left(1 - \frac{y^3}{2R}\right)\Xi\,dy,$$

we conclude from this that

$$\frac{1}{r}\int_{\mathcal{D}\times[0,r]} |\Psi|\,dy \le \frac{1}{2R}\int_{\mathcal{D}\times[0,2R]} |\Psi|\,dy + \int_{\mathcal{D}\times[0,r]} y^3\left(\frac{1}{r} - \frac{1}{2R}\right)|\Xi|\,dy + \int_{\mathcal{D}\times[r,2R]} y^3\left(1 - \frac{y^3}{2R}\right)|\Xi|\,dy.$$

Using the obvious inequality

$$\frac{1}{\sqrt{r}}\int_{\mathcal{D}\times[0,r]}|\overline{\mathbb{S}}|\,dy \le c\|\overline{\mathbb{S}}\|_{L^2(\mathcal{V})} \le c\|\mathbb{S}\|_{L^2(\Omega)},$$

and invoking estimate (8.6) we obtain the inequality

$$\begin{aligned}\frac{1}{\sqrt{r}}\int_{\mathcal{D}\times[0,r]}\chi(p+\varrho(\bar u^3)^2)\,dy &\le c\|\mathbb{S}\|_{L^2(\Omega)} + \sqrt{r}\int_{\mathcal{D}\times[r,2R]}(p+\varrho(\bar u^3)^2)\,dy\\ &+ c\sqrt{r}\int_{\mathcal{D}\times[0,r]} y^3\left(\frac{1}{r}-\frac{1}{2R}\right)|\Xi|\,dy + c\sqrt{r}\int_{\mathcal{D}\times[r,2R]}\left(1-\frac{y^3}{2R}\right)|\Xi|\,dy,\end{aligned}$$

which along with the inequalities

$$\sqrt{r}y^3\left(\frac{1}{r}-\frac{1}{2R}\right) \le \sqrt{y^3} \quad \text{for } 0 < y^3 < r,$$

$$\sqrt{r}\left(1-\frac{y^3}{2R}\right) \le \sqrt{y^3} \quad \text{for } r < y^3 < 2R$$

implies the estimate

$$\begin{aligned}\frac{1}{\sqrt{r}}\int_{\mathcal{D}\times[0,r]}\chi(p+\varrho(\bar u^3)^2)\,dy &\le c\|\mathbb{S}\|_{L^2(\Omega)} + \sqrt{r}c\int_{\mathcal{D}\times[r,2R]}(p+\varrho(\bar u^3)^2)\,dy\\ &+ c\int_{\mathcal{V}}\sqrt{y^3}|\Xi|\,dy.\end{aligned}$$

From this and (8.7) we obtain

$$\begin{aligned}&\frac{1}{\sqrt{r}}\int_{\mathcal{D}\times[0,r]}\chi(p+\varrho(\bar u^3)^2)\,dy\\ &\quad\le c\|\mathbb{S}\|_{L^2(\Omega)} + c\int_{\mathcal{V}}\sqrt{y^3}(1+|\overline{\mathbf{F}}|+p+\varrho|\bar{\mathbf{u}}|^2)\,dy.\end{aligned}\tag{8.8}$$

Next note that for $\iota < 1/2$ any nonnegative function $f \in L^1(\mathcal{V})$, satisfies the inequality

$$\int_{\mathcal{D}\times[0,2R]}(y^3)^{-\iota} f\,dy \le \sup_{(0,2R)}\frac{c(\iota)}{\sqrt{r}}\int_{\mathcal{D}\times[0,r]} f\,dy + c(\iota,R)\int_{\mathcal{D}\times[0,2R]} f\,dy,$$

which together with (8.8) yields the estimate

$$\begin{aligned}&\int_{\mathcal{D}\times[0,2R]}(y^3)^{-\iota}\chi(p+\varrho(\bar u^3)^2)\,dy\\ &\quad\le c\|\mathbb{S}\|_{L^2(\Omega)} + c\int_{\mathcal{V}}\sqrt{y^3}(1+p+\varrho|\bar{\mathbf{u}}|^2+|\overline{\mathbf{F}}|)\,dy.\end{aligned}$$

After return to the original Cartesian coordinates x we finally obtain

$$\int_{\mathcal{U}} d^{-\iota}\tilde\chi(p+\varrho|\mathbf{u}_n|^2)\,dx \le c\|\mathbb{S}\|_{L^2(\Omega)} + c\int_{\Omega}\sqrt{d}(1+|\overline{\mathbf{F}}|+p+\varrho|\mathbf{u}|^2)\,dx.$$

Here $\tilde\chi = \chi\circ\mathcal{O}$ and $r \le R$. It remains to note that the functions $\tilde\chi$ form a partition of unity in the domain $\Omega\setminus\Omega_R$ and the lemma follows. □

COROLLARY 8.3. *Inequality* (8.4) *can be rewritten in the equivalent form*

$$\int_{\{0<d<r\}} d(x)^{-\iota}(\varrho|\mathbf{u}_n|^2 + p)\,dx \le c\int_\Omega d^{1/2}\varrho|\mathbf{u}|^2\,dx + c\int_{\Omega_r} p\,dx + c(\|\mathbb{S}\|_{L^2(\Omega)} + \|\mathbf{F}\|_{L^1(\Omega)} + 1). \tag{8.9}$$

The second step. Our next task is to estimate for the convolution of p and the green function of domain Ω near the boundary. To this end fix an arbitrary standard neighborhood $\mathcal{U}$ and corresponding diffeomorphism $\mathcal{O}:\mathcal{U}\mapsto\mathcal{V}$. For any $t\in\mathcal{V}$, set $\mathfrak{r}_- = |y-t|$,

$$\mathfrak{r}_+ = \sqrt{(y^1-t^1)^2 + (y^2-t^2)^2 + (y^3+t^3)^2},$$
$$\mathfrak{r} = \sqrt{(y^1-t^1)^2 + (y^2-t^2)^2 + (t^3)^2}.$$

LEMMA 8.4. *Under the assumptions of Theorem* 8.1 *there exists a constant c depending only on Ω such that*

$$\int_{\mathcal{V}}\left(\frac{1}{\mathfrak{r}_-} - \frac{1}{\mathfrak{r}_+}\right)\chi y^3 p\,dy \le c\int_\Omega d^{-\iota}\varrho(\nabla d\cdot\mathbf{u})^2\,dx + c\|\mathbb{S}\|_{L^2(\Omega)} + c\int_\Omega d^\iota(1 + |\mathbf{F}| + p + \varrho|\mathbf{u}|^2)\,dx. \tag{8.10}$$

PROOF. Introduce the auxiliary potentials given by the formulae

$$\Phi_t(y) = \mathfrak{r}_- + \mathfrak{r}_+,\quad \Phi_0(y) = h_b\left(\frac{y^3}{t^3}\right)\mathfrak{r},\quad \Phi = \Phi_t - 2\Phi_0,$$

in which

$$h(z) = 1 - bz \quad\text{when } 0\le z\le b^{-1},\qquad h(z) = 0\quad\text{when } z > b^{-1},$$

a positive constant $b > 4$ will be specified below. Introduce also the covariant vector field $\bar\varphi$ defined by the equalities

$$\bar\varphi_1(y) = \frac{\partial\Phi}{\partial y^1},\quad \bar\varphi_2(y) = \frac{\partial\Phi}{\partial y^2},\quad \bar\varphi_3(y) = \frac{\partial\Phi_t}{\partial y^3}. \tag{8.11}$$

It is clear that $\bar\varphi$ vanishes for $y^3 = 0$. Substituting $\chi(y)\bar\varphi(y)$ into (8.3) we obtain the integral identity

$$\int_{\mathcal{V}}\chi(\Psi_0(y) + \Psi_1(y))\,dy + \int_{\mathcal{V}}(\Theta_0(y) + \Theta_1(y))\,dy = 0, \tag{8.12}$$

where

$$\Psi_0 = \frac{\partial \bar{\varphi}_j}{\partial y^i} \varrho \bar{u}^j \bar{u}^i g^{1/2}, \quad \Psi_1 = p g^{1/6} \operatorname{div} \bar{\varphi}, \tag{8.13}$$

$$\Theta_0 = -\frac{\partial \bar{\varphi}_j}{\partial y^i} \overline{\mathbb{S}}^{ij} g^{1/2}, \tag{8.14}$$

$$\begin{aligned} \Theta_1 = {} & g^{1/2} (\varrho \bar{u}^i \bar{u}^j - \overline{\mathbb{S}}^{ij}) \left(\frac{\partial \chi}{\partial y^i} \bar{\varphi}_j - \chi \Gamma^k_{ij} \bar{\varphi}_k \right) \\ & + p \nabla (\chi g^{1/6}) \bar{\varphi} + g^{1/2} \chi \overline{\mathbf{F}} \bar{\varphi}. \end{aligned} \tag{8.15}$$

The further considerations are based on the following proposition.

PROPOSITION 8.5. *There are absolute positive constants b and c such that for all $y, t \in \mathbb{R}^2 \times \mathbb{R}^+$,*

$$|\bar{\varphi}| \le c \frac{y^3}{t^3}, \quad \left| \frac{\partial \bar{\varphi}_i}{\partial y^j} \right| \le c \left(\frac{1}{\mathfrak{r}_-} + \frac{1}{\mathfrak{r}_+} + \frac{1}{t^3} \right), \quad \operatorname{div} \bar{\varphi} \ge c^{-1} \frac{y^3}{t^3} \left(\frac{1}{\mathfrak{r}_-} - \frac{1}{\mathfrak{r}_+} \right), \tag{8.16}$$

and the quadratic form

$$\mathfrak{S}(y,t) z \cdot z := \sum_{i,j=1}^{3} \frac{\partial^2 \Phi_t}{\partial y^i \partial y^j} z^i z^j - 2 \sum_{i,j=1}^{2} \frac{\partial^2 \Phi_0}{\partial y^i \partial y^j} z^i z^j, \quad z \in \mathbb{R}^3,$$

is nonnegative.

PROOF. Note that the potentials Φ_i are homogeneous functions. Moreover, they are invariant with rotation around the vertical axis and shift in the horizontal direction. Therefore it suffices to prove the proposition for

$$y = (y^1, 0, y^3), \quad t = (0,0,1), \quad \mathfrak{r}_\pm = \sqrt{1 + (y^1)^2 + (y^3)^2 \pm 2y^3}, \\ \mathfrak{r} = \sqrt{1 + (y^1)^2}.$$

In this case the first two inequalities (8.16) are obviously true. In order to prove the third note that

$$\operatorname{div} \bar{\varphi} = \frac{2}{\mathfrak{r}_-} + \frac{2}{\mathfrak{r}_-} - h(y^3) \frac{4}{\mathfrak{r}} + \frac{2(y^1)^2}{\mathfrak{r}^3} \ge \frac{2}{\mathfrak{r}_-} + \frac{2}{\mathfrak{r}_-} - h(y^3) \frac{4}{\mathfrak{r}}.$$

Hence inequality (8.16) is trivial for $y^3 > b^{-1}$ when $h(y^3) = 0$. In the strip $0 < y^3 < b^{-1} < 1/4$ we have the identity

$$\operatorname{div} \bar{\varphi} - y^3 \left(\frac{1}{\mathfrak{r}_-} - \frac{1}{\mathfrak{r}_+} \right) \ge \frac{1}{\mathfrak{r}} \left(4by^3 + \frac{1}{\mathfrak{r}_-} + \frac{3}{\mathfrak{r}_+} - 4 \right) = \frac{y^3}{\mathfrak{r}} (4b + O(1)),$$

in which the quantity $O(1)$ is bounded by an absolute constant. It follows from this that inequalities (8.16) hold true for all sufficiently large b. In order to prove the nonnegativity of the quadratic form (8.5) note that under the above assumptions its coefficients are given by the formulae

$$\mathfrak{S}_{11} = \frac{(y^3-1)^2}{\mathfrak{r}_-^3} + \frac{(y^3+1)^2}{\mathfrak{r}_+^3} - 2h(y^3)\frac{1}{\mathfrak{r}^3}, \quad \mathfrak{S}_{22} = \frac{1}{\mathfrak{r}_-} + \frac{1}{\mathfrak{r}_+} - 2h(y^3)\frac{1}{\mathfrak{r}},$$
$$\mathfrak{S}_{33} = \frac{(y^1)^2}{\mathfrak{r}_-^3} + \frac{(y^1)^2}{\mathfrak{r}_+^3}, \quad \mathfrak{S}_{13} = -\frac{y^1(y^3-1)}{\mathfrak{r}_-^3} - \frac{y^1(y^3+1)}{\mathfrak{r}_+^3},$$

$\mathfrak{S}_{12} = \mathfrak{S}_{23} = 0$. It is clear that the quadratic form $\mathfrak{S}$ is defined positive when $h = 0$. On the other hand, in the strip $0 < y^3 < b^{-1}$, where h is not equal to zero, we have the inequality

$$\mathfrak{S}_{22} = \frac{1}{\mathfrak{r}}\left(2by^3 + \frac{\mathfrak{r}}{\mathfrak{r}_-} + \frac{\mathfrak{r}}{\mathfrak{r}_+} - 2\right),$$
$$\mathfrak{S}_{11}\mathfrak{S}_{33} - \mathfrak{S}_{13}^2 = \frac{2(y^1)^2(\mathfrak{r}_-^3 + \mathfrak{r}_+^3)}{\mathfrak{r}_-^3\mathfrak{r}_+^3\mathfrak{r}^3}\left(by^3 + \frac{2\mathfrak{r}^3}{\mathfrak{r}_-^3 + \mathfrak{r}_-^3} - 1\right),$$

which yields the positivity of $\mathfrak{S}$ for all sufficiently large b. □

Let us turn to the proof of Lemma 8.4. Assume that the vector field $\bar{\varphi}$ meets all requirements of Proposition 8.5. Our task is to estimate Ψ_i from below and Θ_i from above. The expression for $\bar{\varphi}$ implies the identity

$$\frac{\partial\bar{\varphi}_\mu}{\partial y^\tau}\varrho\bar{u}^\mu\bar{u}^\tau = \varrho\mathfrak{S}\bar{\mathbf{u}}\cdot\bar{\mathbf{u}} - \frac{4}{t^3}h'\left(\frac{y^3}{t^3}\right)\left(\frac{y^1-t^1}{\mathfrak{r}}\bar{u}^1 + \frac{y^2-t^2}{\mathfrak{r}}\bar{u}^2\right)\bar{u}^3\varrho$$

which along with Proposition 8.5 yields the estimate

$$\frac{\partial\bar{\varphi}_\mu}{\partial y^\tau}\varrho\bar{u}^\mu\bar{u}^\tau \geq -\frac{c}{t^3}\varrho|\bar{\mathbf{u}}||\bar{u}^3| \geq -\frac{c}{t^3}(y^3)^\iota\varrho|\bar{\mathbf{u}}|^2 - \frac{c}{t^3}(y^3)^{-\iota}\varrho|\bar{u}^3|^2.$$

Combining this result with inequalities (8.16) we obtain

$$\int_{\mathcal{V}}\chi(\Psi_0(y) + \Psi_1(y))\,dy \geq \frac{c}{t^3}\int_{\mathcal{V}}\left(\frac{1}{\mathfrak{r}_-} - \frac{1}{\mathfrak{r}_+}\right)y^3p\chi\,dy$$
$$- \frac{c}{t^3}\int_{\mathcal{V}}((y^3)^\iota\varrho|\bar{\mathbf{u}}|^2 + (y^3)^{-\iota}\varrho|\bar{u}^3|^2)\chi\,dy. \quad (8.17)$$

Next estimates (8.14) and (8.16) implies the inequality

$$|\Theta_0| \leq c\left(\frac{1}{\mathfrak{r}_-} + \frac{1}{\mathfrak{r}_+} + \frac{1}{t^3}\right)|\bar{\mathbb{S}}|,$$

which together with the Cauchy inequality implies the estimate

$$\int_{\mathcal{V}}|\Theta_0|\,dy \leq c\left(\int_{\mathcal{V}}(\mathfrak{r}_-^{-2} + \mathfrak{r}_+^{-2} + (t^3)^{-2})\,dy\int_{\mathcal{V}}|\bar{\mathbb{S}}|^2\,dy\right)^{1/2}$$
$$\leq \frac{c}{t^3}\|\bar{\mathbb{S}}\|_{L^2(\Omega)}. \quad (8.18)$$

In its turn, from (8.15), (8.16), and the obvious inequality

$$|\Theta_1| \le \frac{cy^3}{t^3}(|\overline{\mathbb{S}}| + \varrho|\bar{\mathbf{u}}|^2 + p + |\overline{\mathbf{F}}| + 1),$$

we obtain the estimate

$$\int_{\mathcal{V}} |\Theta_1|\,dy \le \frac{c}{t^3}\|\overline{\mathbb{S}}\|_{L^2(\Omega)} + \frac{c}{t^3}\int_{\mathcal{V}} y^3(\varrho|\bar{\mathbf{u}}|^2 + p + |\overline{\mathbf{F}}| + 1)\,dy. \tag{8.19}$$

It remains to note that substituting (8.17)–(8.19) into (8.12) leads to the desired estimate

$$\begin{aligned}\int_{\mathcal{V}}\left(\frac{1}{\mathfrak{r}_-} - \frac{1}{\mathfrak{r}_+}\right) y^3 p\chi\,dy &\le c\int_{\mathcal{V}} (y^3)^{-\iota}\varrho|\bar{u}^3|^2\chi\,dy + c\|\overline{\mathbb{S}}\|_{L^2(\Omega)} \\ &\quad + \int_{\mathcal{V}} y^3(\varrho|\bar{\mathbf{u}}|^2 + p + |\overline{\mathbf{F}}| + 1)\,dy \\ &\le c\int_{\Omega} d^{-\iota}\varrho|\bar{\mathbf{u}}_n|^2\,dx + c\|\overline{\mathbb{S}}\|_{L^2(\Omega)} \\ &\quad + c\int_{\Omega} d(\varrho|\mathbf{u}|^2 + p + |\mathbf{F}| + 1)\,dx. \quad \square\end{aligned}$$

Lemmas 8.2 and 8.4 imply the following estimate for the convolution of the Green function $\mathfrak{G}$ of domain Ω and the function p.

LEMMA 8.6. *Let R be a diameter of standard neighborhood depending only on Ω. Then there exist positive constants c, r depending only on Ω such that for any $z \in \Omega \setminus \Omega_{R/2}$,*

$$\begin{aligned}\int_{\Omega} d(x)\mathfrak{G}(x,z)p(x)\,dx &\le c\|\overline{\mathbb{S}}\|_{L^2(\Omega)} + c\int_{\Omega} d^{\iota}\varrho|\mathbf{u}|^2\,dx \\ &\quad + c\int_{\Omega_r} p\,dy + c\int_{\Omega}(|\mathbf{F}| + 1)\,dx.\end{aligned}$$

PROOF. Recalling Lemma 8.4 and Corollary 8.3 we get

$$\begin{aligned}\int_{\mathcal{V}}\left(\frac{1}{\mathfrak{r}_-} - \frac{1}{\mathfrak{r}_+}\right)\chi y^3 p\,dy &\le c\int_{\Omega} d^{\iota}\varrho|\mathbf{u}|^2\,dx \\ &\quad + c\int_{\Omega_r} p\,dx + c(\|\overline{\mathbb{S}}\|_{L^2(\Omega)} + \|\mathbf{F}\|_{L^1(\Omega)} + 1).\end{aligned}$$

Next note that for all $x, z \in \mathcal{U}$ and $y = y(x), t = y(z) \in \mathcal{V}$,

$$c^{-1}\mathfrak{G}(x,z) \le \left(\frac{1}{\mathfrak{r}_-} - \frac{1}{\mathfrak{r}_+}\right) \le c\,\mathfrak{G}(x,z).$$

Hence the estimate

$$\begin{aligned}\int_{\mathcal{U}} \mathfrak{G}(x,z)d(x)p(x)\tilde{\chi}(x)\,dx &\le c\int_{\Omega} d^{\iota}(\varrho|\mathbf{u}|^2\,dx \\ &\quad + c\int_{\Omega_r} p\,dx + c(\|\overline{\mathbb{S}}\|_{L^2(\Omega)} + \|\mathbf{F}\|_{L^1(\Omega)} + 1)\end{aligned}$$

holds true for all $z \in \mathcal{U}$. Now choose an arbitrary point $z \in \Omega \setminus \Omega_{R/2}$. Since the functions $\tilde{\chi}$ form the partition of unity in the domain $\Omega \setminus \Omega_R$ we have

$$\int_{\Omega\setminus\Omega_R} \mathfrak{G}(x,z)d(x)p(x)\,dx \le c\int_\Omega d^\iota(\varrho|\mathbf{u}|^2\,dx \\ + c\int_{\Omega_r} p\,dx + c(\|\overline{\mathbb{S}}\|_{L^2(\Omega)} + \|\mathbf{F}\|_{L^1(\Omega)} + 1).$$

It remains to note that $x \in \Omega_R$ and $z \in \Omega \setminus \Omega_{R/2}$, $\mathfrak{G}(x,z) \le c$ and hence

$$\int_{\Omega_R} \mathfrak{G}(x,z)d(x)p(x)\,dx \le c\int_\Omega d^\iota\varrho|\mathbf{u}|^2\,dx + c\int_{\Omega_r} p\,dx \\ + c(\|\overline{\mathbb{S}}\|_{L^2(\Omega)} + \|\mathbf{F}\|_{L^1(\Omega)} + 1). \quad \square$$

The third step. In order to complete the proof of Theorem 8.1 it remains to deduce the internal estimates for the Green potential of p. They result from the following

LEMMA 8.7. *Under the assumptions of Theorem 8.1 there is $r > 0$, depending only on Ω such that for any $z \in \Omega_{R/2}$,*

$$\int_\Omega \mathfrak{G}(x,z)p(x)\,dx \le c\|\mathbb{S}\|_{L^2(\Omega)} + c\int_\Omega d^\iota\varrho|\mathbf{u}|^2\,dx \\ + c\int_\Omega (p + 1 + |\mathbf{F}|)\,dx. \tag{8.20}$$

PROOF. Recall that $\Omega_{R/2} = \{\text{dist}\,(x, \partial\Omega) > r/2\}$. Choose an arbitrary $z \in \Omega_{R/2}$ and set $t = R/4$. Since $\mathfrak{G}(x,z) \le c|x-z|^{-1}$, we have

$$\int_\Omega \mathfrak{G}(x,z)p\,dx \le c\int_{B(z,t)} \frac{p\,dx}{|x-z|} + \frac{c}{t}\int_{\Omega\setminus B(z,t)} p\,dx \quad \text{and} \\ \int_\Omega \mathfrak{G}(x,z)|\overline{\mathbb{S}}|\,dx \le c\|\overline{\mathbb{S}}\|_{L^2(\Omega)}.$$

Next we use the identity

$$\int_{B(z,t)} \frac{\varrho}{|x-z|}(I - \mathbf{n}\otimes\mathbf{n}) : \mathbf{u}\otimes\mathbf{u}\,dx + \int_{B(z,t)} \frac{2}{|x-z|}p\,dx \\ = \int_{B(z,t)} \frac{1}{|x-z|}(I - \mathbf{n}\otimes\mathbf{n}) : \mathbb{S}\mathbf{u}\,dx \\ + \frac{1}{t}\int_{B(z,t)} (\varrho\mathbf{u}^2 + 3p - \operatorname{Tr}\mathbb{S})\,dx + \int_{B(z,t)} \varrho\left(\mathbf{n} - \frac{1}{t}(x-z)\right)\cdot\mathbf{F}\,dx = 0,$$

with $\mathbf{n} = (x-z)/|x-z|$, which yields the estimate

$$\int_{B(z,t),} \frac{\varrho}{|x-z|}(I - \mathbf{n}\otimes\mathbf{n}) : \mathbf{u}\otimes\mathbf{u}\,dx + \int_\Omega \frac{p}{|x-z|}\,dx \\ \le \frac{1}{t}\int_{B(y,t)} \varrho|\mathbf{u}|^2\,dx + c\int_\Omega p\,dx + c(\|\mathbb{S}\|_{L^2(\Omega)} + \|\mathbf{F}\|_{L^1(\Omega)} + 1). \tag{8.21}$$

Since $(I - \mathbf{n} \otimes \mathbf{n}) : \mathbf{u} \otimes \mathbf{u} \geq 0$ and we have $t^{-1} \leq 16d(x)R^{-2}$ for all $x \in B(z,t)$, the desired inequality (8.20) is the straightforward consequence of (8.21). □

In conclusion we note that the statement of Theorem 8.1 is a consequence of Lemma 8.6, 8.7 and Corollary 8.3.

Theorem 8.1 and Lemma 4.4 imply the following corollary which plays the key role in the proof of Theorem 7.6.

COROLLARY 8.8. *Let all the assumptions of Theorem 8.1 be satisfied. Then for any $0 < \iota < 1/2$ and $v \in H_0^{1,2}(\Omega)$,*

$$\int_\Omega d(x)p(x)|v|^2\,dx \leq c\biggl(\int_\Omega d(x)^\iota \varrho|\mathbf{u}|^2\,dx + \int_{\Omega_r} p\,dx + \|\mathbb{S}\|_{L^2(\Omega)} + \|\mathbf{F}\|_{L^1(\Omega)} + 1\biggr)\|v\|^2_{H^{1,2}(\Omega)}. \tag{8.22}$$

8.1. *Proof of Theorem 7.6*

We are now in a position to complete the proof of Theorem 7.6. It is based on the following lemma, which gives the auxiliary estimate for the pressure.

LEMMA 8.9. *Under the assumptions Theorem 7.6 for any $t \geq 0$ $1 < \sigma < 3/2$,*

$$\|p\|_{L^\sigma(\Omega_t)} \leq c(\|\varrho|\mathbf{u}|^2\|_{L^\sigma(\Omega_t)} + \sqrt{\mathbf{D}} + \|\mathbf{F}\|_{L^1(\Omega_t)} + \mathbf{M}). \tag{8.23}$$

PROOF. Recall that for any $\psi \in L^{\sigma/(\sigma-1)}(\Omega_t)$ with $\int_{\Omega_t} \psi\,dx = 0$ there exists a vector field φ satisfying the conditions

$$\Omega_t : \operatorname{div}\varphi = \psi, \qquad \partial\Omega_t : \varphi = 0$$
$$\|\nabla\varphi\|_{L^{\sigma/(\sigma-1)}(\Omega_t)} \leq c(\Omega_r)\|\psi\|_{L^{\sigma/(\sigma-1)}(\Omega_t)}, \qquad |\varphi| \leq c\|\psi\|_{L^{\sigma/(\sigma-1)}(\Omega_t)}.$$

Multiplying both the sides of relation (8.1) by φ and integrating the result over Ω_t we obtain the integral identity

$$-\int_{\Omega_t} ((\varrho\mathbf{u} \otimes \mathbf{u} - \mathbb{S}) : \nabla\varphi + \mathbf{F}\varphi) = \int_{\Omega_r} p\psi\,dx,$$

which along with the obvious inequality

$$\int_{\Omega_t} (|\varrho\mathbf{u} \otimes \mathbf{u} : \nabla\varphi| + |\mathbb{S}||\nabla\varphi|)\,dx \leq (\|\varrho|\mathbf{u}|^2\|_{L^\sigma(\Omega_t)} + \sqrt{\mathbf{D}})\|\nabla\varphi\|_{L^{\sigma/(\sigma-1)}(\Omega_t)}$$

yields the estimate

$$\int_{\Omega_t} p\psi\,dx \leq c(\|\varrho|\mathbf{u}|^2\|_{L^\sigma(\Omega_t)} + \sqrt{\mathbf{D}} + 1)c\|\psi\|_{L^{\sigma/(\sigma-1)}(\Omega_t)}.$$

Thus we get

$$\|p - (\text{ meas } \Omega_t)^{-1} \int_{\Omega_t} p\,dx\|_{L^\sigma(\Omega_r)} \leq (\|\varrho|\mathbf{v}|^2\|_{L^\sigma(\Omega_t)} + \sqrt{\mathbf{D}} + \|\mathbf{F}\|_{L^1(\Omega_t)}).$$

It remains to note that for any $\delta > 0$,

$$\int_{\Omega_t} p\,dx \leq \delta\|p\|_{L^\sigma(\Omega_r)} + c(\delta)\int_\Omega \varrho\,dx \leq c\delta\|p\|_{L^\sigma(\Omega_t)} + c(\delta)\mathbf{M}. \quad \square$$

Let us turn to the proof of Theorem 7.6. Choose a number σ satisfying the inequalities

$$1 < \sigma < 6/(10 - 3\gamma) \quad \text{for } 4/3 < \gamma < 2, \quad 1 < \sigma < 3/2 \quad \text{for } 2 < \gamma, \tag{8.24}$$

and set

$$\alpha = \frac{4\sigma - 3}{3\gamma - 2}, \quad \beta = \frac{\gamma\sigma - 2\sigma + 1}{3\gamma - 2}, \quad \tau = \frac{3\gamma - 2\sigma - \gamma\sigma}{3\gamma - 2}, \quad \iota = \frac{\alpha}{\sigma + \tau}. \tag{8.25}$$

It follows from (8.24) that

$$\alpha + \beta + \tau = 1, \quad (2\sigma - 2\alpha)/\beta = 6, \quad \alpha/\sigma < 1/2. \tag{8.26}$$

From this and the Holder inequality we obtain

$$\begin{aligned}&\Big(\int_\Omega (d^\iota\varrho|\mathbf{u}|^2)^\sigma\,dx\Big)^{1/\sigma}\\ &\quad\leq \Big(\int_\Omega d^{-\iota}\varrho\,dx\Big)^\kappa \Big(\int_\Omega dp|\mathbf{u}|^2\,dx\Big)^a \Big(\int_\Omega |\mathbf{u}|^6\,dx\Big)^{2(1-a)/6},\end{aligned} \tag{8.27}$$

where $\kappa = \tau/\sigma$, $a = \alpha/\sigma$. It is easily seen that $\iota = (5\gamma - 4)^{-1} > s$ for $\sigma = 1$. Hence we can choose σ so close to 1 that $\iota \leq s$. For such choice of σ, inequality (8.27) yields the estimate

$$\|d^\iota\varrho|\mathbf{u}|^2\|_{L^\sigma(\Omega)} \leq c\mathbf{M}^\kappa \Big(\int_\Omega dp|\mathbf{u}|^2\,dx\Big)^a \mathbf{D}^{(1-a)}. \tag{8.28}$$

By virtue of weak energy estimate (7.24), we have $\|\mathbb{S}\|_{L^2(\Omega)} \leq \sqrt{\mathbb{D}}$. From this, Corollary 8.8 and Lemma 8.9 we obtain

$$\begin{aligned}\|dp|\mathbf{u}|^2\|_{L^1(\Omega)} &\leq c(\|d^\iota\varrho|\mathbf{u}|^2\|_{L^\sigma(\Omega)} + \|p\|_{L^\sigma(\Omega_r)} + \sqrt{\mathbf{D}} + 1)\mathbf{D}\\ &\leq c(\|d^\iota\varrho|\mathbf{u}|^2\|_{L^\sigma(\Omega)} + \|\varrho|\mathbf{u}|^2\|_{L^\sigma(\Omega_r)} + \sqrt{\mathbf{D}} + 1 + \|\mathbf{F}\|_{L^1(\Omega)} + \mathbf{M})\mathbf{D}\\ &\leq c(\|d^\iota\varrho|\mathbf{u}|^2\|_{L^\sigma(\Omega)} + \sqrt{\mathbf{D}} + 1 + \|\mathbf{F}\|_{L^1(\Omega)} + \mathbf{M})\mathbf{D}.\end{aligned}$$

Substituting this inequality into (8.28) leads to the estimate

$$\mathcal{K} \leq c(\mathcal{K} + \mathbf{D}^{1/2} + \|\mathbf{F}\|_{L^1(\Omega)} + \mathbf{M} + 1)^a\mathbf{D}, \tag{8.29}$$

where $\mathcal{K} = \|d^\iota \varrho |\mathbf{u}|^2\|_{L^\sigma(\Omega)}$. Since $\iota \le s$, we have

$$\mathbf{K} \le c\|d^s \varrho |\mathbf{u}|^2\|_{L^\sigma(\Omega)} \le c\mathcal{K}.$$

From this and the weak energy estimate (7.24) we conclude that

$$\mathbf{D} \le c\sqrt{\mathcal{K}} + c\|\mathbf{F}\|_{L^1(\Omega)}. \tag{8.30}$$

Combining inequalities (8.29) and (8.30) we get

$$\mathcal{K} \le c(\mathcal{K} + \|\mathbf{F}\|_{L^1(\Omega)} + \mathbf{M} + 1)^a(\sqrt{\mathcal{K}} + \|\mathbf{F}\|_{L^1(\Omega)}).$$

Since, by virtue of (8.26), the exponent a is less than $1/2$, we finally obtain estimate

$$\|d^s \varrho |\mathbf{u}|^2\|_{L^\sigma(\Omega)} + \mathbf{D} \le \mathcal{K} + \mathbf{D} \le C(\Omega, s, \gamma, \mathbf{M}, \|\mathbf{F}\|_{L^1(\Omega)}). \tag{8.31}$$

To complete the proof of estimate (7.25) we note that the estimate for $\|p\|_{L^\sigma(\Omega_t)}$ follows from (8.31) and Lemma 8.9, and the estimate for $\|p\|_{L^1}(\Omega)$ follows from (8.31), estimate for $\|p\|_{L^\sigma(\Omega_t)}$, and inequality (8.9). □

9. Outflow–inflow problem

In this section we consider the questions on the local existence and uniqueness of the outflow–inflow problem in a smooth domain. For simplicity we restrict our considerations to the case of isothermal flow without mass and volume forces. The extension of the results on the case of barotropic flows is obvious. Note also that in the local theory the heat transfer does not create principal mathematical difficulties and the result also holds true for the Navier–Stokes–Fourier equations. Under these assumptions the problem can be formulated as follows. Let $\Omega \Subset \mathbb{R}^3$ be a bounded domain with the boundary $\partial\Omega \in C^\infty$, and $\mathbf{U} : \partial\Omega \mapsto \mathbb{R}^3$ be a given smooth vector field satisfying the compatibility condition

$$\int_{\partial\Omega} \mathbf{U} \cdot \mathbf{n}\, ds = 0. \tag{9.1}$$

The problem is to find the velocity field $\mathbf{u}$ and the density distribution ϱ satisfying the following equation and the boundary conditions:

$$\begin{aligned} &\Delta\mathbf{u} + \lambda\nabla \operatorname{div}\mathbf{u} = k\varrho\mathbf{u}\cdot\nabla\mathbf{u} + \sigma\nabla\varrho \quad \text{in } \Omega,\\ &\operatorname{div}(\varrho\mathbf{u}) = 0 \quad \text{in } \Omega,\\ &\mathbf{u} = \mathbf{U} \quad \text{on } \partial\Omega, \qquad \varrho = 1 \quad \text{on } \Sigma_{\text{in}}. \end{aligned}$$

Before formulation of the main results we write the governing equation in more transparent form using the change of unknown functions proposed in [48]. To do so we introduce *the* *effective viscous pressure* $q = \sigma\varrho - \lambda \operatorname{div} \mathbf{u}$, and rewrite equations (9.2) in the equivalent form

$$\Delta \mathbf{u} - \nabla q = k\varrho \mathbf{u}\nabla \mathbf{u} \quad \text{in } \Omega, \tag{9.2a}$$

$$\operatorname{div} \mathbf{u} = \frac{1}{\lambda}(\sigma\varrho - q) \quad \text{in } \Omega, \tag{9.2b}$$

$$\mathbf{u} = \mathbf{U} \quad \text{on } \partial\Omega, \tag{9.2c}$$

$$\mathbf{u} \cdot \nabla\varrho + \frac{\sigma}{\lambda}\varrho^2 = \frac{q\varrho}{\lambda} \quad \text{in } \Omega, \tag{9.2d}$$

$$\varrho = 1 \quad \text{on } \Sigma_{\text{in}}. \tag{9.2e}$$

We assume that $\lambda \gg 1$ and $k \ll 1$. In such a case, the *approximate solutions* to problem (9.2) can be chosen in the form $(1, \mathbf{u}_0, q_0)$, where $(\mathbf{u}_0, q_0)$ is a solution to the boundary value problem for the Stokes equations,

$$\begin{aligned} &\Delta \mathbf{u}_0 - \nabla q_0 = 0, \quad \operatorname{div} \mathbf{u}_0 = 0 \quad \text{in } \Omega, \\ &\mathbf{u}_0 = \mathbf{U} \quad \text{on } \partial\Omega, \quad \Pi q_0 = q_0. \end{aligned} \tag{9.3}$$

In our notations Π is the projector,

$$\Pi u = u - \frac{1}{\operatorname{meas} \Omega} \int_\Omega u \, dx.$$

Equations (9.3) can be obtained as the limit of equations (9.2) for the passage $\lambda \to \infty$, $k \to 0$. It follows from the standard elliptic theory that for the boundary $\partial\Omega \in C^\infty$, we have $(\mathbf{u}_0, q_0) \in C^\infty(\Omega)$. We look for solutions to problem (9.2) in the form

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{v}, \quad \varrho = 1 + \varphi, \quad q = q_0 + \sigma + \pi + \lambda m, \tag{9.4}$$

with the unknown functions $\Theta = (\mathbf{v}, \pi, \varphi)$ and the unknown constant m. Substituting (9.4) into (9.2) we obtain the following boundary problem for the vector-function Θ,

$$\begin{aligned} \Delta \mathbf{v} - \nabla\pi &= k\mathcal{B}(\varrho, \mathbf{u}, \mathbf{u}) \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{v} &= \sigma_\lambda\varphi - \Psi(\Theta) - m \quad \text{in } \Omega, \\ \mathbf{u} \cdot \nabla\varphi + \sigma_\lambda\varphi &= \Psi_1(\Theta) + \varrho m \quad \text{in } \Omega, \\ \mathbf{v} = 0 \quad \text{on } \partial\Omega, \quad \varphi &= 0 \quad \text{on } \Sigma_{\text{in}}, \quad \Pi\pi = \pi, \end{aligned} \tag{9.5a}$$

where

$$B(\varrho, \mathbf{v}, \mathbf{w}) = \varrho\mathbf{v}\nabla\mathbf{w}, \quad \varrho_\lambda = \varrho/\lambda,$$

$$\Psi_1[\Theta] = \varrho\Psi[\mu] - \sigma_\lambda\varphi^2, \quad \Psi[\Theta] = \frac{q_0 + \pi}{\lambda},$$

the vector field $\mathbf{u}$ and the function ϱ are given by (9.4). Finally, we specify the constant m. In our framework, in contrast to the case of homogeneous boundary problem, the solution to such a problem is not trivial since this problem is connected with the difficult question of mass control in outflow and/or inflow problems. Recall that the absence of mass control is the main obstacle for proving the global solvability of inhomogeneous boundary problems for compressible Navier–Stokes equations, we refer to [34] for a discussion. In order to cope with this difficulty we write the compatibility condition in a sophisticated form, which allows us to control the total mass of the gas. To this end we introduce the auxiliary function ζ satisfying the equations

$$-\operatorname{div}(\mathbf{u}\zeta) + \sigma_\lambda \zeta = \sigma_\lambda \quad \text{in } \Omega, \qquad \zeta = 0 \quad \text{on } \Sigma_{\text{out}}, \tag{9.5b}$$

and fix the constant m as follows

$$m = \varkappa \int_\Omega (\Psi_1[\Theta]\zeta - \Psi[\Theta])\,dx, \qquad \varkappa = \left(\int_\Omega (1 - \zeta - \zeta\varphi)\,dx\right)^{-1}. \tag{9.5c}$$

In this way the auxiliary function ζ becomes an integral part of the solution to the problem. Now, our aim is to prove the existence and uniqueness of solutions to problem (9.5).

9.1. *Existence result*

For an arbitrary bounded domain $\Omega \subset \mathbb{R}^3$ with a Lipschitz boundary, we introduce the Banach spaces

$$X^{s,r} = H^{s,r}(\Omega) \cap H^{1,2}(\Omega), \qquad Y^{s,r} = H^{s+1,r}(\Omega) \cap H^{2,2}(\Omega),$$
$$Z^{s,r} = \mathcal{H}^{s-1,r}(\Omega) \cap L^2(\Omega)$$

equipped with the norms

$$\|u\|_{X^{s,r}} = \|u\|_{H^{s,r}(\Omega)} + \|u\|_{H^{1,2}(\Omega)}, \qquad \|u\|_{Y^{s,r}} = \|u\|_{H^{1+s,r}(\Omega)} + \|u\|_{H^{2,2}(\Omega)},$$
$$\|u\|_{Z^{s,r}} = \|u\|_{\mathcal{H}^{s-1,r}(\Omega)} + \|u\|_{L^2(\Omega)}.$$

It can be easily seen that the embeddings $Y^{s,r} \hookrightarrow X^{s,r} \hookrightarrow Z^{s,r}$ are compact and for $sr > 3$, each of the spaces $X^{s,r}$ and $Y^{s,r}$ is a commutative Banach algebra. Denote by E the closed subspace of the Banach space $Y^{s,r}(\Omega)^3 \times X^{s,r}(\Omega)^2$ in the following form

$$E = \{\vartheta = (\mathbf{v}, \pi, \varphi) : \mathbf{v} = 0 \text{ on } \partial\Omega, \varphi = 0 \text{ on } \Sigma_{\text{in}}, \Pi\pi = \pi\}, \tag{9.6}$$

and denote by $\mathcal{B}_\tau \subset E$ the closed ball of radius τ centered at 0. Next, note that for $sr > 3$, elements of the ball $\mathcal{B}_\tau$ satisfy the inequality

$$\|\mathbf{v}\|_{C^1(\Omega)} + \|\pi\|_{C(\Omega)} + \|\varphi\|_{C(\Omega)} \le c_e(r, s, \Omega)\|\Theta\|_E \le c_e\tau, \tag{9.7}$$

where the norm in E is defined by

$$\|\Theta\|_E = \|\mathbf{v}\|_{Y^{s,r}(\Omega)} + \|\pi\|_{X^{s,r}(\Omega)} + \|\varphi\|_{X^{s,r}(\Omega)}.$$

THEOREM 9.1. *Assume that the surface $\partial\Omega$ and given vector field $\mathbf{U}$ satisfy the emergent field condition* (**H1**)–(**H3**). *Furthermore, let σ^*, R be constants given by Theorem 5.7,* *and let positive numbers r, s, σ_λ, r_0 meet all requirements of this theorem and satisfy the inequalities*

$$1/2 < s \le 1, \qquad sr > 3, \sigma_\lambda > \sigma^*. \tag{9.8}$$

Then there exists $\tau_0 \in (0, R]$, depending only on $\mathbf{U}, \Omega, r, s, \sigma_\lambda$, such that for all

$$\tau \in (0, \tau_0], \quad \lambda^{-1}, k \in (0, \tau^2], \tag{9.9}$$

problem (9.5), with $\mathbf{u}_0$ given by (9.3), has a solution $\Theta \in B_\tau$. Moreover, the auxiliary function ζ and the constants $\varkappa, m$ admit the estimates

$$\|\zeta\|_{X^{s,r}} + |\varkappa| \le c, \quad |m| \le c\tau < 1, \tag{9.10}$$

where the constant c depends only on $\mathbf{U}, \Omega, r, s$ and σ_λ.

The proof is based on the following lemma which is a straightforward consequence of the classical results on solvability of first boundary value problem for the Stokes equations (see [13]) and the interpolation theory.

LEMMA 9.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $\partial\Omega \in C^2$ and $(F, G) \in \mathcal{H}^{s-1,r}(\Omega) \times H^{s,r}(\Omega)$ $(0 \le s \le 1, 1 < r < \infty)$ Then the boundary value problem*

$$\begin{aligned} &\Delta \mathbf{v} - \nabla\pi = F, \quad \operatorname{div} \mathbf{v} = \Pi G \quad \text{in } \Omega, \\ &\mathbf{u} = 0 \text{ on } \partial\Omega, \quad \Pi\pi = \pi, \end{aligned} \tag{9.11}$$

has a unique solution $(\mathbf{v}, \pi) \in H^{s+1,r}(\Omega) \times H^{s,r}(\Omega)$ such that

$$\|\mathbf{v}\|_{H^{s+1,r}(\Omega)} + \|\pi\|_{H^{s,r}(\Omega)} \le c(\Omega, r, s)(\|F\|_{\mathcal{H}^{s-1,r}(\Omega)} + \|G\|_{H^{s,r}(\Omega)}). \tag{9.12}$$

In particular, we have

$$\|\mathbf{v}\|_{Y^{s,r}} + \|\pi\|_{X^{s,r}} \le c(\Omega, r, s)(\|F\|_{Z^{s,r}} + \|G\|_{X^{s,r}}).$$

PROOF. Note that, by virtue of Theorem 6.1 in [13], for any $\mathbf{F} \in \mathcal{H}^{s-1,r}(\Omega)$ and $G \in H^{s,r}(\Omega)$ with $s = 0, 1$, problem (9.11) has a unique solution $\mathbf{v}, \pi$ satisfying inequality

$$\|\mathbf{v}\|_{H^{s+1,r}(\Omega)} + \|\pi\|_{H^{s,r}(\Omega)} \le c(\Omega, r, s)(\|\mathbf{F}\|_{\mathcal{H}^{s-1,r}(\Omega)} + \|G\|_{H^{s,r}(\Omega)}).$$

Thus the relation $(F, G) \mapsto (\mathbf{v}, \pi)$ determines the linear operator $T : \mathcal{H}^{s-1,r}(\Omega) \times H^{s,r}(\Omega) \mapsto H^{s+1,r}(\Omega) \times H^{s,r}(\Omega)$. Therefore, Lemma 9.2 is a consequence of Lemma 4.1. □

Let us turn to the proof of Theorem 9.1. We solve problem (9.5) by an application of the Schauder fixed-point theorem in the following framework. Fix $\sigma_\lambda > \sigma^*$. Next choose an arbitrary element $\Theta \in \mathcal{B}_\tau$. Since $\tau < R$, Theorem 5.7 implies that the problem

$$\mathbf{u} \cdot \nabla\varphi_1 + \sigma_\lambda\varphi_1 = \Psi_1[\Theta] + m\varrho \quad \text{in } \Omega, \quad \varphi_1 = 0 \quad \text{on } \Sigma_{\text{in}} \tag{9.13}$$

has a unique solution satisfying the inequality

$$\|\varphi_1\|_{X^{s,r}} \le c(\Omega, \mathbf{U}, \sigma, r, s)(\|\Psi_1[\vartheta]\|_{X^{s,r}} + |m|). \tag{9.14}$$

Next, define $\mathbf{v}_1$ and π_1 to be the solutions of the boundary problem for the Stokes equations

$$\begin{aligned} \Delta\mathbf{v}_1 - \nabla\pi_1 &= k\,\mathcal{B}(\varrho, \mathbf{u}, \mathbf{u}) \equiv F[\Theta] \quad \text{in } \Omega \\ \operatorname{div}\mathbf{v}_1 &= \Pi(\sigma_\lambda\varphi_1 - \Psi[\Theta] - m) \quad \text{in } \Omega, \\ \mathbf{v}_1 &= 0 \quad \text{on } \partial\Omega, \quad \pi_1 - \Pi\pi_1 = 0, \end{aligned} \tag{9.15}$$

where m is given by (9.5c). By Lemma 9.2, this problem admits a unique solution such that

$$\|\mathbf{v}_1\|_{Y^{s,r}} + \|\pi_1\|_{X^{s,r}} \le c(\|F[\Theta]\|_{Z^{s,r}} + |\Psi[\Theta]|_{X^{s,r}} + \|\varphi_1\|_{X^{s,r}} + |m|). \tag{9.16}$$

Equations (9.13), (9.15), (9.5c), define the mapping $\Xi : \Theta \to \Theta_1 = (\mathbf{v}_1, \pi_1, \varphi_1)$. We claim that for a suitable choice of the constant τ, Ξ is a weakly continuous automorphism of the ball $\mathcal{B}_\tau$. We begin with the estimates for nonlinear operators present in (9.13). Fix an arbitrary $\Theta \in \mathcal{B}_\tau$. We have

$$\|\Psi[\Theta]\|_{X^{s,r}} \le \frac{c}{\lambda}(\|q_0\|_{C^1(\Omega)} + \|\pi\|_{X^{s,r}}) \le c/\lambda \le c\tau^2. \tag{9.17}$$

Since, under assumptions of Theorem 9.1, $X^{s,r}(\Omega)$ is a Banach algebra and $\|\varrho\|_{X^{s,r}} \le c + \|\varphi\|_{X^{s,r}} \le$ const., we conclude from this and (9.14) that

$$\|\varphi_1\|_{X^{s,r}} \le c/\lambda + c\tau^2 + c|m| \le c\tau^2 + c|m|. \tag{9.18}$$

Recall that the operator $\mathcal{B}$ constitutes a cubic polynomial of $\mathbf{u}$, $\nabla\mathbf{u}$ and ϱ, which along with the inequalities $k < \tau^2 < 1$ yields

$$\|\mathcal{B}(\varrho, \mathbf{u}, \mathbf{u})\|_{X^{s,r}} \le ck(1 + \|\varrho\|_{X^{s,r}} + \|\mathbf{u}\|_{Y^{s,r}})^3 \le c\tau^2 \quad \text{in } \mathcal{B}_\tau. \tag{9.19}$$

Inequalities (9.9) and (9.19) imply

$$\|F[\vartheta]\|_{Z^{s,r}} \le c\tau^2(1 + \tau) \quad \text{in } \mathcal{B}_\tau. \tag{9.20}$$

Combining inequalities (9.17) and (9.18) we get the estimate

$$\|\sigma_\lambda\varphi_1 + \Psi[\Theta]\|_{X^{s,r}} \le c\tau^2 + c|m|.$$

From this, (9.20), (9.16) and Lemma 9.2 we finally obtain

$$\|\mathbf{v}_1\|_{Y^{s,r}} + |\pi_1|_{X^{s,r}} \le c\tau^2 + c|m|. \tag{9.21}$$

It remains to estimate m. Recall that the vector field $\mathbf{u}$ and parameter σ_λ meet all requirements of Theorem 5.7. Therefore, problem (9.5b) has the unique solution $\zeta \in$

$H^{s,r}(\Omega)$. In particular, inequalities (5.21) yield the estimate $\|\zeta\|_{X^{s,r}} \le c$. Since $sr > 3$, by virtue of the embedding theory the embedding $H^{s,r}(\Omega) \hookrightarrow C^\beta(\Omega)$ is bounded for some $\beta \in (0, 1)$. Hence estimates (5.17) and (5.22) for $rs > 3$ yield

$$\|\zeta\|_{C^\beta(\Omega)} + \|\zeta\|_{H^{1,r_0}(\Omega)} \le C(\mathbf{U}, \Omega, \sigma_\lambda). \tag{9.22}$$

Recalling that $\operatorname{div} \mathbf{u} = \operatorname{div} \mathbf{v}$, we obtain $|\operatorname{div} \mathbf{u}| \le c\tau$. From this and the maximum principle (5.22) we conclude that

$$\|\zeta\|_{C(\Omega)} \le (1 - \sigma_\lambda^{-1} c\tau)^{-1} \le (1 - c\tau)^{-1}, \tag{9.23}$$

which leads to the following estimate

$$|1 - \zeta| \le c\tau(1 - c\tau)^{-1}.$$

Now we can estimate the right-hand side of (9.5c). Rewrite the first integral in the form

$$\int_\Omega (1 - \zeta - \zeta\varphi)\, dx = \int_\Omega (1 - \zeta)^+\, dx - \int_\Omega ((1 - \zeta)^- \zeta\varphi)\, dx.$$

We have

$$|(1 - \zeta)^- + \zeta\varphi)| \le c_e\tau + c\tau(1 - c\tau)^{-1}.$$

On the other hand, we have $\|(1 - \zeta)^+\|_{C^\beta(\Omega)} \le c(\mathbf{U}, \Omega, \sigma)$ and $(1 - \zeta)^+ = 1$ on Σ_{out}. Hence

$$\int_\Omega (1 - \zeta)^+\, dx > \kappa(\mathbf{U}, \Omega, \sigma) > 0.$$

Thus, we get

$$\varkappa^{-1} \ge \kappa(1 - c\kappa^{-1}\tau(1 - c\tau)^{-1}).$$

In particular, there is a positive τ_0 depending only on $\mathbf{U}$, Ω and σ_λ, such that

$$|\varkappa^{\pm 1}| \le c \quad \text{for all } \tau \le \tau_0.$$

Repeating these arguments and using inequality (9.17), and relation (9.5c), we arrive at $|m| \le c\tau^2$. Combining this estimate with (9.18) and (9.21), we finally obtain $\|\Theta_1\|_{X^{s,r}} \le c\tau^2$. Choose sufficiently small $\tau_0 = \tau_0(\mathbf{U}, \Omega, \sigma_\lambda)$, so that $c\tau_0^2 < \tau_0$. Thus, for all $\tau \le \tau_0$, Ξ maps the ball $\mathcal{B}_\tau$ into itself. Let us show that Ξ is weakly continuous. Choose an arbitrary sequence $\Theta_n \in \mathcal{B}_\tau$ such that $\Theta_n = (\mathbf{v}_n, \pi_n, \varphi_n)$ converges weakly in E to some Θ. Since the ball $\mathcal{B}_\tau$ is closed and convex, Θ belongs to $\mathcal{B}_\tau$. Let us consider the corresponding sequences of the elements $\Theta_{1,n} = \Xi(\Theta_n) \in \mathcal{B}_\tau$ and functions ζ_n. There are subsequences $\{\Theta_{1,j}\} \subset \{\Theta_{1,n}\}$ and $\{\zeta_j\} \subset \{\zeta_n\}$ such that $\Theta_{1,j}$ converges weakly in E to some element $\Theta_1 \in \mathcal{B}_\tau$ and ζ_j converges weakly in $X^{s,r}$ to some function $\zeta \in X^{s,r}$. Since the embedding $E \hookrightarrow C(\Omega)^5$ is compact, we have $\Theta_n \to \Theta$, $\Theta_{1j} \to \Theta_1$ in $C(\Omega)^5$, and

$$\nabla\zeta_j \rightharpoonup \nabla\zeta \quad \text{weakly in } L^{r_0}(\Omega), \qquad \zeta_j \to \zeta \quad \text{in } C(\Omega).$$

Substituting Θ_j and $\Theta_{1,j}$ into equations (9.13), (9.15), (9.5c) and letting $j \to \infty$ we obtain that the limits Θ and Θ_1 also satisfy (9.13), (9.15), (9.5c). Thus, we get $\Theta_1 = \Xi(\vartheta)$. Since for given Θ, a solution to equations (9.13), (9.15) is unique, we conclude from this that all weakly convergent subsequences of $\Theta_{1,n}$ have the unique limit Θ_1. Therefore, the whole sequence $\Theta_{1,n} = \Xi(\Theta_n)$ converges weakly to $\Xi(\vartheta)$. Hence the mapping $\Xi : \mathcal{B}_\tau \mapsto \mathcal{B}_\tau$ is weakly continuous and, by virtue of the Schauder fixed-point theorem, there is $\Theta \in \mathcal{B}(\tau)$ such that $\Theta = \Xi(\vartheta)$.

It remains to prove that Θ is given by a solution to problem (9.5a). For $\Theta_1 = \Theta$, the only difference between problems (9.5a) and (9.15), (9.5c) is the presence of the projector Π on the right-hand side of (9.15). Hence, it suffices to show that

$$\Pi(\sigma_\lambda \varphi - \Psi[\Theta] - m) = \sigma_\lambda \varphi - \Psi[\Theta] - m. \tag{9.24}$$

To this end we note that φ is a strong continuous solution to the transport equation

$$\mathbf{u} \cdot \nabla \varphi + \sigma_\lambda \varphi = \Psi_1[\Theta] + m\varrho.$$

Multiplying both the sides by ζ and integrating by parts we obtain

$$\sigma_\lambda \int_\Omega \varphi\, dx = \int_\Omega \zeta(\Psi_1[\Theta] + m\varrho)\, dx.$$

On the other hand, equality (9.5c) reads

$$\int_\Omega \zeta(\Psi_1[\Theta] + m(1 + \varphi))\, dx = \int_\Omega (\Psi[\Theta] + m)\, dx.$$

Combining these equalities and noting that $1 + \varphi = \varrho$ we obtain

$$\int_\Omega (\sigma_\lambda \varphi - \Psi[\Theta] - m)\, dx = 0$$

which yields (9.24) and the proof of Theorem 9.1 is completed. □

9.2. *Uniqueness*

In this paragraph we prove that under the assumptions of Theorem 9.1 a solution to problem (9.5) is unique, in the ball $\mathcal{B}_\tau$. Assume, contrary to our claim that there exist two different solutions $(\Theta_i, \zeta_i, m_i) \in E \times X^{s,r} \times \mathbb{R}$, $i = 1, 2$, with $\Theta_i \in \mathcal{B}_\tau$ to problem (9.5) with $\Theta_i \in \mathcal{B}_\tau$. Recall that they, together with the constants $\varkappa_i$, satisfy the inequalities

$$|m_i| + \|\Theta_i\|_E \le c\tau, \quad |\varkappa_i| + \|\zeta\|_{X^{s,r}} \le c, \tag{9.25}$$

where the constant c depends only on $\mathbf{U}$, Ω, r, s, and σ_λ. We denote $\mathbf{u}_i = \mathbf{u}_0 + \mathbf{v}_i$, $i = 1, 2$, the corresponding solutions to (9.2) $i = 1, 2$. Now set

$$\mathbf{w} = \mathbf{v}_1 - \mathbf{v}_2, \quad \omega = \pi_1 - \pi_2, \quad \psi = \varphi_1 - \varphi_2, \quad \xi = \zeta_1 - \zeta_2,$$
$$n = m_1 - m_2.$$

It follows from (9.5) that these quantities satisfy the linear equations

$$
\begin{aligned}
\mathbf{u}_1\nabla\psi + \sigma_\lambda\psi = -\mathbf{w}\cdot\nabla\varphi_2 + b_{11}\psi + b_{12}\omega + b_{13}n \quad &\text{in } \Omega,\\
\Delta\mathbf{w} - \nabla\omega = k\mathcal{C}_1(\psi, \mathbf{w}) \quad &\text{in } \Omega,\\
\operatorname{div}\mathbf{w} = b_{21}\psi + b_{22}\omega + b_{23}n \quad &\text{in } \Omega,\\
-\operatorname{div}(\mathbf{u}_1\xi) + \sigma_\lambda\xi = \operatorname{div}(\zeta_2\mathbf{w}) + \quad &\text{in } \Omega,\\
\mathbf{w} = 0 \quad \text{on } \partial\Omega, \qquad \psi = 0 \quad \text{on } \Sigma_{\text{in}}, \qquad \xi = 0 \quad &\text{on } \Sigma_{\text{out}},\\
\omega - \Pi\omega = 0, \qquad n = \varkappa\int_\Omega (b_{31}\psi + b_{32}\omega + b_{34}\xi)\,dx&.
\end{aligned}
\tag{9.26}
$$

Here the coefficients are given by the formulae

$$
\begin{aligned}
&b_{11} = \Psi[\Theta_1] + m_2 - \sigma_\lambda(\varphi_1 + \varphi_2),\\
&b_{12} = \lambda^{-1}\varrho_2, \quad b_{13} = \varrho_1, \quad b_{21} = \sigma_\lambda, \quad b_{22} = -1/\lambda, \quad b_{23} = -1,\\
&b_{31} = \zeta_1\Psi[\Theta_1] - \sigma_\lambda\zeta_1(\varphi_1 + \varphi_2) + m_2\zeta_2,\\
&b_{32} = \zeta_1 b_{12} - b_{22}, \quad b_{34} = \Psi_1[\Theta_2] + m_2(1 + \varphi_1),
\end{aligned}
\tag{9.27}
$$

and the operator $\mathcal{C}$ is defined by the equalities

$$
\mathcal{C}(\psi, \mathbf{w}) = \mathcal{B}(\psi, \mathbf{u}_1, \mathbf{u}_1) + \mathcal{B}(\varrho_2, \mathbf{w}, \mathbf{u}_1) + \mathcal{B}(\varrho_2, \mathbf{u}_2, \mathbf{w}).
$$

We consider relations (9.26) as the system of equations and boundary conditions for unknowns $\mathbf{w}$, ψ, ξ, and n. The next step is crucial for further analysis. We replace equations (9.26) by an integral identity, which leads to the notion of a very weak solution of problem (9.26). To this end choose an arbitrary function $(\mathbf{H}, G, F, M) \in C^\infty(\Omega)^6$ such that $G - \Pi G = 0$, and consider the auxiliary boundary value problems

$$
\mathcal{L}^*\varsigma = F, \quad \mathcal{L}\upsilon = M \quad \text{in } \Omega, \quad \varsigma = 0 \quad \text{on } \Sigma_{\text{out}}, \quad \upsilon = 0 \quad \text{on } \Sigma_{\text{in}}. \tag{9.28}
$$

$$
\Delta\mathbf{h} - \nabla g = \mathbf{H}, \quad \operatorname{div}\mathbf{h} = \Pi G \quad \text{in } \Omega, \quad \mathbf{h} = 0 \quad \text{on } \partial\Omega, \quad \Pi g = g, \tag{9.29}
$$

where $\mathcal{L} =: \mathbf{u}\nabla + \sigma_\lambda$. Since under the assumptions of Theorem 9.1, $\mathbf{u}$ and σ_λ meet all requirements of Theorem 5.7, each of problems (9.28) has a unique solution, such that

$$
\|\varsigma\|_{H^{s,r}(\Omega)} \le c\|F\|_{H^{s,r}(\Omega)}, \qquad \|\upsilon\|_{H^{s,r}(\Omega)} \le c\|M\|_{H^{s,r}(\Omega)}, \tag{9.30}
$$

where c depends only on $\mathbf{U}$, Ω, r, s, and σ_λ. On the other hand, by virtue of Lemma 9.2, problem (9.29) has the unique solution satisfying the inequality

$$
\|\mathbf{h}\|_{H^{1+s,r}(\Omega)} + \|g\|_{H^{s,r}(\Omega)} \le c\|\mathbf{H}\|_{H^{1+s,r}(\Omega)} + c\|G\|_{H^{s,r}(\Omega)}. \tag{9.31}
$$

Recall that $\mathbf{w} \in H^{2,r_0}(\Omega)^3 \cap C^1(\Omega)^3$ vanishes on $\partial\Omega$, and $(\omega, \psi, \xi) \in H^{1,r_0}(\Omega)^3 \cap C(\Omega)^3$, where $1 < r_0 < \infty$ is the exponent in the definition of $X^{s,r}$. Multiplying both sides of the first equation in system (9.26) by ς, both sides of the fourth equation in (9.26) by υ, integrating the results over Ω and using the Green formula for the Stokes equations we obtain the system of integral equalities

$$\begin{aligned}
&\int_\Omega \psi F\,dx = \int_\Omega (-\mathbf{w}\cdot\nabla\varphi_2 + b_{11}\psi + b_{12}\omega + b_{13}n)\varsigma\,dx,\\
&\int_\Omega \mathbf{w}\mathbf{H}\,dx + \int_\Omega \omega G\,dx = \int_\Omega (b_{21}\psi + b_{22}\omega + b_{23}n)g\,dx\\
&\quad + \int_\Omega k\mathcal{C}(\mathbf{w},\psi)\mathbf{h}\,dx, \qquad \int_\Omega \xi M\,dx = \int_\Omega \operatorname{div}(\zeta_2\mathbf{w})\upsilon\,dx.
\end{aligned}\tag{9.32}$$

Next, since $\operatorname{div}(\varrho_2\mathbf{u}_2) = 0$, we have

$$\begin{aligned}
&\int_\Omega (\mathcal{B}(\varrho_2,\mathbf{w},\mathbf{u}_1) + \mathcal{B}(\varrho_2,\mathbf{u}_2,\mathbf{w}))\cdot\mathbf{h}\,dx\\
&\quad = \int_\Omega \varrho_2\mathbf{w}\cdot(\nabla\mathbf{u}_1\mathbf{h} - (\nabla\mathbf{h})^*\mathbf{u}_2)\,dx.
\end{aligned}$$

On the other hand, integration by parts gives

$$\int_\Omega \operatorname{div}(\zeta_2\mathbf{w})\upsilon\,dx = -\int_\Omega \zeta_2\mathbf{w}\nabla\upsilon\,dx.$$

Using these identities and recalling the duality pairing we can collect relations (9.32), together with the expression for n, in one integral identity

$$\begin{aligned}
&\int_\Omega \mathbf{w}(\mathbf{H} - k\varrho_2\nabla\mathbf{u}_1\cdot\mathbf{h} + k\varrho_2(\nabla\mathbf{h})^*\mathbf{u}_2)\,dx\\
&\qquad - \mathfrak{B}(\mathbf{w},\varphi_2,\varsigma) - \mathfrak{B}(\mathbf{w},\upsilon,\zeta_2) + \langle\omega, G - b_{12}\varsigma - b_{22}g - \varkappa b_{32}\rangle\\
&\quad + \langle\psi, F - b_{11}\varsigma - b_{21}g - \varkappa b_{31} - k\mathbf{u}_1\cdot\nabla\mathbf{u}_1\cdot\mathbf{h}\rangle\\
&\qquad + \langle\xi, M - \varkappa b_{34}\rangle + n - n\langle 1, b_{13}\varsigma + b_{23}g\rangle = 0.
\end{aligned}\tag{9.33}$$

Here, the trilinear form $\mathfrak{B}$ is defined by the equality

$$\mathfrak{B}(\mathbf{w},\varphi_2,\varsigma) = -\int_\Omega \varsigma\mathbf{w}\cdot\nabla\varphi_2\,dx.$$

Note, that relations (9.33) are well defined for all $\mathbf{w}\in\mathcal{H}_0^{1-s,r'}(\Omega)$ and $\psi,\xi\in\mathbb{H}^{-s,r'}(\Omega)$. It is obviously true for all terms, possibly except of $\mathfrak{B}$. Well-posedness of the form $\mathfrak{B}$ follows the next lemma, the proof is at the end of this section. The lemma is given in $\mathbb{R}^d$, for our application $d = 3$.

LEMMA 9.3. *Let $\Omega\subset\mathbb{R}^d$ be a bounded domain with the Lipschitz boundary, let exponents s and r satisfy the inequalities $sr > d$, $1/2\le s\le 1$ and $\varphi,\varsigma\in H^{s,r}(\Omega)\cap H^{1,r_0}(\Omega)$,*

$\mathbf{w} \in \mathcal{H}_0^{1-s,r'}(\Omega) \cap H_0^{1,r_0}(\Omega)$, $1 < r_0 < \infty$. *Then there is a constant c depending only on* s, r *and* Ω, *such that the trilinear form*

$$\mathfrak{B}(\mathbf{w}, \varphi, \varsigma) = -\int_\Omega \varsigma \mathbf{w} \cdot \nabla\varphi \, dx$$

satisfies the inequality

$$|\mathfrak{B}(\mathbf{w}, \varphi, \varsigma)| \le c\|\mathbf{w}\|_{\mathcal{H}_0^{1-s,r'}(\Omega)} \|\varphi\|_{H^{s,r}(\Omega)} \|\varsigma\|_{H^{s,r}(\Omega)}, \tag{9.34}$$

and can be continuously extended to $\mathfrak{B} : \mathcal{H}_0^{1-s,r'}(\Omega)^d \times H^{s,r}(\Omega)^2 \mapsto \mathbb{R}$. *In particular, we have* $\varsigma\nabla\varphi \in \mathcal{H}^{s-1,r}(\Omega)$ *and* $\|\varsigma\nabla\varphi\|_{H^{1-s,r}(\Omega)} \le c\|\varphi\|_{H^{s,r}(\Omega)}\|\varsigma\|_{H^{s,r}(\Omega)}$.

Thus, relations (9.33) are well defined for all $(\mathbf{w}, \psi, \omega, \xi) \in \mathcal{H}_0^{1-s,r'}(\Omega)^3 \times \mathbb{H}^{-s,r'}(\Omega)^3$. Equalities (9.33) along with equations (9.28), (9.29) are called the very weak formulation of problem (9.26). Note that the notion of *very weak solutions* to incompressible Stokes equations was introduced in [13], see also [12] for generalizations. The natural question is the uniqueness of solutions to such weak formulation. The following theorem, which is the second main result of this section, guarantees the uniqueness of very weak solutions for sufficiently small τ.

THEOREM 9.4. *Let* s, r, *and parameters* λ, k, σ_λ, *and positive number* τ *meet all requirements of Theorem 9.1 and the solutions* (Θ_i, ζ_i, m_i), $i = 1, 2$, *to problem* (9.5) *belong to* $\mathcal{B}_\tau \times X^{s,r} \times \mathbb{R}$. *Furthermore, assume that for any* $(\mathbf{H}, G, F, M) \in C^\infty(\Omega)^6$ *and for* $(\varsigma, \upsilon, \mathbf{h}, g)$ *satisfying* (9.28), (9.29), *the elements* $(\mathbf{w}, \omega, \psi, \xi) \in \mathcal{H}_0^{1-s,r'}(\Omega)^3 \times \mathbb{H}^{-s,r'}(\Omega)^3$ *and the constant n satisfy identity* (9.33). *Then* $\mathbf{w} = 0$, $\psi = \xi = n = 0$.

PROOF. The proof is based upon two auxiliary lemmas, the first lemma establishes the bounds for coefficients of problem (9.26). □

LEMMA 9.5. *Under the assumptions of Theorem 9.4, all the coefficients of identity* (9.33) *satisfy the inequalities* $\|b_{ij}\|_{X^{s,r}} \le c$, *furthermore*

$$\begin{aligned} &\|b_{12}\|_{X^{s,r}} + \|b_{22}\|_{X^{s,r}} + \|b_{11}\|_{X^{s,r}} \le c\tau, \\ &\|b_{31}\|_{X^{s,r}} + \|b_{32}\|_{X^{s,r}} \le c\tau. \end{aligned} \tag{9.35}$$

PROOF. Since $X^{s,r}$ is a Banach algebra, estates (9.35) follows from formulae (9.27). □

In order to formulate the second auxiliary result we introduce the following denotations.

$$\begin{aligned} &\mathfrak{I}_1 = \langle\psi, b_{11}\varsigma\rangle + \langle\omega, b_{12}\varsigma\rangle, \quad \mathfrak{I}_2 = \langle\psi, b_{21}g\rangle + \langle\omega, b_{22}g\rangle, \\ &\mathfrak{I}_3 = \varkappa(\langle\psi, b_{31}\rangle + \langle\omega, b_{32}\rangle + \langle\xi, b_{34}\rangle), \ \mathfrak{I}_4 = \langle\psi, \mathbf{u}_1 \cdot \nabla\mathbf{u}_1 \cdot \mathbf{h}\rangle, \\ &\mathfrak{I}_5 = \int_\Omega \varrho_2 \mathbf{w} \cdot (\nabla\mathbf{u}_1 \, \mathbf{h} - \nabla h^* \mathbf{u}_2) \, dx, \\ &\mathfrak{G} = \|\mathbf{w}\|_{\mathcal{H}_0^{1-s,r'}(\Omega)} + \|\psi\|_{\mathbb{H}^{-s,r'}(\Omega)} + \|\omega\|_{\mathbb{H}^{-s,r'}(\Omega)} + \|\xi\|_{\mathbb{H}^{-s,r'}(\Omega)}, \\ &\mathfrak{Q} = \|\mathbf{H}\|_{\mathcal{H}^{s-1,r}(\Omega)} + \|G\|_{H^{s,r}(\Omega)} + \|F\|_{H^{s,r}(\Omega)} + \|M\|_{H^{s,r}(\Omega)}. \end{aligned}$$

LEMMA 9.6. *Under the assumptions of Theorem 9.4, there is a constant c, depending only on* $\mathbf{U}$, Ω, *s*, *r*, *and* σ, *such that*

$$\mathfrak{I}_1 \le c\tau \mathfrak{Q}\,\mathfrak{G} \tag{9.36}$$

$$\mathfrak{I}_2 \le c\mathfrak{Q}(\tau\mathfrak{G} + \|\psi\|_{\mathbb{H}^{-s,r'}(\Omega)}) \tag{9.37}$$

$$\mathfrak{I}_3 \le c\tau\,\mathfrak{G}, \quad \mathfrak{I}_4 + \mathfrak{I}_5 \le c\mathfrak{Q}\,\mathfrak{G}. \tag{9.38}$$

PROOF. We have

$$\langle\psi, b_{11}\varsigma\rangle + \langle\omega, b_{12}\varsigma\rangle + \langle\mathfrak{d}, b_{10}\varsigma\rangle \le \|b_{11}\varsigma\|_{H^{s,r}(\Omega)}\|\psi\|_{\mathbb{H}^{-s,r'}(\Omega)} + \|b_{12}\varsigma\|_{H^{s,r}(\Omega)}\|\omega\|_{\mathbb{H}^{-s,r'}(\Omega)}.$$

Recall that for $rs > 3$, $H^{s,r}(\Omega)$ is a Banach algebra. From this, estimate (9.30), and inequalities (9.35) we obtain

$$\begin{aligned}
&\|b_{11}\varsigma\|_{H^{s,r}(\Omega)}\|\psi\|_{\mathbb{H}^{-s,r'}(\Omega)} + \|b_{12}\varsigma\|_{H^{s,r}(\Omega)}\|\omega\|_{\mathbb{H}^{-s,r'}(\Omega)} \\
&\quad\le c\|\varsigma\|_{H^{s,r}(\Omega)}(\|b_{11}\|_{H^{s,r}(\Omega)}\|\psi\|_{\mathbb{H}^{-s,r'}(\Omega)} + \|b_{12}\|_{H^{s,r}(\Omega)}\|\omega\|_{\mathbb{H}^{-s,r'}(\Omega)} \\
&\quad\le c\tau\|F\|_{H^{s,r}(\Omega)}(\|\psi\|_{\mathbb{H}^{-s,r'}(\Omega)} + \|\omega\|_{\mathbb{H}^{-s,r'}(\Omega)}),
\end{aligned}$$

which gives (9.36). Repeating these arguments and using inequality (9.30) we obtain the estimates for $\mathfrak{I}_2$ and $\mathfrak{I}_3$. Next, we have

$$\|\mathbf{u}_1\cdot\nabla\mathbf{u}_1\cdot\mathbf{h}\|_{H^{s,r}(\Omega)} \le c\|\mathbf{u}_1\|_{H^{s,r}(\Omega)}\|\mathbf{u}_1\|_{H^{1+s,r}(\Omega)}\|\mathbf{h}\|_{H^{1+s,r}(\Omega)} \le c\|\mathbf{H}\|_{H^{1+s,r}(\Omega)}$$

which gives the estimate for $\mathfrak{I}_4$. Since the embeddings $H^{s,r}(\Omega) \hookrightarrow C(\Omega)$, $H^{1+s,r}(\Omega) \hookrightarrow C^1(\Omega)$ are bounded, we have

$$\varrho_2|\nabla\mathbf{u}_1||\mathbf{h}| + \varrho_2|\mathbf{u}_2||\nabla\mathbf{h}| \le c\|\mathbf{h}\|_{H^{1+s,r}(\Omega)},$$

which leads to the inequality

$$\mathfrak{I}_5 \le c\|\mathbf{h}\|_{H^{1+s,r}(\Omega)}\|\mathbf{w}\|_{L^1(\Omega)} \le c(\|\mathbf{H}\|_{\mathcal{H}^{s-1,r}(\Omega)} + \|G\|_{H^{s,r}(\Omega)})\|\mathbf{w}\|_{\mathcal{H}^{1-s,r'}(\Omega)},$$

and the proof of Lemma 9.6 is completed. □

Let us return to the proof of Theorem 9.4. It follows from the duality principle that the theorem is proved provided we show that, under the assumptions of Theorem 9.1, the following inequality holds

$$\begin{aligned}
&\sup_{\mathfrak{Q}(\mathbf{H},G,F,M)=1} (\langle\mathbf{w},\mathbf{H}\rangle + \langle\omega, G\rangle + \langle\psi, F\rangle + \langle\xi, M\rangle) + |n| \\
&\quad\le c\tau\,(\,\mathfrak{G}(\mathbf{w},\omega,\psi,\xi) + |n|)+,
\end{aligned} \tag{9.39}$$

where the constant c depends only on Ω, $\mathbf{U}$ and r, s, σ_λ. Therefore, our task is to estimate step by step all terms on the left-hand side of (9.39). We begin with an estimate for the term $\langle\psi, F\rangle$. To this end, take $\mathbf{H} = \mathbf{h} = 0$, $G = g = 0$, $M = \upsilon = 0$, and rewrite identity (9.33) in the form

$$\langle\psi, F\rangle = \mathfrak{B}(\mathbf{w}, \varphi_2, \varsigma) + \mathfrak{I}_1 + \mathfrak{I}_3 + n\langle 1, b_{13}\varsigma\rangle - n.$$

By virtue of Lemma 9.3 and estimate (9.31) we have

$$\mathfrak{B}(\mathbf{w}, \varphi_2, \varsigma) \le c\tau\|\mathbf{w}\|_{\mathcal{H}_0^{1-s,r'}(\Omega)}\|\varsigma\|_{H^{s,r}(\Omega)} \le c\tau\|\mathbf{w}\|_{H^{1-s,r'}(\Omega)}\|F\|_{H^{s,r}(\Omega)}. \tag{9.40}$$

On the other hand, Lemma 9.5 and inequality (9.30) yield $|\langle 1, b_{13}\varsigma\rangle| \le c\|F\|_{H^{s,r}(\Omega)}$. From this and (9.36), (9.38) we finally obtain

$$\langle\psi, F\rangle \le |n| + c\|F\|_{H^{s,r}(\Omega)}\,(\,\tau\mathfrak{G} + |n|\,). \tag{9.41}$$

Moreover, by virtue of the duality principle

$$\|\psi\|_{\mathbb{H}^{-s,r'}(\Omega)} = \sup_{\|F\|_{H^{s,r}(\Omega)}=1} \big|\langle\psi, F\rangle\big|,$$

we have the following estimate for ψ

$$\|\psi\|_{\mathbb{H}^{-s,r'}(\Omega)} \le c\tau\mathfrak{G} + c|n|. \tag{9.42}$$

Let us estimate $\mathbf{w}$ and ω. Substituting $F = \varsigma = 0$ and $M = \upsilon = 0$ into (9.33) we obtain

$$\langle\mathbf{w}, \mathbf{H}\rangle + \langle\omega, G\rangle = \mathfrak{I}_2 + \mathfrak{I}_3 + k\mathfrak{I}_4 + k\mathfrak{I}_5 + n\langle 1, b_{23}g\rangle - n.$$

Next we have

$$|n\langle 1, b_{23}g\rangle| \le c(\|\mathbf{H}\|_{\mathcal{H}^{s-1,r}(\Omega)} + \|G\|_{H^{s,r}(\Omega)})|n|.$$

This inequality together with estimates (9.37)–(9.38) and inequality $k \le \tau^2$ imply

$$\langle\mathbf{w}, \mathbf{H}\rangle + \langle\omega, G\rangle \le |n| + c\tau\mathfrak{Q}\,\mathfrak{G} +$$
$$c\mathfrak{Q}(\|\psi\|_{\mathbb{H}^{-s,r'}(\Omega)} + |n|).$$

Combining this result with (9.42) we obtain

$$\langle\mathbf{w}, \mathbf{H}\rangle + \langle\omega, G\rangle \le |n| + c\tau\mathfrak{Q}\,\mathfrak{G} + c\mathfrak{Q}\,|n|, \tag{9.43}$$

where $\mathfrak{Q} = \mathfrak{Q}(\mathbf{H}, G, 0, 0)$. For $G = 0$ and by the duality principle

$$\|\mathbf{w}\|_{\mathcal{H}_0^{1-s,r'}(\Omega)} = \sup_{\|\mathbf{H}\|_{\mathcal{H}^{s-1,r}(\Omega)}=1} \langle\mathbf{H}, \mathbf{w}\rangle,$$

we conclude from this that

$$\|\mathbf{w}\|_{\mathcal{H}_0^{1-s,r'}(\Omega)} \le c|n| + c\tau\mathfrak{G}. \tag{9.44}$$

Next, substituting $\mathbf{H} = \mathbf{h} = 0$, $G = g = F = \varsigma = 0$ into identity (9.33), we arrive at

$$\langle \xi, M \rangle = \mathfrak{B}(\mathbf{w}, \zeta_2, \upsilon) + \mathfrak{I}_3 - n.$$

Lemma 9.3 and (9.30) give the estimate for the first term

$$|\mathfrak{B}(\mathbf{w}, \zeta_2, \upsilon)| \le c\|\mathbf{w}\|_{\mathcal{H}_0^{1-s,r'}(\Omega)} \|\upsilon\|_{H^{s,r}(\Omega)} \le c\|\mathbf{w}\|_{\mathcal{H}_0^{1-s,r'}(\Omega)} \|M\|_{H^{s,r}(\Omega)}.$$

From this and estimates (9.38), (9.30), we obtain

$$\langle \xi, M \rangle \le c\tau \mathfrak{Q}\, \mathfrak{G} + c\mathfrak{Q}\, \|\mathbf{w}\|_{\mathcal{H}_0^{1-s,r'}(\Omega)} + |n|.$$

Combining this result with inequality (9.44) we arrive at

$$\langle \xi, M \rangle \le c\mathfrak{Q}(\tau \mathfrak{G} + |n|) + c|n|. \tag{9.45}$$

Finally, choosing all test functions in (9.33) equal to 0 we obtain $n = \mathfrak{I}_3$ which together with (9.38) yields

$$|n| \le c\tau \mathfrak{Q}\, \mathfrak{G}. \tag{9.46}$$

From (9.41), (9.43), (9.45), combined with (9.46), it follows (9.39) and the proof of Theorem 9.4 is completed. □

9.3. *Proof of Lemma 9.3*

Since $\partial\Omega$ belongs to the class C^1, functions φ, ς have the extensions $\overline{\varphi}, \overline{\varsigma} \in H^{s,r}(\Omega) \cap H^{1,2}(\Omega)$, such that $\overline{\varphi}, \overline{\varsigma}$ are compactly supported in $\mathbb{R}^d$ and

$$\|\overline{\varphi}\|_{H^{s,r}(\mathbb{R}^d)} \le c\|\varphi\|_{H^{s,r}(\Omega)}, \quad \|\overline{\varsigma}\|_{H^{s,r}(\mathbb{R}^d)} \le c\|\varsigma\|_{H^{s,r}(\Omega)}.$$

By virtue of Definition 4.2 and inequality (4.5), function $\mathbf{w}$ has the extension by 0 outside Ω, denoted by $\overline{\mathbf{w}}$, such that

$$\|\overline{\mathbf{w}}\|_{H^{1-s,r'}(\mathbb{R}^d)} \le c\|\mathbf{w}\|_{\mathcal{H}_0^{1-s,r'}(\Omega)}.$$

Obviously we have

$$\mathfrak{B}(\mathbf{w}, \varphi, \varsigma) = -\int_{\mathbb{R}^d} \overline{\mathbf{w}} \cdot \nabla\overline{\varphi}\, \overline{\varsigma} dx.$$

The following multiplicative inequality is due to Mazja [38]. For all $s > 0$, $r > 1$ and $rs < d$,

$$\|uv\|_{H^{s,r}(\mathbb{R}^d)} \le c(r, s, d)(\|v\|_{H^{s,s/d}(\mathbb{R}^d)} + \|v\|_{L^\infty(\mathbb{R}^d)})\|u\|_{H^{s,r}(\mathbb{R}^d)}. \tag{9.47}$$

By virtue of (9.47), we have

$$\|\overline{\mathbf{w}}\,\overline{\varsigma}\|_{H^{1-s,r'}(\mathbb{R}^d)} \le c\|\overline{\mathbf{w}}\|_{H^{1-s,r'}(\mathbb{R}^d)}(\|\overline{\varsigma}\|_{H^{1-s,d/(1-s)}(\mathbb{R}^d)} + \|\overline{\varsigma}\|_{L^\infty(\mathbb{R}^d)}).$$

On the other hand, since $r^{-1} - (s - (1 - s))/d \le (1 - s)/d$ for $sr > d$, embedding inequality (4.7) yields

$$\|\overline{\varsigma}\|_{H^{1-s,d/(1-s)}(\mathbb{R}^d)} \le c\|\overline{\varsigma}\|_{H^{s,r}(\mathbb{R}^d)}, \quad \|\overline{\varsigma}\|_{L^\infty(\mathbb{R}^d)} \le c\|\overline{\varsigma}\|_{H^{s,r}(\mathbb{R}^d)}.$$

Thus we get

$$\|\overline{\mathbf{w}}\,\overline{\varsigma}\|_{H^{1-s,r'}(\mathbb{R}^d)} \le c\|\overline{\mathbf{w}}\|_{H^{1-s,r'}(\mathbb{R}^d)}\|\overline{\varsigma}\|_{H^{s,r}(\mathbb{R}^d)}.$$

It is well known that elements of the fractional Sobolev spaces can be represented via Liouville potentials

$$\overline{\mathbf{w}\varsigma} = (1-\Delta)^{-(1-s)/2}w, \qquad \overline{\varphi} = (1-\Delta)^{-s/2}\phi,$$

with

$$\|w\|_{L_{r'}(\mathbb{R}^d)} \le c\|\overline{\mathbf{w}\varsigma}\|_{H^{1-s,r'}(\mathbb{R}^d)}, \qquad \|\phi\|_{L_r(\mathbb{R}^d)} \le c\|\overline{\varphi}\|_{H^{1-s,r'}(\mathbb{R}^d)}.$$

Thus we get

$$\begin{aligned}\mathfrak{B}(\mathbf{w},\varphi,\varsigma) &= -\int_{\mathbb{R}^d}(1-\Delta)^{-(1-s)/2}w\cdot\nabla(1-\Delta)^{-s/2}\phi\,dx\\ &= -\int_{\mathbb{R}^d} w\cdot\nabla(1-\Delta)^{-1/2}\phi\,dx.\end{aligned}$$

Since the Riesz operator $(1-\Delta)^{-1/2}\nabla$ is bounded in $L^r(\mathbb{R}^d)$, we conclude from this and the Holder inequality that

$$|\mathfrak{B}(\mathbf{w},\varphi,\varsigma)| \le c\|w\|_{L^{r'}(\mathbb{R}^d)}\|\phi\|_{L^r(\mathbb{R}^d)} \le c\|\mathbf{w}\|_{H^{1-s,r'}(\Omega)}\|\varphi\|_{H^{s,r}(\Omega)}\|\varsigma\|_{H^{s,r}(\Omega)},$$

and the lemma follows. □

10. Proof of Theorem 5.5

Our strategy is the following. First we show that in the vicinity of each point $P \in \Sigma_{\text{in}} \cup \Gamma$ there exist normal coordinates (y_1, y_2, y_3) such that $\mathbf{u}\nabla_x = \mathbf{e}_1\nabla_y$. Hence problem of existence of solutions to the transport equation in the neighborhood of $\Sigma_{\text{in}} \cap \Gamma$ is reduced to boundary problem for the model equation $\partial_{y_1}\varphi + \sigma\varphi = f$ in a parabolic domain. Next we prove that the boundary value problem for the model equations has a unique solution in fractional Sobolev space, which leads to the existence and uniqueness of solutions in the neighborhood of the inlet set. Using the existence of local solution we reduce problem (5.11) to the problem for modified equation, which does not require the boundary data. Application of well-known results on solvability of elliptic–hyperbolic equations in the case $\Gamma = \emptyset$ finally gives the existence and uniqueness of solutions to problems (5.11) and (5.18).

LEMMA 10.1. *Assume that the C^2-manifold $\Sigma = \partial B$ and the vector field $\mathbf{U} \in C^2(\Sigma)^3$ satisfy conditions* (**H1**)–(**H3**). *Let $\mathbf{u} \in C^1(\mathbb{R}^3)^3$ be a compactly supported vector field such that*

$$\mathbf{u} = \mathbf{U} \quad \text{on } \Sigma, \qquad \mathbf{u} = 0 \quad \text{on } S,$$

and denote $M = \|\mathbf{u}\|_{C^1(\mathbb{R}^3)}$. Then there is $a > 0$, depending only on M and Σ, with the properties:

(**P1**) *For any point $P \in \Gamma$ there exists a mapping $y \to \mathbf{x}(y)$ which takes diffeomorphically the cube $Q_a = [-a, a]^3$ onto a neighborhood $\mathcal{O}_P$ of P and satisfies the equations*

$$\partial_{y_1}\mathbf{x}(y) = \mathbf{u}(\mathbf{x}(y)) \quad \text{in } Q_a, \quad \mathbf{x}(0, y_2, 0) \in \Gamma \cap \mathcal{O}_P \quad \textit{for } |y_2| \le a, \tag{10.1}$$

and the inequalities

$$\|\mathbf{x}\|_{C^1(Q_a)} + \|\mathbf{x}^{-1}\|_{C^1(\mathcal{O}_P)} \le C_M, \qquad |\mathbf{x}(y)| \le C_M|y|, \tag{10.2}$$

where $C_M = 3(1 + M^{-1})(M^2 + C_\Gamma^2 + 2)^{1/2}$ and C_Γ is the constant in condition (**H2**).
(**P2**) *There is a C^1 function $\Phi(y_1, y_2)$ defined in the square $[-a, a]^2$ such that $\Phi(0, y_2) =$ 0, and*

$$\mathbf{x}(\{y_3 = \Phi\}) = \Sigma \cap \mathcal{O}_P, \qquad \mathbf{x}(\{y_3 > \Phi\}) = \Sigma \cap \mathcal{O}_P. \tag{10.3}$$

Moreover Φ is strictly decreasing in y_1 for $y_1 < 0$, is strictly increasing in y_1 for $y_1 > 0$, and satisfies the inequalities

$$C^- y_1^2 \le \Phi(y_1, y_2) \le C^+ y_1^2, \tag{10.4}$$

where the constants $C^- = |\mathbf{U}(P)|N^-/12$ and $C^+ = 12|\mathbf{U}(P)|N^+$ depend only on $\mathbf{U}$ *and Σ, where $N^\pm$ are defined in Condition* (**H2**).
(**P3**) *Introduce the sets*

$$\Sigma_{\text{in}}^y = \{(y_2, y_3) : |y_2| \le a, 0 < y_3 < \Phi(-a, y_2)\},$$
$$\Sigma_{\text{out}}^y = \{(y_2, y_3) : |y_2| \le a, 0 < y_3 < \Phi(a, y_2)\}.$$

For every $(y_2, y_3) \in \Sigma_{\text{in}}^y$ (resp. $(y_2, y_3) \in \Sigma_{\text{out}}^y$), the equation $y_3 = \Phi(y_1, y_2)$ has a unique negative (resp. positive) solution $y_1 = a^-(y_2, y_3)$, (resp. $y_1 = a^+(y_2, y_3)$) such that

$$|\partial_{y_j} a^\pm(y_2, y_3)| \le C/\sqrt{y_3},$$
$$|a^\pm(y_2, y_3) - a^\pm(y_2', y_3')| \le C(|y_2 - y_2'| + |y_3 - y_3'|)^{1/2}. \tag{10.5}$$

(**P4**) *Denote by $G_a \subset Q_a$ the domain*

$$G_a = \{y \in Q_a : \Phi(y_1, y_2) < y_3 < \Phi(-a, y_2)\}, \tag{10.6}$$

and by $B_P(\rho)$ the ball $|x - P| \le \rho$. Then we have the inclusions

$$B_P(\rho_c) \subset \mathbf{x}(G_a) \subset \mathcal{O}_P \subset B_P(R_c), \tag{10.7}$$

where the constants $\rho_c = a^2 C_M^{-1} C^-$, $R_c = aC_M$.

The next lemma constitutes the existence of the normal coordinates in the vicinity of points of the inlet Σ_{in}.

LEMMA 10.2. *Let vector fields* $\mathbf{u}$ *and* $\mathbf{U}$ *meet all requirements of Lemma 10.1 and $U_n = -\mathbf{U}(P) \cdot \mathbf{n} > N > 0$. Then there is $b > 0$, depending only on N, Σ and $M = \|\mathbf{u}\|_{C^1(\Omega)}$, with the following properties. There exists a mapping $y \to \mathbf{x}(y)$,* *which takes diffeomorphically the cube $Q_b = [-b, b]^3$ onto a neighborhood $\mathcal{O}_P$ of P and satisfies the equations*

$$\partial_{y_3}\mathbf{x}(y) = \mathbf{u}(\mathbf{x}(y)) \quad \text{in } Q_b, \quad \mathbf{x}(y_1, y_2, 0) \in \Sigma \cap \mathcal{O}_P \quad \textit{for } |y_2| \le a, \tag{10.8}$$

and the inequalities

$$\|\mathbf{x}\|_{C^1(Q_b)} + \|\mathbf{x}^{-1}\|_{C^1(\mathcal{O}_P)} \le C_{M,N} \qquad |\mathbf{x}(y)| \le C_M|y|, \tag{10.9}$$

where $C_{M,N} = 3(1 + N^{-1})(M^2 + 2)^{1/2}$. The inclusions

$$B_P(\rho_i) \cap \Omega \subset \mathbf{x}(Q_b \cap \{y_3 > 0\}) \subset B_P(R_i) \cap \Omega, \tag{10.10}$$

hold true for $\rho_i = C_{M,N}^{-1} b$ and $R_i = C_{M,N} b$.

Model equation. Assume that the function $\Phi : [-a, a]^2 \mapsto \mathbb{R}$ and the constant $a > 0$ meet all requirements of Lemma 10.1. Recall that for each y satisfying the conditions $\Phi(y_1, y_2) < y_3 < \Phi(-a, y_2)$ (resp. $\Phi(y_1, y_2) < y_3 < \Phi(a, y_2)$), equation $y_3 = \Phi(y_1, y_2)$ has the solutions $y_1 = a^-(y_2, y_3)$ (resp. $y_1 = a^+(y_3, y_3)$). The functions $a^\pm$ vanish for $y_3 = 0$ and satisfy the inequalities

$$\begin{aligned} &-a < a^-(y_2, y_3) \le 0 \le a^+(y_2, y_3) \le a, \\ &|a^\pm(z_2, y_3) - a^\pm(y_2, y_3)| \le c|z_2 - y_2|, \\ &|a^\pm(y_2, z_3) - a^\pm(y_2, y_3)| \le c|\sqrt{z_3} - \sqrt{y_3}|, \end{aligned} \tag{10.11}$$

where c depends only on Σ, and $\mathbf{U}$. We assume that the functions $a^\pm$ are extended on the rectangle $[-a, a] \times [0, a]$ by the equalities $a^\pm(y_2, y_3) = \pm a$ for $y^3 > \Phi(\pm a, y_2)$. It is clear that the extended functions satisfy (10.11) and

$$Q_a^\phi := \{y_3 > \Phi(y_1, y_2)\} = \{y : a^-(y_2, y_3) \le y_1 \le a^+(y_2, y_3)\}.$$

Let us consider the boundary value problem

$$\begin{aligned} &\partial_{y_1}\varphi(y) + \sigma\varphi(y) = f(y) \quad \text{in } Q_a^\phi, \\ &\varphi(y) = 0 \quad \text{for } y_1 = a^-(y_2, y_3). \end{aligned} \tag{10.12}$$

LEMMA 10.3. *For any $f \in L^r(Q_a)$, $1 < r \le \infty$, problem* (10.12) *has a unique generalized solution such that*

$$\|\varphi\|_{L^r(Q_a^\phi)} \le \sigma^{-1}\|f\|_{L^r(Q_a)}, \quad 1 \le r \le \infty. \tag{10.13}$$

Moreover, if $\sigma > 1$, exponents r, s, α satisfy the inequalities (5.16) *and f belongs to the space $H^{s,r}(Q_a^\phi) \cap L^\infty(Q_a^\phi)$, then a solution to problem* (10.12) *admits the estimate*

$$\|\varphi\|_{H^{s,r}(Q_a^\phi)} \le c(a, r, s, \alpha)(\sigma^{-1}\|f\|_{H^{s,r}(Q_a^\phi)} + \sigma^{-1+\alpha}\|f\|_{L^\infty(Q_a^\phi)}). \tag{10.14}$$

Let us consider the following boundary value problem

$$\begin{aligned} &\partial_{y_3}\varphi(y) + \sigma\varphi(y) = f(y) \quad \text{in } Q_a^+ = [-a, a]^2 \times [0, a], \\ &\varphi(y) = 0 \quad \text{for } y_3 = 0. \end{aligned} \tag{10.15}$$

LEMMA 10.4. *Let $\sigma > 1$, and exponents r, s, α satisfy (5.16). Then problem (10.15) has a unique solution satisfying the inequality*

$$\|\varphi\|_{H^{s,r}(Q_a^+)} \le c(r, s, \alpha, a)(\sigma^{-1}\|f\|_{H^{s,r}(Q_a^+)} + \sigma^{-1+\alpha}\|f\|_{L^\infty(Q_a^+)}). \tag{10.16}$$

PROOF. The proof of Lemma 10.3 can be used also in this case. □

Local existence results. It follows from the conditions of Theorem 5.5 that the vector field $\mathbf{u}$ and the manifold Σ satisfy all assumptions of Lemma 10.1. Therefore, there exist positive numbers a, ρ_c and R_c, depending only on Σ and $\|\mathbf{u}\|_{C^1(\Omega)}$, such that for all $P \in \Gamma$, the canonical diffeomorphism $\mathbf{x} : Q_a \mapsto \mathcal{O}_P$ is well defined and meet all requirements of Lemma 10.1. Fix an arbitrary point $P \in \Gamma$ and consider the boundary value problem

$$\mathbf{u} \cdot \nabla\varphi + \sigma\varphi = f \quad \text{in } \mathcal{O}_P, \qquad \varphi = 0 \quad \text{on } \Sigma_{\text{in}} \cap \mathcal{O}_P. \tag{10.17}$$

LEMMA 10.5. *Suppose that the exponents s, r, α, satisfy condition (5.16) and $\|u\|_{C^1(\Omega)} \le M$. Then for any $f \in C^1(\Omega)$ and $\sigma > 1$, problem (10.17) has a unique solution satisfying the inequalities*

$$\begin{aligned} |\varphi|_{s,r,B_P(\rho_c)} &\le c(\sigma^{-1+\alpha}\|f\|_{C(B_P(R_c))} + \sigma^{-1}|f|_{s,r,B_P(R_c)}), \ \|\varphi\|_{C(B_P(\rho_c))} \\ &\le \sigma^{-1}\|f\|_{C(B_P(R_c))}, \end{aligned} \tag{10.18}$$

where the constant c depends only on Σ, M, s, r, α, and ρ_c is determined by Lemma 10.1.

PROOF. We transform equation (10.18) using the normal coordinates (y_1, y_2, y_3) given by Lemma 10.1. Set $\overline{\varphi}(y) = \varphi(\mathbf{x}(y))$ and $\overline{f}(y) = f(\mathbf{x}(y))$. Next note that equation (10.1) implies the identity $\mathbf{u}\nabla_x\varphi = \partial_{y_1}\overline{\varphi}(y)$. Therefore the function $\overline{\varphi}(y)$ satisfies the following equation and boundary conditions

$$\partial_{y_1}\overline{\varphi} + \sigma\overline{\varphi} = \overline{f} \quad \text{in } Q_a \cap \{y_3 > \Phi\}, \qquad \overline{\varphi} = 0 \quad \text{for } y_3 = \Phi(y_1, y_2),\ y_1 < 0. \tag{10.19}$$

It follows from Lemma 10.3 that for all $\sigma > 1$, problem (10.19) has a unique solution $\overline{\varphi} \in H^{s,r}(G_a)$ satisfying the inequality

$$\begin{aligned} &|\overline{\varphi}|_{s,r,G_a} \le c(\sigma^{-1+\alpha}\|\overline{f}\|_{C(Q_a)} + \sigma^{-1}|\overline{f}|_{s,r,Q_a}), \\ &\|\overline{\varphi}\|_{C(G_a)} \le \sigma^{-1}\|f\|_{C(Q_a)}, \end{aligned} \tag{10.20}$$

where the domain G_a is defined by (10.6). It remains to note that, by estimate (10.2), the mappings $\mathbf{x}^{\pm 1}$ are uniformly Lipschitz, which along with inclusions (10.7) implies the estimates

$$|\varphi|_{s,r,B_P(\rho_c)} \le c|\overline{\varphi}|_{s,r,G_a}, \qquad |\overline{f}|_{s,r,Q_a} \le c|f|_{s,r,B_P(R_c)}.$$

Combining these results with (10.20) we finally obtain (10.18) and the lemma follows. □

In order to formulate the similar result for interior points of inlet we introduce the set

$$\Sigma_{\text{in}}' = \big\{x \in \Sigma_{\text{in}} : \quad \text{dist}\,(x, \Gamma) \geq \rho_c/3), \tag{10.21}$$

where the constant ρ_c is given by Lemma 10.1. It is clear that

$$\inf_{P \in \Sigma_{\text{in}}'} \mathbf{U}(P) \cdot \mathbf{n}(P) \geq N > 0,$$

where the constant N depends only on M, $\mathbf{U}$, and Σ. It follows from Lemma 10.2 that there are positive numbers b, ρ_i, and R_i such that for each $P \in \Sigma_{\text{in}}'$, the canonical diffeomorphism $\mathbf{x} : Q_b \mapsto \mathcal{O}_P$ is well defined and satisfies the hypotheses of Lemma 10.2. The following lemma gives the local existence and uniqueness of solutions to the boundary value problem

$$\mathbf{u} \cdot \nabla\varphi + \sigma\varphi = f \quad \text{in } \mathcal{O}_P, \qquad \varphi = 0 \quad \text{on } \Sigma_{\text{in}} \cap \mathcal{O}_P. \tag{10.22}$$

LEMMA 10.6. *Suppose that the exponents s, r, α satisfy condition* (5.16). *Then for any $f \in C^1(\Omega)$, $\sigma > 1$ and $P \in \Sigma_{\text{in}}'$, problem* (10.17) *has a unique solution satisfying the inequalities*

$$\begin{aligned} |\varphi|_{s,r,B_P(\rho_i)} \leq \quad & c(\sigma^{-1+\alpha}\|f\|_{C(B_P(R_i))} + \sigma^{-1}|f|_{s,r,B_P(R_i)}), \\ & \|\varphi\|_{C(B_P(\rho_i))} \leq \sigma^{-1}\|f\|_{C(B_P(R_i))}. \end{aligned} \tag{10.23}$$

where c depends on Σ, M, $\mathbf{U}$ and exponents s, r, α.

PROOF. Using the normal coordinates given by Lemma 10.2 we rewrite equation (10.22) in the form.

$$\partial_{y_3}\overline{\varphi} + \sigma\overline{\varphi} = \overline{f} \quad \text{in } Q_b, \qquad \overline{\varphi} = 0 \quad \text{for } y_3 = 0.$$

Applying Lemma 10.3 and arguing as in the proof of Lemma 10.5 we obtain (10.23). □

Existence of solutions near inlet. The next step is based on the well-known geometric lemma (see Ch. 3 in [27]).

LEMMA 10.7. *Suppose that a given set $A \subset \mathbb{R}^d$ is covered by balls such that each point $x \in A$ is the center of a certain ball $B_x(r(x))$ of radius $r(x)$. If $\sup r(x) < \infty$, then from the system of the balls $\{B_x(r(x))\}$ it is possible to select a countable system $B_{x_k}(r(x_k))$ covering the entire set A and having multiplicity not greater than a certain number $n(d)$ depending only on the dimension d.*

The following lemma gives the dependence of the multiplicity of radii of the covering balls.

LEMMA 10.8. *Assume that a collection of balls $B_{x_k}(r) \subset \mathbb{R}^3$ of constant radius r has the multiplicity n_r. Then the multiplicity of the collections of the balls $B_{x_k}(R)$, $r < R$, is bounded by the constant $27(R/r)^3 n_r$.*

PROOF. Let n_R be a multiplicity of the system $\{B_{x_k}(R)\}$. This means that at least n_R balls, say $B_{x_1}(R), \dots, B_{x_{n_R}}(R)$, have the common point P. In particular, we have $B_{x_i}(r) \subset B_P(3R)$ for all $i \le n_R$. Introduce the counting function $\iota(x)$ for the collection of balls $B_{x_i}(r)$, defined by

$$\iota(x) = \operatorname{card}\{i : x \in B_{x_i}(r), 1 \le i \le n_r\}.$$

Note that $\iota(x) \le n_r$. We have

$$\begin{aligned}\frac{4\pi}{3} n_R r^3 &= \sum_{i=1}^{n_R} \operatorname{meas} B_{x_i}(r) \\ &= \int_{\cup_i B_{x_i}(r)} \iota(x)\, dx \le n_r \int_{\cup_i B_{x_i}(r)} dx \le \frac{4\pi}{3}(3R)^3 n_r,\end{aligned}$$

and the lemma follows. □

We are now in a position to prove the local existence and uniqueness of solution for the first boundary value problem for the transport equation in the neighborhood of the inlet. Let Ω_t be the t-neighborhood of the set Σ_{in},

$$\Omega_t = \{x \in \Omega : \operatorname{dist}(x, \Sigma_{\text{in}}) < t\}.$$

LEMMA 10.9. *Let $t = \min\{\rho_c/2, \rho_i/2\}$ and $T = \max\{R_c, R_i\}$, where the constants ρ_α, R_α are defined by Lemmas 10.1 and 10.2. Then there exists a constant C depending only on M, Σ and σ, such that for any $f \in C^1(\Omega)$, the boundary value problem*

$$\mathbf{u} \cdot \nabla\varphi + \sigma\varphi = f \quad \text{in } \Omega_t, \qquad \varphi = 0 \quad \text{on } \Sigma_{\text{in}} \tag{10.24}$$

has a unique solution satisfying the inequalities

$$|\varphi|_{s,r,\Omega_t} \le C(\sigma^{-1+\alpha}\|f\|_{C(\Omega_T)} + \sigma^{-1}|f|_{s,r,\Omega_T}), \quad \|\varphi\|_{C(\Omega_t)} \le \sigma^{-1}\|f\|_{C(\Omega_T)}. \tag{10.25}$$

PROOF. It follows from Lemma 10.7 that there is a covering of the characteristic manifold Γ by the finite collection of balls $B_{P_i}(\rho_c/4)$, $1 \le i \le m$, $P_i \in \Gamma$, of the multiplicity n. The cardinality m of this collection does not exceed $4n(\rho_c)^{-1}L$, where L is the length of Γ. Obviously, the balls $B_{P_i}(\rho_c)$ cover the set

$$\mathcal{V}_\Gamma = \{x \in \Omega : \operatorname{dist}(x, \Gamma) < \rho_c/2\}.$$

By virtue of Lemma 10.5, in each of such balls the solution to problem (10.24) satisfies inequalities (10.18), which leads to the estimate

$$\begin{aligned}|\varphi|^r_{s,r,\mathcal{V}_\Gamma} &\le \sum_i |\varphi|^r_{s,r,B_{P_i}(\rho_c)} \\ &\le c\sigma^{-1+\alpha}\sum_i \|f\|^r_{C(B_{P_i}(R_c))} + c\sigma^{-1}\sum_i |f|^r_{s,r,B_{P_i}(R_c)},\end{aligned} \tag{10.26}$$

where c depends only on M, Σ and $\mathbf{U}$. By Lemma 10.8, the multiplicity of the system of balls $B_{P_i}(R_c)$ is bounded from above by $12^3(R_c/\rho_c)^3$, which along with the inclusion $\cup_i B_{P_i}(R_c) \subset \Omega_T$ yields

$$\sum_{i=1}^{m} |f|^r_{s,r,B_{P_i}(R_c)} \le 12^3 (R_c/\rho_c)^3 |f|^r_{s,r,\Omega_T}.$$

Obviously we have

$$\sum_i \|f\|^r_{C(B_{P_i}(R_c))} \le m\|f\|^r_{C(\Omega_T)} \le 4n(\rho_c)^{-1} L \|f\|^r_{C(\Omega_T)}.$$

Combining these results with (10.26) we obtain the estimates for solution to problem (10.24) in the neighborhood of the characteristic manifold Γ,

$$|\varphi|_{s,r,\mathcal{V}_\Gamma} \le c\sigma^{-1+\alpha}\|f\|_{C(\Omega_T)} + c\sigma^{-1}|f|_{s,r,\Omega_T}. \tag{10.27}$$

Our next task is to obtain the similar estimate in the neighborhood of the compact $\Sigma'_{\text{in}} \subset \Sigma_{\text{in}}$. To this end, we introduce the set

$$\mathcal{V}_{\text{in}} = \{x \in \Omega :\ \text{dist}\,(x, \Sigma'_{\text{in}}) < \rho_i/2\},$$

where Σ'_{in} is given by (10.21). By virtue of Lemma 10.7, there exists the finite collection of balls $B_{P_k}(\rho_i/4)$, $1 \le k \le m$, $P_k \in \Sigma'_{\text{in}}$, of the multiplicity n which covers Σ'_{in}. Obviously $m \le 16n(\rho_i)^{-2}\text{meas}\,\Sigma_{\text{in}}$, and the balls $B_{P_k}(\rho_i)$ cover the set $\mathcal{V}_{\text{in}}$. From this and Lemma 10.6 we conclude that

$$\begin{aligned}|\varphi|^r_{s,r,\mathcal{V}_{\text{in}}} &\le \sum_k |\varphi|^r_{s,r,B_{P_k}(\rho_i)}\\ &\le c\sigma^{-1+\alpha}\sum_k \|f\|^r_{C(B_{P_k}(R_i))} + c\sigma^{-1}\sum_k |f|^r_{s,r,B_{P_k}(R_i)}.\end{aligned}$$

By virtue of Lemma 10.8, the multiplicity of the system of balls $B_{P_i}(R_i)$ is not greater than $12^3(R_i/\rho_i)^3$, which yields

$$\sum_i |f|^r_{s,r,B_{P_i}(R_i)} \le 12^3 (R_i/\rho_i)^3 |f|^r_{s,r,\Omega_T}.$$

Obviously we have

$$\sum_k \|f\|^r_{C(B_{P_k}(R_i))} \le m\|f\|^r_{C(\Omega_T)} \le 16n(\rho_k)^{-2}\text{meas}\,\Sigma_{\text{in}}\|f\|^r_{C(\Omega_T)}.$$

Thus we get

$$|\varphi|_{s,r,\mathcal{V}_{in}} \le c\sigma^{-1+\alpha}\|f\|_{C(\Omega_T)} + c\sigma^{-1}|f|_{s,r,\Omega_T}. \tag{10.28}$$

Since $\mathcal{V}_\Gamma$ and $\mathcal{V}_{\text{in}}$ cover Ω_t, this inequality along with inequalities (10.27) yields (10.25), and the lemma follows. □

Partition of unity. Let us turn to the analysis of the general problem

$$\mathcal{L}\varphi := \mathbf{u}\cdot\nabla\varphi + \sigma\varphi = f \quad \text{in } \Omega, \qquad \varphi = 0 \quad \text{on } \Sigma_{\text{in}}. \tag{10.29}$$

Recall that by virtue of Theorem 5.3, for any $f \in C^1(\Omega)$, problem (10.29) has a unique strong solution defined in neighborhood Ω_t of the inlet Σ_{in}. On the other hand, Theorem 5.1 guarantees the existence and uniqueness of the bounded weak solution to problem (10.29). The following lemma shows that both the solutions coincide in Ω_t.

LEMMA 10.10. *Under the assumptions of Theorem 5.5 each bounded generalized solution to problem* (10.29) *coincides in Ω_t with the local solution φ_t.*

PROOF. Let $\varphi \in L^\infty(\Omega)$ be a weak solution to problem (10.29). Recall that each point $P \in \Gamma$ has a canonical neighborhood $\mathcal{O}_P := \mathbf{x}(Q_a)$, where canonical diffeomorphism $\mathbf{x} : Q_a \mapsto \mathcal{O}_P$ is defined by Lemma 10.1. Choose an arbitrary function $\zeta \in C^1(\Omega)$ vanishing on Σ_{in} and outside of $\mathcal{O}_P$ and set

$$\overline{\varphi}(y) = \varphi(\mathbf{x}(y)), \quad \overline{f}(y) = f(\mathbf{x}(y)), \quad \overline{\zeta}(y) = \zeta(\mathbf{x}(y)), \quad y \in Q_a \cap \{y_3 > \Phi\}.$$

By definition of the weak solution to the transport equation we have

$$\int_{\mathcal{O}_P\cap\Omega} (\sigma\varphi\zeta - \varphi\,\mathrm{div}(\zeta\mathbf{u}) - f\zeta)\,dx = 0.$$

Direct calculations lead to the identity $\mathrm{div}_x(\zeta\mathbf{u}) = \det\mathfrak{F}^{-1}\,\mathrm{div}_y(\overline{\zeta}\det\mathfrak{F}\,\mathfrak{F}^{-1}\overline{\mathbf{u}})$, in which the notation $\mathfrak{F}$ stands for the Jacobi matrix $\mathfrak{F} = D_y\mathbf{x}(y)$. On the other hand, equation (10.1) implies the equality $\mathfrak{F}^{-1}\overline{\mathbf{u}} = \mathbf{e_1}$. From this we conclude that

$$\int_{Q_a\cap\{y_3>\Phi\}} \left((\det\mathfrak{F}\,\overline{\zeta})(\sigma\overline{\varphi} - \overline{f}) - \overline{\varphi}\frac{\partial}{\partial y_1}(\det\mathfrak{F}\,\overline{\zeta}) \right) dy = 0.$$

Recall that, by Lemma 10.1, $\partial_{y_1}\mathfrak{F}$ is continuous and $\det\mathfrak{F}$ is strictly positive in the cube Q_a. Setting $\xi = \det\mathfrak{F}\zeta$ we conclude that the integral identity

$$\int_{Q_a\cap\{y_3>\Phi\}} \left(\xi(\sigma\overline{\varphi} - \overline{f}) - \overline{\varphi}\frac{\partial\xi}{\partial y_1} \right) dy = 0$$

holds true for all functions $\xi \in C_0(Q_a)$ having continuous derivative $\partial_{y_1}\zeta \in C(Q_a)$ and vanishing for $y_3 = \Phi(y_1, y_2)$, $y_1 < 0$. Since $\overline{f}$ is continuously differentiable, $\overline{\varphi}$ belongs to the class $C^1_{\text{loc}}(Q_a) \cap \{y_3 > \Phi\}$, and satisfies equations (10.19). On the other hand, $\overline{\varphi_t}$ also satisfies (10.19). Obviously, all solutions to problem (10.19) coincides in the domain G_a and hence $\overline{\varphi}_t = \overline{\varphi}$ in this domain. Recalling that $B_P(\rho_c) \subset \mathbf{x}(G_a)$ we obtain that $\varphi_t = \varphi$ in the ball $B_P(\rho_c)$. The same arguments show that for any $P \in \Sigma'_{\text{in}}$, the function φ_t is equal to φ in the ball $B_P(\rho_i)$. It remains to note that the balls $B_P(\rho_c)$ and $B_P(\rho_i)$ cover Ω_t and the lemma follows. □

Now we split the weak solution $\varphi \in L^\infty(\Omega)$ to problem (10.29) into two parts, namely the local solution φ_t and the remainder vanishing near the inlet. To this end fix a function $\Lambda \in C^\infty(\mathbb{R})$ such that

$$0 \le \Lambda' \le 3, \quad \Lambda(u) = 0 \quad \text{for } u \le 1 \quad \text{and} \quad \Lambda(u) = 1 \quad \text{for } u \ge 3/2, \tag{10.30}$$

and introduce the one-parametric family of smooth functions

$$\chi_t(x) = \frac{1}{t^3} \int_{\mathbb{R}^3} \Theta\left(\frac{2(x-y)}{t}\right) \Lambda\left(\frac{\text{dist }(y, \Sigma_{\text{in}})}{t}\right) dy \tag{10.31}$$

where $\Theta \in C^\infty(\mathbb{R}^3)$ is a standard mollifying kernel supported in the unit ball. It follows that

$$\begin{aligned} &\chi_t(x) = 0 \quad \text{for dist } (x, \Sigma_{\text{in}}) \le t/2, \quad \chi_t(x) = 1 \quad \text{for dist } (x, \Sigma_{\text{in}}) \ge 2t, \\ &|\partial^l \chi_t(x)| \le \varpi(l) t^{-l} \quad \text{for all } l \ge 0, \end{aligned} \tag{10.32}$$

where $\varpi(l)$ is a constant. Now fix a number $t = t(\Sigma, M)$ satisfying all assumptions of Lemma 10.9 and set

$$\varphi(x) = (1 - \chi_{t/2}(x))\varphi_t(x) + \phi(x). \tag{10.33}$$

By virtue of (10.32) and Lemma 10.10, the function $\phi \in L^\infty(\Omega)$ vanishes in $\Omega_{t/4}$ and satisfies in a weak sense the equations

$$\mathbf{u}\nabla\phi + \sigma\phi = \chi_{t/2} f + \varphi_t \mathbf{u} \nabla \chi_{t/2} =: F \quad \text{in } \Omega, \qquad \phi = 0 \quad \text{on } \Sigma_{\text{in}}.$$

Next introduce new vector field $\tilde{\mathbf{u}}(x) = \chi_{t/8}(x)\mathbf{u}(x)$. It easy to see that $\chi_{t/8} = 1$ on the support of ϕ and hence the function ϕ is also a weak solution to the modified transport equation

$$\tilde{\mathcal{L}}\phi := \tilde{\mathbf{u}}\nabla\phi + \sigma\phi = F \quad \text{in } \Omega. \tag{10.34}$$

The advantage of such an approach is that the topology of integral lines of the modified vector field $\tilde{\mathbf{u}}$ drastically differs from the topology of integral lines of $\mathbf{u}$. The corresponding inlet, outgoing set, and characteristic set have the other structure and $\tilde{\Sigma}_{\text{in}} = \emptyset$. In particular, equation (10.34) does not require boundary conditions. Finally note that the C^1-norm of the modified vector fields has the majorant

$$\|\tilde{\mathbf{u}}\|_{C^1(\Omega)} \le M(1 + 16\varpi(1)t^{-1}), \tag{10.35}$$

where $\varpi(1)$ is a constant from (10.32). The following lemma constitutes the existence and uniqueness of solutions to the modified equation.

LEMMA 10.11. *Suppose that*

$$\begin{aligned} &\sigma > \sigma^*(M, \Sigma) + 1, \quad \sigma^* = 4M(1 + 16\varpi(1)t^{-1}) + 1, \\ &\quad M = \|\mathbf{u}\|_{C^1(\Omega)}, \end{aligned} \tag{10.36}$$

and $0 \le s \le 1$, $r > 1$. *Then for any* $F \in H^{s,r}(\Omega) \cap L^\infty(\Omega)$, *equation* (10.34) *has a unique weak solution* $\phi \in H^{s,r}(\Omega) \cap L^\infty(\Omega)$ *such that*

$$\|\phi\|_{H^{s,r}(\Omega)} \le c\sigma^{-1}\|F\|_{H^{s,r}(\Omega)}, \quad \|\phi\|_{L^\infty(\Omega)} \le \sigma^{-1}\|F\|_{L^\infty(\Omega)}, \tag{10.37}$$

where c depends only on r.

PROOF. Without any loss of generality we can assume that $F \in C^1(\Omega)$. By virtue of (10.35) and (10.36), the vector field $\tilde{\mathbf{u}}$ and σ meet all requirements of Lemma 5.3. Hence equation (10.34) has a unique solution $\phi \in H^{1,\infty}(\Omega)$. For $i = 1, 2, 3$ and $\tau > 0$, define the finite difference operator

$$\delta_{i\tau}\phi = \frac{1}{\tau}(\phi(x + \tau\mathbf{e}_i) - \phi(x)).$$

It is easy to see that

$$\tilde{\mathbf{u}}\nabla\delta_{i\tau}\phi + \sigma\delta_{i\tau}\phi = \delta_{i\tau}F - \delta_{i\tau}\tilde{\mathbf{u}}\nabla\phi(x + \tau\mathbf{e}_i) \quad \text{in } \Omega \cap (\Omega - \tau\mathbf{e}_i). \tag{10.38}$$

Next introduce the function $\eta \in C^\infty(\mathbb{R})$ such that $\eta' \geq 0$, $\eta(u) = 0$ for $u \leq 1$ and $\eta(u) = 1$ for $u \geq 1$, and set $\eta_h(x) = \eta(\text{dist }(x, \partial\Omega)/h)$. Since $\tilde{\Sigma}_{\text{in}} = \emptyset$, the inequality

$$\limsup_{h\to 0} \int_\Omega g\tilde{\mathbf{u}} \cdot \nabla\eta_h(x)\, dx \leq 0 \tag{10.39}$$

holds true for all nonnegative functions $g \in L^\infty(\Omega)$. Choosing $h > \tau$, multiplying both the sides of equation (10.38) by $\eta_h|\delta_{i\tau}\phi|^{r-2}\delta_{i\tau}\phi$ and integrating the result over $\Omega \cap (\Omega - \tau\mathbf{e}_i)$ we obtain

$$\begin{aligned}\int_{\Omega\cap(\Omega-\tau\mathbf{e}_i)} &\eta_h|\delta_{i\tau}\phi|^r \left(\sigma - \frac{1}{r}\operatorname{div}\tilde{\mathbf{u}}\right) dx - \int_{\Omega\cap(\Omega-\tau\mathbf{e}_i)} |\delta_{i\tau}\phi|^r\tilde{\mathbf{u}}\nabla\eta_h\, dx \\ &= \int_{\Omega\cap(\Omega-\tau\mathbf{e}_i)} (\delta_{i\tau}F - \delta_{i\tau}\tilde{\mathbf{u}}\nabla\phi(x + \tau\mathbf{e}_i))\eta_h|\delta_{i\tau}\phi|^{r-2}\delta_{i\tau}\phi\, dx.\end{aligned}$$

Letting $\tau \to 0$ and then $h \to 0$ and using inequality (10.39) we obtain

$$\int_\Omega |\partial_{x_i}\phi|^r \left(\sigma - \frac{1}{r}\operatorname{div}\tilde{\mathbf{u}}\right) dx \leq \int_\Omega (\partial_{x_i}F - \partial_{x_i}\tilde{\mathbf{u}}\nabla\phi)|\partial_{x_i}\phi|^{r-2}\partial_{x_i}\phi\, dx. \tag{10.40}$$

Next note that

$$\sum_i \partial_{x_i}\tilde{\mathbf{u}}\nabla\phi\big|\partial_{x_i}\phi\big|^{r-2}\partial_{x_i}\phi \leq 3\|\tilde{\mathbf{u}}\|_{C^1(\Omega)} \sum_i |\partial_{x_i}\phi|^r.$$

On the other hand, since $1/r + 3 \leq 4$, inequalities (10.35) and (10.36) imply

$$\sigma - \left(\frac{1}{r} + 3\right)\|\tilde{\mathbf{u}}\|_{C^1(\Omega)} \geq \sigma - \sigma^* \geq 1.$$

From this we conclude that

$$\begin{aligned}(\sigma - \sigma^*)\sum_i \int_\Omega |\partial_{x_i}\phi|^r\, dx &\leq \sum_i \int_\Omega |\partial_{x_i}\phi|^{r-1}|\partial_{x_i}F|\, dx \\ &\leq \left(\sum_i \int_\Omega |\partial_{x_i}\phi|^{r/(r-1)}\, dx\right)^{(r-1)/r} \left(\sum_i \int_\Omega |\partial_{x_i}F|^r\, dx\right)^{1/r},\end{aligned}$$

which leads to the estimate

$$\|\nabla\phi\|_{L^r(\Omega)} \leq c(r)\sigma^{-1}\|\nabla F\|_{L^r(\Omega)} \quad \text{for } \sigma > \sigma^*(M, r). \tag{10.41}$$

Next multiplying both the sides of (10.34) by $|\phi|^{r-2}\eta_h$ and integrating the result over Ω we get the identity

$$\int_\Omega (\sigma - r^{-1}\operatorname{div}\tilde{\mathbf{u}})\eta_h|\phi|^r\,dx - \int_\Omega |\phi|^r\tilde{\mathbf{u}}\nabla\eta_h\,dx = \int_\Omega F\eta_h|\phi|^{r-2}\phi\,dx.$$

The passage $h \to 0$ gives the inequality

$$\int_\Omega (\sigma - r^{-1}\operatorname{div}\tilde{\mathbf{u}})|\phi|^r\,dx \le \int_\Omega |F||\phi|^{r-1}\,dx.$$

Recalling that $\sigma - r^{-1}\operatorname{div}\tilde{\mathbf{u}} \ge \sigma - \sigma^*$ we finally obtain

$$\|\phi\|_{L^r(\Omega)} \le c(r)\sigma^{-1}\|F\|_{L^r(\Omega)}. \tag{10.42}$$

Inequalities (10.41) and (10.42) imply estimate (10.37) for $s = 0, 1$. Hence for $\sigma > \sigma^*$, the linear operator $\tilde{\mathfrak{L}}^{-1} : F \mapsto \phi$ is continuous in the Banach spaces $H^{0,r}(\Omega)$ and $H^{1,r}(\Omega)$ and its norm does not exceed $c(r)\sigma^{-1}$. Recall that $H^{s,r}(\Omega)$ is the interpolation space $[L^r(\Omega), H^{1,r}(\Omega)]_{s,r}$. From this and Lemma 4.1 we conclude that inequality (10.37) is fulfilled for all $s \in [0, 1]$, which completes the proof. □

PROOF OF THEOREM 5.5. *Fix $\sigma > \sigma^*$, where the constant σ^* depends only on Σ, $\mathbf{U}$ and $\|\mathbf{u}\|_{C^1(\Omega)}$, and it is defined by* (10.36). *Without any loss of generality we can assume that $f \in C^1(\Omega)$. The existence and uniqueness of a weak bounded solution for $\sigma > \sigma^*$, follows from Lemma* 5.1. *Therefore, it suffices to prove estimate* (5.17) *for $\|\varphi\|_{H^{s,r}(\Omega)}$. Since $H^{s,r}(\Omega) \cap L^\infty(\Omega)$ is the Banach algebra, representation* (10.33) *together with inequality* (10.32) *implies*

$$\|\varphi\|_{H^{s,r}(\Omega)} \le c(1 + t^{-1})(\|\varphi_t\|_{H^{s,r}(\Omega_t)} + \|\varphi_t\|_{L^\infty(\Omega_t)}) + c\|\phi\|_{H^{s,r}(\Omega)}. \tag{10.43}$$

On the other hand, Lemma 10.11 *along with* (10.34) *yields*

$$\|\phi\|_{H^{s,r}(\Omega)} \le c\sigma^{-1}\|F\|_{H^{s,r}(\Omega)} \le c\sigma^{-1}\|\chi_{t/2}f\|_{H^{s,r}(\Omega)} + \sigma^{-1}\|\varphi_t\mathbf{u}\nabla\chi_{t/2}\|_{H^{s,r}(\Omega)}.$$

The first terms on the right-hand side is bounded,

$$\|\chi_{t/2}f\|_{H^{s,r}(\Omega)} \le c(1 + t^{-1})\|f\|_{H^{s,r}(\Omega)}.$$

In order to estimate the second term we note that, by virtue of (10.32), $\|\mathbf{u}\nabla\chi_{t/2}\|_{C^1(\Omega)} \le cM(1 + t^{-2})$ *which gives*

$$\|\varphi_t\mathbf{u}\nabla\chi_{t/2}\|_{H^{s,r}(\Omega)} \le cM(1 + t^{-2})(\|\varphi_t\|_{H^{s,r}(\Omega_t)} + \|\varphi_t\|_{L^\infty(\Omega_t)}).$$

Substituting the obtained estimates into (10.43) *we arrive at the inequality*

$$\begin{aligned}\|\varphi\|_{H^{s,r}(\Omega)} \le c(M+1)(1+t^{-2})\big((1+\sigma^{-1})\|\varphi_t\|_{H^{s,r}(\Omega)} + \|\varphi_t\|_{L^\infty(\Omega_t)} \\ + \sigma^{-1}\|f\|_{H^{s,r}(\Omega_t)} + \sigma^{-1}\|f\|_{L^\infty(\Omega)}\big),\end{aligned}$$

which along with (10.25) *leads to the estimate*

$$\|\varphi\|_{H^{s,r}(\Omega)} \le c(\sigma^{-1}\|f\|_{H^{s,r}(\Omega)} + \sigma^{-1+\alpha}\|f\|_{L^\infty(\Omega)}), \tag{10.44}$$

which holds true for all $\sigma > \sigma^* + 1$. It remains to note that c and σ^* depend only on Σ, $\mathbf{U}$, $\|\mathbf{u}\|_{C^1(\Omega)}$, r, s, α and do not depend on σ, and the theorem follows.

A. Appendix. Proof of Lemmas 10.1 and 10.2

Proof of Lemma 10.1. We start with the proof of (**P1**). Choose the Cartesian coordinate system (x_1, x_2, x_3) associated with the point P and satisfying Condition (**H1**). Let us consider the Cauchy problem.

$$\partial_{y_1}\mathbf{x} = \mathbf{u}(\mathbf{x}(y)) \quad \text{in } Q_a, \qquad \mathbf{x}|_{y_1=0} = \mathbf{x}_0(y_2) + y_3\mathbf{e}_3. \tag{A.1}$$

Here the function $\mathbf{x}_0$ is given by condition (**H2**). Without loss of generality we can assume that $0 < a < k < 1$. It follows from (**H1**) that for any such a, problem (A.1) has a unique solution of class $C^1(Q_a)$. Next note that, by virtue of condition (**H1**), for $y_1 = 0$, we have

$$|\mathbf{x}(y)| \le (C_\Gamma + 1)|y|, \qquad |\mathbf{u}(\mathbf{x}(y)) - \mathbf{u}(0)| \le M(C_\Gamma + 1)|y|. \tag{A.2}$$

Denote by $\mathfrak{F}(y) = D_y\mathbf{x}(y)$. The calculations show that

$$\mathfrak{F}_0 := \mathfrak{F}(y)\Big|_{y_1=0} = \begin{pmatrix} u_1 & \Upsilon'(y_2) & 0 \\ u_2 & 1 & 0 \\ u_3 & \partial_{y_2}F(\Upsilon(y_2), y_2) & 1 \end{pmatrix}$$

and

$$\mathfrak{F}(0) = \begin{pmatrix} U & \Upsilon'(0) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which along with (A.2) implies

$$\|\mathfrak{F}(0)^{\pm 1}\| \le C_M/3, \qquad \|\mathfrak{F}_0(y) - \mathfrak{F}(0)\| \le ca. \tag{A.3}$$

Differentiation of (A.1) with respect to y leads to the ordinary differential equation for $\mathfrak{F}$

$$\partial_{y_1}\mathfrak{F} = D_y\mathbf{u}(\mathbf{x})\mathfrak{F}, \qquad \mathfrak{F}\Big|_{y_1=0} = \mathfrak{F}_0.$$

Noting that $\partial_{y_1}\|\mathfrak{F} - \mathfrak{F}_0\| \le \|\partial_{y_1}\mathfrak{F}\|$ we obtain

$$\partial_{y_1}\|\mathfrak{F} - \mathfrak{F}_0\| \le M(\|\mathfrak{F} - \mathfrak{F}_0\| + \|\mathfrak{F}_0\|),$$

and hence $\|\mathfrak{F} - \mathfrak{F}_0\| \le c(M)\|\mathfrak{F}_0\|a$. Combining this result with (A.3) we finally arrive at

$$\|\mathfrak{F}(y) - \mathfrak{F}(0)\| \le ca. \tag{A.4}$$

From this and the implicit function theorem we conclude that there is a positive constant a, depending only on M and Σ, such that the mapping $x = \mathbf{x}(y)$ takes diffeomorphically the cube Q_a onto some neighborhood of the point P, and satisfy inequalities (10.2).

Let us turn to the proof of (**P2**). We begin with the observation that the manifold $\mathbf{x}(\Sigma \cap \mathcal{O}_P)$ is defined by the equation

$$\Phi_0(y) := x_3(y) - F(x_1(y), x_2(y)) = 0, \quad y \in Q_a.$$

Let us show that Φ_0 is strictly monotone in y_3 and has the opposite signs on the faces $y_3 = \pm a$. To this end note that the formula for $\mathfrak{F}(0)$ along with (A.4) implies the estimates

$$|\partial_{y_3} x_3(y) - 1| + |\partial_{y_3} x_1(y)| + |\partial_{y_3} x_2(y)| \le ca \quad \text{in } Q_{a_1}.$$

Thus we get

$$1 - ca \le \partial_{y_3}\Phi_0(y) = \partial_{y_3} x_3(y) - \partial_{x_i} F(x_1, x_2)\partial_{y_3} x_i(y) \le 1 + ca. \tag{A.5}$$

On the other hand, by (A.4), we have the inequality $|x_3(y)| \le ca|y|$, which along with (10.2) yields the estimate

$$|\Phi_0(y)| \le |x_3(y)| + |F(x(y))| \le ca|y| + KC_M|y|^2 \le ca^2 \quad \text{for } y_3 = 0. \tag{A.6}$$

Combining (A.5) and (A.6), we conclude that there exists a positive a depending only on M and Σ, such that the inequalities

$$1/2 \le \partial_{y_3}\Phi_0(y) \le 2, \qquad \pm\Phi_0(y_1, y_2, \pm a) > 0, \tag{A.7}$$

hold true for all $y \in Q_a$. Therefore, the equation $\Phi_0(y) = 0$ has a unique solution $y_3 = \Phi(y_1, y_2)$ in the cube Q_a, this solution vanishes for $y_1 = y_3 = 0$. By the implicit function theorem, the function Φ belongs to the class $C^1([-a, a]^2)$.
It remains to prove that Φ admits the both-side estimates (10.3). Note that inequality (10.2) implies the estimate $|\mathbf{u}(\mathbf{x}(y)) - U\mathbf{e}_1| \le M|\mathbf{x}(y)| \le MC_M a$. Therefore, we can choose $a = a(M, \Sigma)$ sufficiently small, such that

$$2U/3 \le u_1 \le 4U/3, \qquad C_\Gamma|u_2| \le U/3.$$

Recall that $x_1(y) - \Upsilon(x_2(y))$ vanishes at the plane $y_1 = 0$ and

$$\partial_{y_1}[x_1(y) - \Upsilon(x_2(y))] = u_1(y) - \Upsilon'(x_2(y))u_2(y).$$

Since $|\Upsilon'| \le C_\Gamma$, we obtain from this that

$$|y_1|U/3 \le \big|x_1(y) - \Upsilon(x_2(y))\big| \le |y_1|5U/3 \quad \text{for } y \in Q_a. \tag{A.8}$$

Equations (10.1) implies the identity

$$\partial_{y_1}\Phi_0(y) \equiv \nabla F_0(\mathbf{x}(y)) \cdot \mathbf{u}(x(y)) = \nabla F_0(\mathbf{x}(y)) \cdot \mathbf{U}(x(y)) \quad \text{for } \Phi_0(y) = 0.$$

Combining this result with (5.14) and (A.8), we finally obtain the inequality,

$$|y_1|N^- U/3 \le |\partial_{y_1}\Phi_0(y)| \le |y_1|N^+ U/5, \tag{A.9}$$

which along with estimate (A.7) and the identity $\partial_{y_1}\Phi = -\partial_{y_1}\Phi_0(\partial_{y_3}\Phi_0)^{-1}$ yields (10.4). Since the term $x_1(y) - \Upsilon(x_2(y))$ is positive for positive y_1, the function Φ is increasing in y_1 for $y_1 > 0$ and is decreasing for $y_1 < 0$, which implies the existence of the functions

$a^{\pm}$. Next, the identities $\partial_{y_i} a^{\pm} = -\partial_{y_i}\Phi_0/\partial_{y_1}\Phi_0$, $i = 2, 3$, and estimate (A.9) yield the inequality

$$|\partial_{y_i} a^{\pm}(y)| \le c|y_1|^{-1}.$$

On the other hand, for $y_1 = a^{\pm}(y_2, y_3)$, we have $y_3 = |\Phi(y_1, y_2)| \ge c y_1^2$ and hence $|y_1|^{-1} \le c y_3^{-1/2}$, which implies the first estimates in (10.5). The second estimate is obvious.
In order to prove inclusions (10.7) note that $\Phi(-a, y_2) \ge a^2 C^-$ and hence $B_0(r) \cap \{y_3 > \Phi\} \subset G_a \subset Q_a$ for $r = a^2 C^-$. But estimate (10.2) implies that $B_P(\rho_c) \subset \mathbf{x}(B_0(r))$ for $\rho_c = r C_M^{-1}$, which yields the first inclusion in (10.7). It remains to note that the second is a consequence of (10.2) and the lemma follows.

Proof of Lemma 10.2. The proof simulates the proof of the Lemma 10.1. Choose the local Cartesian coordinates (x_1, x_2, x_3) centered at P such that in new coordinates $\mathbf{n} = \mathbf{e}_3$. By the smoothness of Σ, there is a neighborhood $\mathcal{O} = [-k, k]^2 \times [-t, t]$ such that the manifold $\Sigma \cap \mathcal{O}$ is defined by the equation

$$x_3 = F(x_1, x_2), \quad F(0, 0) = 0, \quad |\nabla F(x_1, x_2)| \le K(|x_1| + |x_2|).$$

The constants k, t and K depend only on Σ. Let us consider the initial value problem

$$\partial_{y_3}\mathbf{x} = \mathbf{u}(\mathbf{x}(y)) \quad \text{in } Q_a, \qquad \mathbf{x}|_{y_3=0} = (y_1, y_2, F(y_1, y_2)). \tag{A.10}$$

Without loss of generality we can assume that $0 < b < k < 1$. It follows from (**H1**) that for any such b, problem (A.10) has a unique solution of class $C^1(Q_b)$. Next, note that for $y_3 = 0$ we have

$$|\mathbf{x}(y)| \le (K + 1)|y|, \qquad |\mathbf{u}(\mathbf{x}(y)) - \mathbf{u}(0)| \le M(K + 1)|y|. \tag{A.11}$$

Denote by $\mathfrak{F}(y) = D_y\mathbf{x}(y)$. The calculations show that

$$\mathfrak{F}_0 := \mathfrak{F}(y)\Big|_{y_3=0} = \begin{pmatrix} 1 & 0 & u_1 \\ 0 & 1 & u_2 \\ 0 & 0 & u_3 \end{pmatrix} \qquad \mathfrak{F}(0) = \begin{pmatrix} 1 & 0 & u_1(P) \\ 0 & 1 & u_2(P) \\ 0 & 0 & U_n \end{pmatrix},$$

which along with (A.11) implies

$$\|\mathfrak{F}(0)^{\pm 1}\| \le C_{M,N}/3, \qquad \|\mathfrak{F}_0(y) - \mathfrak{F}(0)\| \le cb. \tag{A.12}$$

Next, differentiation of (A.10) with respect to y leads to the equation

$$\partial_{y_1}\mathfrak{F} = D_y\mathbf{u}(\mathbf{x})\mathfrak{F}, \qquad \mathfrak{F}\Big|_{y_3=0} = \mathfrak{F}_0.$$

Arguing as in the proof of Lemma 10.1 we obtain $\|\mathfrak{F} - \mathfrak{F}_0\| \le c(M)\|\mathfrak{F}_0\| b$. Combining this result with (A.12) we finally arrive at $\|\mathfrak{F}(y) - \mathfrak{F}(0)\| \le cb$. From this and the implicit function theorem we conclude that there is positive b, depending only on M and Σ, such that the mapping $x = \mathbf{x}(y)$ takes diffeomorphically the cube Q_b onto some neighborhood of the point P, and satisfies inequalities (10.9). Inclusions (10.10) easily follows from (10.9).

B. Appendix. Proof of Lemma 10.3

Existence and uniqueness of solution to problem (10.12) is obvious. Multiplying both the sides of equation (10.12) by $|\varphi|^{r-1}\varphi$ and integrating the result over Q_a^ϕ we obtain the inequality

$$\sigma \int_{Q_a^\phi} |\varphi|^r\, dy \le \int_{Q_a^\phi} |\varphi|^{r-1} |f|\, dy \le \Big(\int_{Q_a^\phi} |\varphi|^r\, dy\Big)^{1-1/r} \Big(\int_{Q_a^\phi} |f|^r\, dy\Big)^{1/r},$$

which yields (10.13) for $r < \infty$. Letting $r \to \infty$ we get (10.13) for $r = \infty$.

Let us turn to the proof of inequality (10.14) and begin with the case $s = 1$. For every $y \in \mathbb{R}^3$, we denote by $Y = (y_2, y_3)$. It is easy to see that the function $\partial_{y_3}\varphi$ has the representation $\partial_{y_3}\varphi = \varphi' + \varphi''$, where

$$\varphi'(y) = -e^{\sigma(y_1 - a^-(Y))}\partial_{y_3} a_-(Y) f(a^-(Y), Y),$$

and φ'' is a solution to boundary value problem

$$\partial_{y_1}\varphi'' + \sigma\varphi'' = \partial_{y_3} f \quad \text{in } Q_a^\phi, \qquad \varphi''(y) = 0 \quad \text{for } y_1 = a^-(y_2, y_3).$$

It follows from (10.13) that $\|\varphi''\|_{L^r(Q_a^\phi)} \le \sigma^{-1}\|f\|_{H^{1,r}(Q_a^\phi)}$. On the other hand, inequalities

$$|\partial a^\pm(Y)| \le c y_3^{-1/2}, \qquad a^+(Y) - a^-(Y) \le c y_3^{1/2},$$

yield the estimate

$$\begin{aligned}\int_{a^-(Y)}^{a^+(Y)} |\varphi'(y_1, Y)|^r\, dy_1 &\le c(r)\sigma^{-1}\|f\|^r_{L^\infty(Q_a^\phi)} y_3^{-r/2}(1 - e^{r\sigma(a^-(Y) - a^+(Y))})\\ &\le c(r,\beta)\sigma^{-1+\beta} y_3^{(\beta - r)/2}\|f\|^r_{L^\infty(Q_a^\phi)},\end{aligned}$$

which holds true for all $\beta \in [0, 1]$. We conclude from this that

$$\|\varphi'\|_{L^r(Q_a^\phi)} \le c(r,\beta)\sigma^{(\beta-1)/r}\|f\|_{L^\infty(Q_a^\phi)} \quad \text{for } r - 2 < \beta < 1.$$

Combining this result with the estimate of φ'' and setting $\alpha = 1 + (\beta - 1)/r$ we finally obtain

$$\|\partial_{y_3}\varphi\|_{L^r(Q_a^\phi)} \le c(r,\alpha)(\sigma^{-1}\|f\|_{H^{1,r}(Q_a^\phi)} + \sigma^{-1+\alpha}\|f\|_{L^\infty(Q_a^\phi)}).$$

The same estimates hold true for $\partial_{y_2}\varphi$ and $\partial_{y_1}\varphi$, which yields (10.14) in the case $s = 1$.

The proof of inequality (10.14) for $0 < s < 1$ is more complicated. By virtue of (10.13), it suffices to estimate the semi-norm $|\varphi|_{s,r,Q_a^\phi}$. For an arbitrary $y, z \in \mathbb{R}^3$, set $Y = (y_2, y_3)$,

$Z = (z_2, z_3)$. Since expression (4.4) for the semi-norm $|\varphi|_{s,r,Q_a^\phi}$ is invariant with respect to the permutation $(Y, Z) \to (Z, Y)$, we have

$$|\varphi|_{s,r,Q_a^\phi} \le (2I)^{1/r}, \qquad I = \int_{D_a} |\varphi(z) - \varphi(y)|^r |z - y|^{-3-rs}\, dxdy, \tag{B.1}$$

where $D_a = \{(y, z) \in (Q_a^\Phi)^2 : a^-(Z) \le a^-(Y)\}$. It is easy to see that

$$\begin{aligned}\varphi(z) - \varphi(y) &= \varphi(z_1, Z) - \varphi(y_1, Z) + \int_{a^-(Z)}^{a^-(Y)} e^{\sigma(x_1-y_1)} f(x_1, Z)\, dx_1 \\ &+ \int_{a^-(Y)}^{y_1} e^{\sigma(x_1-y_1)} (f(x_1, Z) - f(x_1, Y))\, dx_1 = I_1 + I_2 + I_3.\end{aligned} \tag{B.2}$$

Hence our task is to estimate the quantities

$$J_k = \int_{D_a} |I_k|^r |z - y|^{-3-rs}\, dydz, \quad k = 1, 2, 3. \tag{B.3}$$

The evaluation falls naturally into three steps and is based on the following proposition

PROPOSITION B.1. *If $r, s > 0$ and $i \ne j \ne k$, $i \ne k$, then*

$$\int_{[-a,a]^2} |z - y|^{-3+rs}\, dy_i dy_j \le c(r, s)|z_k - y_k|^{-1-rs},$$
$$\int_{[-a,a]} |z - y|^{-3+rs}\, dy_i \le c(r, s)(|z_j - y_j|^2 + |z_k - y_k|^2)^{(-2-rs)/2}.$$

PROOF. The left-hand side of the first equality is equal to

$$\begin{aligned}&|z_k - y_k|^{-1-rs} \int_{[-a,a]^2} \left(1 + \frac{|z_i - y_i|^2 + |z_j - y_j|^2}{|z_k - y_k|^2}\right)^{(-2-rs)/2} \frac{dy_i dy_j}{|z_k - y_k|^2} \\ &\quad \le c(r, s)|z_k - y_k|^{-1-rs} \int_{\mathbb{R}^2} (1 + |y_i|^2 + |y_j|^2)^{-(3+rs)/2}\, dy_i dy_j,\end{aligned}$$

which yields the first estimate. Repeating these arguments gives the second, and the proposition follows. □

THE FIRST STEP. We begin with the observation that, by virtue of the extension principle, the right-hand side f has an extension over $\mathbb{R}^3$, which vanishes outside the cube Q_{3a} and satisfies the inequalities

$$\|f\|_{H^{s,r}(\mathbb{R}^3)} \le c(a, r, s)\|f\|_{H^{s,r}(Q_a)}, \qquad \|f\|_{L^\infty(\mathbb{R}^3)} \le \|f\|_{L^\infty(Q_a)}. \tag{B.4}$$

Next recall that $a^-(Z) \le y_1, z_1 \le a$ for all $(y,z) \in D_a$. From this and Proposition B.1 we obtain

$$\begin{aligned}
&\int_{D_a} |I_1|^r |z-y|^{-3-rs}\, dydz \\
&\quad \le \int_{[-a,a]^2} \Big\{ \int_{a^-(Z)}^{a} \int_{a^-(Z)}^{a} |\varphi(z_1, Z) - \varphi(y_1, Z)|^r \\
&\qquad \times \Big\{ \int_{[-a,a]^3} |z-y|^{-3-rs} dy_2 dy_3 dz_2 \Big\}\, dy_1 dz_1 \Big\}\, dZ \\
&\quad \le \int_{[-a,a]^2} \Big\{ \int_{[a^-(Z),a]^2} |\varphi(z_1, Z) - \varphi(y_1, Z)|^r |y_1 - z_1|^{-1-rs}\, dy_1 dz_1 \Big\}\, dZ.
\end{aligned}$$

Since the right-hand side of this inequality is invariant with respect to the permutation $(y_1, z_1) \to (z_1, y_1)$, we have

$$J_1 \le c(r,s) \int_{[-a,a]^2} \Big\{ \int_{D(Z)} |\varphi(z_1, Z) - \varphi(y_1, Z)|^r |y_1 - z_1|^{-1-rs}\, dy_1 dz_1 \Big\}\, dZ, \tag{B.5}$$

where $D(Z) = \{(y_1, z_1) : a^-(Z) \le z_1 \le y_1 \le a\}$. Now our task is to estimate the integral over $D(Z)$. Note that for all $(y_1, z_1) \in D(Z)$, we have the identity

$$\begin{aligned}
\varphi(y_1, Z) - \varphi(z_1, Z) = &\int_{a^-(Z)}^{z_1} e^{\sigma(t-z_1)} (f(t+\xi, Z) - f(t, Z))\, dt \\
&+ \int_{a^-(Z)-\xi}^{a^-(Z)} e^{\sigma(t-z_1)} f(t+\xi, Z)\, dt \;:=\; I_{11} + I_{12},
\end{aligned} \tag{B.6}$$

where $\xi = y_1 - z_1$.

Since f is extended over $\mathbb{R}^3$, we have the estimate

$$\begin{aligned}
&\int_{D(Z)} |I_{11}|^r |y_1 - z_1|^{-1-rs}\, dy_1 dz_1 \\
&\quad \le \int_{a^-(Z)}^{a} \int_0^{2a} \Big| \xi^{-s-1/r} \int_{a^-(Z)}^{z_1} e^{\sigma(t-z_1)} (f(t+\xi, Z) - f(t, Z))\, dt \Big|^r\, dz_1 d\xi \\
&\quad = \int_{a^-(Z)}^{a} \int_0^{2a} |M(z_1, \xi, Z)|^r\, dz_1 d\xi.
\end{aligned}$$

It is easy to see that the function M on the right-hand side satisfies the equation and boundary condition

$$\partial_{z_1} M + \sigma M = K \quad \text{for } z_1 \in (a^-(Z), a), \qquad M = 0 \quad \text{for } z_1 = a^-(Z),$$

where $K(z_1, \xi, Z) = \xi^{-s-1/r}(f(z_1+\xi, Z) - f(z_1, Z))$. Multiplying both the sides of this equation by $|M|^{r-2}M$ and integrating the result over the interval $(a^-(Z), a)$ we arrive at the inequality

$$\sigma \int_{a^-(Z)}^{a} |M|^r \, dz_1 \le \int_{a^-(Z)}^{a} |M|^{r-1} |K| \, dz_1 \le \Big(\int_{a^-(Z)}^{a} |M|^r \, dz_1 \Big)^{1-1/r} \Big(\int_{a^-(Z)}^{a} |K|^r \, dz_1 \Big)^{1/r},$$

which gives

$$\int_{a^-(Z)}^{a} |M(z_1, \xi, Z)|^r \, dz_1 \le \sigma^{-r} \xi^{-1-rs} \int_{a^-(Z)}^{a} |f(z_1 + \xi, Z) - f(z_1, Z)|^r \, dz_1.$$

Recalling that f is extended over $\mathbb{R}^3$ and vanishes outside the cube Q_{3a} we obtain the following estimate for the quantity I_{11} on the right-hand side of (B.6),

$$\begin{aligned} &\int_{[-a,a]^2} \int_{D(Z)} |I_{11}|^r |y_1 - z_1|^{-1-rs} \, dy_1 dz_1 dZ \\ &\quad \le \sigma^{-r} \int_{[-a,a]^2} \int_{a^-(Z)}^{a} \int_0^{2a} \xi^{-1-rs} |f(z_1 + \xi, Z) - f(z_1, Z)|^r \, dz_1 d\xi dZ \\ &\quad \le \sigma^{-r} \int_{\mathbb{R}^4} |f(y_1, Z) - f(z_1, Z)|^r |y_1 - z_1|^{-1-rs} dy_1 dz_1 dZ \\ &\quad \le c\sigma^{-r} \|f\|^r_{L^r(\mathbb{R}^2; H^{r,s}(\mathbb{R}))} \\ &\quad \le c\sigma^{-r} \|f\|^r_{H^{r,s}(\mathbb{R}^3)}. \end{aligned} \tag{B.7}$$

In order to estimate I_{12} note that for any $\beta \in [0, 1]$,

$$\begin{aligned} |I_{12}| &\le \|f\|_{L^\infty(Q_{2a})} e^{\sigma(a^-(Z) - z_1)} \sigma^{-1} (1 - e^{-\sigma\xi}) \\ &\le c(\beta) \|f\|_{L^\infty(Q_{2a})} \sigma^{\beta - 1} e^{\sigma(a^-(Z) - z_1)} \xi^\beta. \end{aligned}$$

Hence for all $\beta \in (s, 1]$ we have

$$\begin{aligned} &\int_{[-a,a]^2} \int_{D(Z)} |I_{12}|^r |y_1 - z_1|^{-1-rs} \, dy_1 dz_1 dZ \\ &\quad \le c\sigma^{(\beta-1)r} \|f\|^r_{L^\infty(Q_{2a})} \int_{[-a,a]^2} \int_{a^-(Z)}^{a} \int_0^{2a} e^{\sigma(a^-(Z) - z_1)} \xi^{-1+r(\beta - s)} \, dZ dz_1 d\xi \\ &\quad \le c(r, s, \lambda) \sigma^{-1+(\beta-1)r} \|f\|^r_{L^\infty(Q_{2a})}. \end{aligned} \tag{B.8}$$

Substituting inequalities (B.7), (B.8) in (B.5), setting $\alpha = \beta - r^{-1}$, and recalling inequalities (B.4) we conclude that the estimate

$$J_1 \le c(r, s, a, \lambda) (\sigma^{-r} \|f\|^r_{H^{r,s}(Q_a)} + \sigma^{-r+\alpha r} \|f\|^r_{L^\infty(Q_a)}) \tag{B.9}$$

holds true for each $\alpha \in (s - r^{-1}, 1 - r^{-1}]$. Since $\sigma > 1$, this estimate is obviously fulfilled for all $\alpha > s$.

THE SECOND STEP. Our next task is to estimate J_2. Recall that, by virtue of (10.11) and Lemma 10.1, for all $(y,z) \in D_a$,

$$a^-(Z) \le a^-(Y) \le y_1 \le a, \quad a^-(Z) \le z_1 \le a, \quad cz_1^2 \le z_3 \le a. \tag{B.10}$$

In particular, for any $\alpha \in [0,1]$, we have

$$\begin{aligned}|I_2| &\le \|f\|_{L^\infty(Q_a)}\sigma^{-1}(1-e^{\sigma(a^-(Z)-a^-(Y))})\\ &\le \|f\|_{L^\infty(Q_a)}\sigma^{-1+\alpha}|a^-(Z)-a^-(Y))|^\alpha.\end{aligned}$$

Since,

$$|a^-(Z)-a^-(Y))| \le c|z_2-y_2| + c|\sqrt{z_3}-\sqrt{y_3}|,$$

we conclude from this that

$$J_2 = \int_{D_a}|I_2|^r|z-y|^{-3-rs} \le c\sigma^{(\alpha-1)r}\|f\|^r_{L^\infty(Q_a)}(J_{21}+J_{22}), \quad \text{where} \tag{B.11}$$

$$J_{21} = \int_{D_a}|z_2-y_2|^{\alpha r}|z-y|^{-3-rs}\,dydz,$$

$$J_{22} = \int_{D_a}|\sqrt{z_3}-\sqrt{y_3}|^{\alpha r}|z-y|^{-3-rs}\,dydz.$$

On the other hand, Proposition B.1 yields the estimate

$$\int_{[-a,a]^4}|z-y|^{-3-rs}dy_1dy_3dz_3 \le c(r,s)|z_2-y_2|^{-1-rs},$$

which leads to the inequality

$$J_{21} \le c(r,s)\int|z_2-y_2|^{-1+r(\alpha-s)} \le c(r,s,\alpha) \quad \text{for all } \alpha \in (s,1]. \tag{B.12}$$

Next note that

$$J_{22} \le \int_a^a dz_1\left\{\int_{cz_1^2}^a dz_3\left\{\int_0^a|\sqrt{z_3}-\sqrt{y_3}|^{\alpha r}\right.\right.$$
$$\left.\left.\left\{\int_{[-a,a]^3}|y-z|^{-3-rs}\,dy_1dy_2dz_2\right\}dy_3\right\}\right\}. \tag{B.13}$$

It follows from Proposition B.1 that the interior integral has the estimate

$$\begin{aligned}&\int_0^a|\sqrt{z_3}-\sqrt{y_3}|^{\alpha r}\left\{\int_{[-a,a]^3}|y-z|^{-3-rs}\,dy_1dy_2dz_2\right\}dy_3\\ &\quad\le \int_0^a|\sqrt{z_3}+\sqrt{y_3}|^{-\alpha r}|z_3-y_3|^{-1+(\alpha-s)r}\,dy_3\\ &\quad= z_3^{r(\alpha/2-s)}\int_0^{a/z_3}(1+\sqrt{t})^{-\alpha r}|1-t|^{-1+(\alpha-s)r}\,dt \le cz_3^{r(\alpha/2-s)}\\ &\qquad\text{for } \alpha/2 < s < \alpha.\end{aligned}$$

Substituting this result in (B.13) we arrive at the inequality

$$J_{22} \le c(r, s, \alpha) \int_a^a |z_1|^{r(\alpha-2s)+2} dz_1 \le c \quad \text{for } s \in (\alpha/2, \alpha), \alpha \in (0, 1),$$
$$(2s - \alpha)r < 3.$$

Combining this result with (B.12) we finally obtain that for all exponents λ, r, and s satisfying the inequalities

$$1 < r < \infty, \quad 0 < s < 1, \quad \alpha/2 < s < \alpha < 1, \quad (2s - \alpha)r < 3, \tag{B.14}$$

the integral J_2 has the estimate

$$J_2 \le c(r, s, \alpha, a)\|f\|^r_{L^\infty(Q_a)} \sigma^{r(\alpha-1)}. \tag{B.15}$$

The third step. We begin with the observation that the function $I_3(y_1, Y, Z)$ defined by relation (B.2) satisfies the equation and boundary condition

$$\partial_{y_1} I_3 + \sigma I_3 = K_3 \quad \text{for } a^-(Y) < y_1 < a, \qquad I_3(a^-(Y), Y, Z) = 0,$$

where $K_3(y_1, Y, Z) = f(y_1, Z) - f(y_1, Y)$. Multiplying both the sides of this equation by $|I_3|^{r-2} I_3$ and integrating the result over the interval $(a^-(Y), a)$ we arrive at the inequality

$$\sigma \int_{a^-(Y)}^a |I_3|^r \, dy_1 \le \int_{a^-(Y)}^a |I_3|^{r-1} |K_3| \, dy_1$$
$$\le \left(\int_{a^-(Y)}^a |I_3|^r \, dy_1 \right)^{1-1/r} \left(\int_{a^-(Y)}^a |K_3|^r \, dy_1 \right)^{1/r},$$

which leads to the estimate

$$\int_{a^-(Y)}^a |I_3|^r \, dy_1 \le \sigma^{-r} \int_{[-a,a]} |f(y_1, Z) - f(y_1, Y)|^r \, dy_1.$$

Since $a^-(Y) \le y_1$ for all $(y, z) \in D_a$, we conclude from this and the inequality

$$\int_{[-a,a]} |z - y|^{-3-rs} dz_1 \le c|Y - Z|^{-2-rs}$$

that

$$J_3 = \int_{D_a} |I_3|^r |z - y|^{-3-rs} dy dz$$
$$\le c\sigma^{-r} \int_{[-a,a]^3} |f(y_1, Z) - f(y_1, Y)|^r |Y - Z|^{-2-rs} dy_1 dY dZ$$
$$\le c\sigma^{-r} \|f\|^r_{L^r(-a,a; H^{s,r}([-a,a]^2))} \le c\sigma^{-r} \|f\|^r_{H^{s,r}(Q_a)}. \tag{B.16}$$

Combining estimates (B.9), (B.15), and (B.16) we conclude that the estimate

$$\|\varphi\|_{s,r Q_a^\phi} \le c(a, r, s, \lambda)(\sigma^{-1} \|f\|_{H^{s,r}(Q_a)} + \sigma^{-r(1-\alpha)} \|f\|_{L^\infty(Q_a)}),$$

holds true for all exponents r, s and α satisfying inequalities (B.14). It remains to note that these inequalities can be written in the form $\max\{2s - 3/r, s\} < \alpha < \min\{2s, 1\}$.

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CHAPTER 6

Positive Solutions for Lotka–Volterra Systems with Cross-Diffusion

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Abstract

This article is concerned with reaction-diffusion systems with nonlinear diffusion effects, which describe competition models and prey–predator models of Lotka-Volterra type in population biology. The system consists of two nonlinear diffusion equations where two unknown functions denote population densities of two interacting species. The main purpose is to discuss the existence and nonexistence of positive steady state solutions to such systems. Here a positive solution corresponds to a coexistence state in population models. We will derive a priori estimates of positive solutions by maximum principle for elliptic equations and employ the degree theory on a positive cone to show the existence of a positive solution. The existence results can be reconsidered from the view-point of bifurcation theory. We will give some information on the direction of bifurcation of positive solutions and their stability properties in terms of some biological coefficients. Moreover, we will also study the existence of multiple positive solutions for a certain class of prey–predator systems with nonlinear diffusion by making one of cross-diffusion coefficients sufficiently large.

Keywords: Reaction-diffusion system, Competition model, Prey–predator model, Cross-diffusion, Degree theory, Bifurcation theory, Maximum principle

AMS Subject Classifications: primary 35K57; secondary 35B32, 35B50, 35J65, 37C25, 92D25

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HANDBOOK OF DIFFERENTIAL EQUATIONS
Stationary Partial Differential Equations, volume 6
Edited by M. Chipot

1. Introduction

1.1. *Problems*

In this article we study positive steady-state solutions for reaction diffusion systems appearing in population biology. One of the typical reaction diffusion systems is as follows:

$$\begin{cases} u_t = \Delta[(d_1 + \alpha_{11}u + \alpha_{12}v)u] + u(a_1 - b_1u - c_1v) & \text{in } \Omega \times (0, \infty), \\ v_t = \Delta[(d_2 + \alpha_{21}u + \alpha_{22}v)v] + v(a_2 - b_2u - c_2v) & \text{in } \Omega \times (0, \infty), \\ Bu = Bv = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(\cdot, 0) = u_0 \geq 0, \quad v(\cdot, 0) = v_0 \geq 0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\boldsymbol{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$; $Bu = u$ (Dirichlet boundary condition) or $Bu = \partial u/\partial \nu$ with outward normal ν on $\partial\Omega$ (Neumann boundary condition); α_{ij} $(i, j = 1, 2)$ are nonnegative constants; a_i, b_i, c_i, d_i $(i = 1, 2)$ are also positive constants. The system (1.1) is known as the Lotka–Volterra competition system with nonlinear diffusion effects. In (1.1), u and v, respectively, represent the population densities of two species which are interacting and migrating in the same habitat Ω. Such a population model was first proposed by Shigesada *et al.* [57] to investigate the habitat segregation phenomena between two competing species.

According to Okubo and Levin [44], the formulation of diffusion is based on the assumption that individual species move under the influence of the following fundamental forces; (i) a dispersive force associated with the random movements of individuals; (ii) an attractive force, which induces directed movement of individual species toward favorable environments; and (iii) population pressure due to interferences between individual species. In (1.1), d_i represents random dispersive force of movement of an individual and α_{ij} describes mutual interferences between individuals due to population pressure. Especially, α_{12} and α_{21} are usually referred to as *cross-diffusion* coefficients, while α_{11} and α_{22} are referred as *self-diffusion* coefficients. The interesting point in (1.1) is that the migration of two competing species is affected by the population pressure in addition to the natural random movement. Many numerical simulations exhibit that the nonlinear dispersive force due to intra- and interspecific interactions brings about a spatial segregation which linear diffusion alone cannot give in the competition system. For details about the background in the field of mathematical biology, see the monograph of Okubo and Levin [44].

In case $Bu = u$ for (1.1), the boundary condition means that the habitat Ω is surrounded by a hostile environment. In case $Bu = \partial u/\partial \nu$, the boundary condition $\partial u/\partial \nu = 0$ is usually called no-flux condition which implies that there exists no migration across the boundary.

The present article is concerned with the steady-state problem of the form

$$\begin{cases} \Delta[\varphi(u, v)u] + uF(u, v) = 0 & \text{in } \Omega, \\ \Delta[\psi(u, v)v] + vG(u, v) = 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \\ u \geq 0, \quad v \geq 0 & \text{in } \Omega, \end{cases} \tag{1.2}$$

where $\varphi(u, v)$, $\psi(u, v)$ are positive functions for $u, v \geq 0$ and F, G are given by

$$F(u, v) = a_1 - b_1 u - c_1 v \quad \text{and} \quad G(u, v) = a_2 - b_2 u - c_2 v.$$

In a competition model, it is standard to assume that a_i, b_i, c_i $(i = 1, 2)$ are all positive constants. If we discuss a prey–predator model for prey u and predator v, we usually assume $a_1, b_1, c_1, c_2 > 0$ and $b_2 < 0$ (a_2 may be negative). We are mainly interested in positive solutions of (1.2); any solution (u, v) of (1.2) is called a positive solution if $u > 0$ in Ω and $v > 0$ in Ω. For the linear diffusion case, i.e, $\varphi(u, v) =$ positive constant and $\psi(u, v) =$ positive constant, there are lots of results on the structure of positive solutions; see, e.g., [4,10,11,14,16–18,30,31,38,45,52] for competition models and see, e.g., [3,12, 15,28,29,39,62] for prey–predator models. However, there are only a few works on the structure of positive solutions of steady-state problems with nonlinear diffusion effects (see [53,55] for competition models, [23,25,43] for prey–predator models and [46,54] for both models).

The purpose of this article is to study the structure of the set of positive solutions of (1.2). We will discuss the existence, uniqueness, multiplicity and stability of positive solutions. Our existence results will be derived using the degree theory. These existence results can be reconsidered from the viewpoint of bifurcation theory. In the course of such studies, we can observe the close relationship between the direction of bifurcation for a branch of positive solutions and their stability properties. It should be noted that homogeneous Dirichlet boundary conditions in (1.2) can be replaced by homogeneous Neumann boundary conditions. For such problems, the structure of nonconstant positive solutions is discussed in [21,32,33,37,40,42] and [61]. See also the article of Ni [41] in this series.

In Section 1 we study the following steady-state problem for a logistic equation with linear diffusion

$$\begin{cases} \Delta w + w(a - w) = 0 & \text{in } \Omega, \\ w \geq 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where a is a positive number. It is well known that (1.3) has a unique positive solution θ_a if and only if $a > \lambda_1$, where λ_1 is the least eigenvalue of $-\Delta$ with zero Dirichlet boundary condition. We will prepare various properties of θ_a, which will be helpful to study positive solutions of reaction diffusion systems with nonlinear diffusion.

As preliminary results, we also collect some necessary results on the degree theory for nonlinear operators on a positive cone established by Amann [1] and Dancer [9]. This theory has been used and developed to study the existence of positive solutions for nonlinear elliptic problems with homogeneous Dirichlet boundary conditions.

We first study the following competition model with cross-diffusion terms:

$$\begin{cases} u_t = \Delta[(1 + \alpha v)u] + u(a - u - cv) & \text{in } \Omega \times (0, \infty), \\ \rho v_t = \Delta[(1 + \beta u)v] + v(b - du - v) & \text{in } \Omega \times (0, \infty), \\ u = v = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(\cdot, 0) = u_0, \quad v(\cdot, 0) = v_0 & \text{in } \Omega, \end{cases} \tag{1.4}$$

where u, v denote the population densities of two competing species, α, β are nonnegative constants, ρ, a, b, c, d are positive constants and u_0, v_0 are nonnegative functions. The steady-state problem for (1.4) corresponds to (1.2) with $\varphi(u, v) = 1 + \alpha v$, $\psi(u, v) = 1 + \beta u$, $F(u, v) = a - u - cv$ and $G(u, v) = b - du - v$. This problem possesses two semi-trivial steady states

$$(\theta_a, 0) \quad \text{for } a > \lambda_1 \quad \text{and} \quad (0, \theta_b) \quad \text{for } b > \lambda_1.$$

The stability properties of these semi-trivial states are discussed in Section 1.3. It will be shown in Proposition 1.5 that

$$(\theta_a, 0) \quad \text{is} \quad \begin{cases} \text{asymptotically stable} & \text{if } \lambda_1\left(\dfrac{d\theta_a - b}{1 + \beta\theta_a}\right) > 0, \\ \text{unstable} & \text{if } \lambda_1\left(\dfrac{d\theta_a - b}{1 + \beta\theta_a}\right) < 0, \end{cases} \tag{1.5}$$

where $\lambda_1(q)$ denotes the least eigenvalue of $-\Delta + q(x)$ with homogeneous Dirichlet boundary condition for $q \in C(\overline{\Omega})$. The same stability result for $(0, \theta_b)$ also holds true with $\lambda_1((d\theta_a - b)/(1 + \beta\theta_a))$ replaced by $\lambda_1((c\theta_b - a)/(1 + \alpha\theta_b))$ in (1.5). Here it is possible to show that, for any fixed $(a, b) \in (\lambda_1, \infty) \times (\lambda_1, \infty)$, $\lambda_1((d\theta_a - b)/(1 + \beta\theta_a))$ is positive for sufficiently large β. This fact together with (1.5) implies that $(\theta_a, 0)$ becomes asymptotically stable owing to large cross-diffusion effect.

The prey–predator model which we want to discuss mainly in this article is given by the following system

$$\begin{cases} u_t = \Delta[(1 + \alpha v)u] + u(a - u - cv) & \text{in } \Omega \times (0, \infty), \\ \rho v_t = \Delta[(1 + \beta u)v] + v(b + du - v) & \text{in } \Omega \times (0, \infty), \\ u = v = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(\cdot, 0) = u_0, \quad v(\cdot, 0) = v_0 & \text{in } \Omega, \end{cases} \tag{1.6}$$

where α, β are nonnegative constants, ρ, a, c, d are positive constants, b is a real constant and u_0, v_0 are nonnegative functions. Here u denotes the population density of prey and v denotes the population density of predator. In certain prey–predator relationships, it is often observed that various kinds of prey species form a huge group to protect themselves from the attacks of predators. So we assume in the second equation of (1.6) for predator species that the population pressure due to the high density of prey species induces repulsive forces for predators and, therefore, the cross-diffusion term of the form $\beta\Delta[uv]$ is added to the second equation.

Similarly to the competition model, the steady-state problem for (1.6) possesses the same semi-trivial steady states; $(\theta_a, 0)$ for $a > \lambda_1$ and $(0, \theta_b)$ for $b > \lambda_1$. Proposition 1.6 in Section 1.4 will assert that

$$(\theta_a, 0) \quad \text{is} \quad \begin{cases} \text{asymptotically stable} & \text{if } \lambda_1\left(\dfrac{-d\theta_a - b}{1 + \beta\theta_a}\right) > 0, \\ \text{unstable} & \text{if } \lambda_1\left(\dfrac{-d\theta_a - b}{1 + \beta\theta_a}\right) < 0, \end{cases} \tag{1.7}$$

while the stability property of $(0, \theta_b)$ is the same as in the competition model.

Section 2 is concerned with positive solutions of the following nonlinear elliptic problem

$$\text{(SP-1)} \qquad \begin{cases} \Delta[(1+\alpha v)u] + u(a-u-cv) = 0 & \text{in } \Omega, \\ \Delta[(1+\beta u)v] + v(b-du-v) = 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \\ u \geq 0, \quad v \geq 0 & \text{in } \Omega, \end{cases}$$

which corresponds to the steady-state problem for (1.4). Our main task is to look for positive solutions of (SP-1). The first step is to introduce a pair of new unknown functions

$$U = (1+\alpha v)u \quad \text{and} \quad V = (1+\beta u)v, \tag{1.8}$$

which induces a one-to-one correspondence between (u, v) with $u \geq 0$, $v \geq 0$ and (U, V) with $U \geq 0$, $V \geq 0$. Then (SP-1) can be rewritten in the form of semi-linear elliptic system

$$\text{(RSP-1)} \qquad \begin{cases} \Delta U + U\left(\dfrac{a-u-cv}{1+\alpha v}\right) = 0 & \text{in } \Omega, \\ \Delta V + V\left(\dfrac{b-du-v}{1+\beta u}\right) = 0 & \text{in } \Omega, \\ U = V = 0 & \text{on } \partial\Omega, \\ U \geq 0, \quad V \geq 0 & \text{in } \Omega, \end{cases}$$

where u and v should be regarded as functions of U and V. We note that (SP-1) is equivalent to (RSP-1). Some a priori estimates of positive solutions of (SP-1) and (RSP-1) will be given in Section 2.1.

Let a and b be regarded as two parameters. In ab-plane, define two curves S_1 and S_2 by

$$\begin{aligned} S_1 &= \left\{(a, b) \in R^2; \lambda_1\left(\frac{d\theta_a - b}{1+\beta\theta_a}\right) = 0, \quad a \geq \lambda_1\right\}, \\ S_2 &= \left\{(a, b) \in R^2; \lambda_1\left(\frac{c\theta_b - a}{1+\alpha\theta_b}\right) = 0, \quad b \geq \lambda_1\right\}. \end{aligned} \tag{1.9}$$

It is possible to show that S_1 (resp. S_2) can be expressed as $b = f(a)$ (resp. $b = g(a)$), where both f and g are strictly monotone increasing functions of class C^1. The stability result (1.5) of $(\theta_a, 0)$ can be stated as follows; $(\theta_a, 0)$ is asymptotically stable if $b < f(a)$ for $a > \lambda_1$ and unstable if $b > f(a)$ for $a > \lambda_1$. Similarly, $(0, \theta_b)$ is asymptotically stable if $b > g(a)$ for $a \geq \lambda_1$ or $b > \lambda_1$ for $0 < a < \lambda_1$ and unstable if $b < g(a)$. See Figure 1.1. As to the dependence of S_1-curve (resp. S_2-curve) on β (resp. α), we can show that S_1-curve (resp. S_2-curve) approaches a line $a = \lambda_1$ (resp. $b = \lambda_1$) as $\beta \to \infty$ ($\alpha \to \infty$). These results will be shown in Section 1.3.

We define a coexistence region in ab-plane by $\{(a, b)$; (SP-1) (or equivalently (RSP-1)) possesses a positive solution for $(a, b)\}$. One can derive useful information on the coexistence region for (RSP-1) by using the degree theory for a suitable nonlinear operator associated with (RSP-1) in Section 2.2. The index formula in Section 1.2 enables us to calculate the fixed point index for the trivial solution and two semi-trivial solutions.

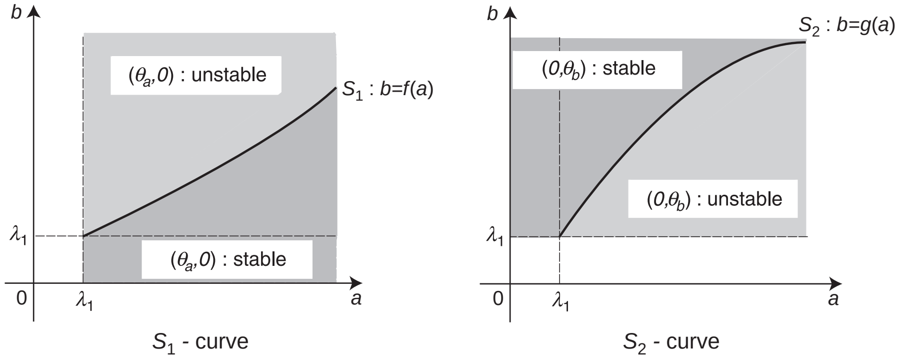

Fig. 1.1. S_1 and S_2 curves.

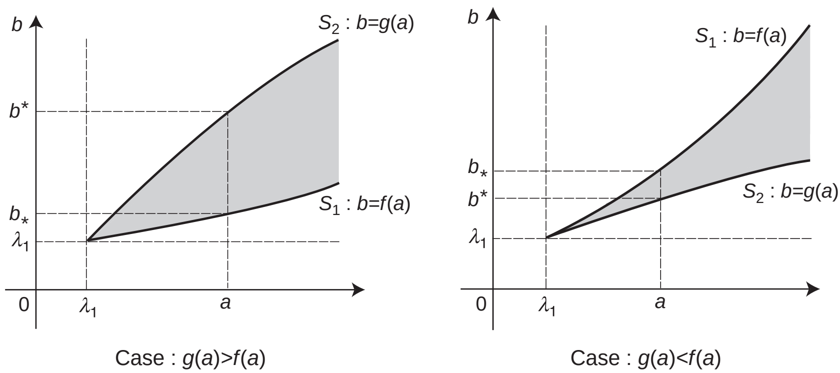

Fig. 1.2. Coexistence region for competition model.

Then we sum up the indices of these solutions and show the existence of positive solutions of (RSP-1) by contradiction argument. It will be proved that (SP-1) possesses a positive solution if (a, b) lies in a region surrounded by S_1 and S_2 curves (Theorem 2.1 and Corollary 2.1). This fact implies that a coexistence region for (SP-1) contains a region surrounded by S_1 and S_2 curves. See Figure 1.2. The same idea has been used by Ryu and Ahn [54] and [55] to show the existence of positive solutions for (1.2) including (SP-1) (and (SP-2)). See also the work of Ruan [53].

The existence result in Section 2.2 can be reconsidered from the viewpoint of bifurcation theory. Let $a > \lambda_1$ be fixed and regard b as a bifurcation parameter. The local bifurcation theory tells us that positive solutions of (RSP-1) bifurcate from $(\theta_a, 0)$ at $b = b_* := f(a)$. That is, a branch of positive solutions bifurcates from $(\theta_a, 0)$ when b crosses S_1 curve. Similarly, we can also observe the bifurcation of positive solutions from $(0, \theta_b)$ at $b = b^* := g(a)$. See Figure 1.2 for geometrical meaning of the above results. Furthermore, it will be proved by the global bifurcation theory that a branch of positive solutions bifurcating from $(\theta_a, 0)$ at $(a, b_*) \in S_1$ can be connected to $(0, \theta_{b^*})$ at $(a, b^*) \in S_2$. These facts enable us to conclude by different method that (RSP-1) has at least one positive solution when (a, b) lies in a region surrounded by S_1 and S_2 curves (Theorem 2.2). Moreover, we will also give detailed analysis of the direction of bifurcations and the stability of positive solutions in Section 2.3.

Finally we will give some comments about effects of cross-diffusion upon the structure of positive solutions for (SP-1). In ab-plane, recall that S_1-curve approaches $a = \lambda_1$ as $\beta \to \infty$ and that S_2-curve approaches $b = \lambda_1$ as $\alpha \to \infty$. Roughly speaking, this fact implies that a coexistence region becomes larger according as α or β tends to ∞. For any fixed $(a, b) \in (\lambda_1, \infty) \times (\lambda_1, \infty)$, (SP-1) with linear diffusion ($\alpha = \beta = 0$) may not admit a positive solution. However, such (a, b) can be contained in a region surrounded by S_1 and S_2 curves provided that α or β is sufficiently large. Thus we can get the existence of positive solutions of SP-1) for large α and β. In this sense, we may state that large cross-diffusion coefficients bring about rich structure of positive solutions for (SP-1).

In Section 3 we treat the following steady-state problem for prey–predator model with cross-diffusion effects

$$\text{(SP-2)} \qquad \begin{cases} \Delta[(1+\alpha v)u] + u(a - u - cv) = 0 & \text{in } \Omega, \\ \Delta[(1+\beta u)v] + v(b + du - v) = 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \\ u \geq 0, \quad v \geq 0 & \text{in } \Omega. \end{cases}$$

This is a nonlinear elliptic system corresponding to (1.6). The method of analysis of (SP-2) is essentially the same as that of (SP-1). Using (U, V) defined by (1.8), we rewrite (SP-2) in the equivalent form

$$\text{(RSP-2)} \qquad \begin{cases} \Delta U + U\left(\dfrac{a - u - cv}{1 + \alpha v}\right) = 0 & \text{in } \Omega, \\ \Delta V + V\left(\dfrac{b + du - v}{1 + \beta u}\right) = 0 & \text{in } \Omega, \\ U = V = 0 & \text{on } \partial\Omega, \\ U \geq 0, \quad V \geq 0 & \text{in } \Omega, \end{cases}$$

where $u = u(U, V)$ and $v = v(U, V)$ are understood to be functions of (U, V).

Define a curve S_3 in ab-plane by

$$S_3 = \left\{(a, b) \in R^2; \lambda_1\left(\frac{-d\theta_a - b}{1 + \beta\theta_a}\right) = 0, a \geq \lambda_1\right\}, \tag{1.10}$$

which can be expressed as $b = \tilde{f}(a)$. Here $\tilde{f}(a)$ is a smooth monotone decreasing (resp. increasing) function with respect to a if $\beta\lambda_1 < d$ (resp. $\beta\lambda_1 > d$). As to (1.7), it can be seen that $(\theta_a, 0)$ is asymptotically stable if $b < \tilde{f}(a)$ and unstable if $b > \tilde{f}(a)$. See Figure 1.3. The other semi-trivial solution $(0, \theta_b)$ is asymptotically stable if $b > g(a)$ and unstable if $b < g(a)$, where g is the same function as the competition model. These facts will be proved in Section 1.4.

After getting some a priori estimates of (SP-2) and (RSP-2) in Section 3.1, we will employ the degree theory for a suitable nonlinear operator on a positive cone. Then one can show that (SP-2) possesses a positive solution if (a, b) lies in a region surrounded by S_2 and S_3 curves (Theorem 3.1 and Corollary 3.1). Moreover, in the case of $N = 1$, it is also possible to prove the uniqueness of positive solutions provided that α and β are small. These results will be proved in Section 3.2.

In Section 3.3 we will discuss the nonexistence of positive solutions of (SP-2). In the case of linear diffusion ($\alpha = \beta = 0$ in (SP-2)), we know a necessary and sufficient condition for the existence of positive solutions. Similarly, for sufficiently small α and β, we will show that (SP-2) has a positive solution if and only if

$$\lambda_1\left(\frac{-d\theta_a - b}{1+\beta\theta_a}\right) < 0 \quad \text{and} \quad \lambda_1\left(\frac{c\theta_b - a}{1+\alpha\theta_b}\right) < 0,$$

which is equivalent to the condition that (a, b) lies in a region surrounded by S_2 and S_3. See Figure 1.4. In this sense one can obtain the exact coexistence region provided that cross-diffusion coefficients α and β are sufficiently small. This is the most different part from the analysis of competition models.

It is difficult to derive the coexistence region for general α and β. In order to get useful information on the coexistence region, it is very important to know the direction of branch of positive solutions bifurcating from semi-trivial solutions. The analysis along this line will be carried out in Section 3.4 similarly as for competition model.

In Section 4 we will discuss a special case of prey–predator model and get better understanding on the structure of positive solutions of (SP-2) such as the coexistence region, the number of positive solutions and their stability properties. For this purpose, our analysis is concentrated on the case where β is sufficiently large. For the sake of simplicity, we set $\alpha = 0$ in (SP-2); then

$$\text{(SP-3)} \qquad \begin{cases} \Delta u + u(a - u - cv) = 0 & \text{in } \Omega, \\ \Delta[(1+\beta u)v] + v(b + du - v) = 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \\ u \geq 0, \quad v \geq 0 & \text{in } \Omega. \end{cases}$$

Our main interest in this section is to know whether (SP-3) has multiple positive solutions or not and whether (SP-3) has a positive solution even if (a, b) lies outside a region surrounded by S_2 and S_3 curves. Regarding $a > \lambda_1$ as a bifurcation parameter we set

$$S = \{(u, v, a);\ (u, v) \text{ is a positive solution of (SP-3) for } a > \lambda_1\}.$$

We will study the global structure of S when β is sufficiently large. Let $b > \lambda_1$ and $d > \beta\lambda_1$. Then we will show that, for some (β, b, c, d), S contains an unbounded S-shaped curve (with respect to a) which bifurcates from $(0, \theta_b, a)$. More precisely, it will be shown that for any $c > 0$, there exists a large number M and an open set

$$\mathcal{O}_1 \subset \left\{(\beta, b, d);\ \beta \geq M, 0 < \frac{d}{\beta} - \lambda_1, b - \lambda_1 \leq \frac{1}{M}\right\}$$

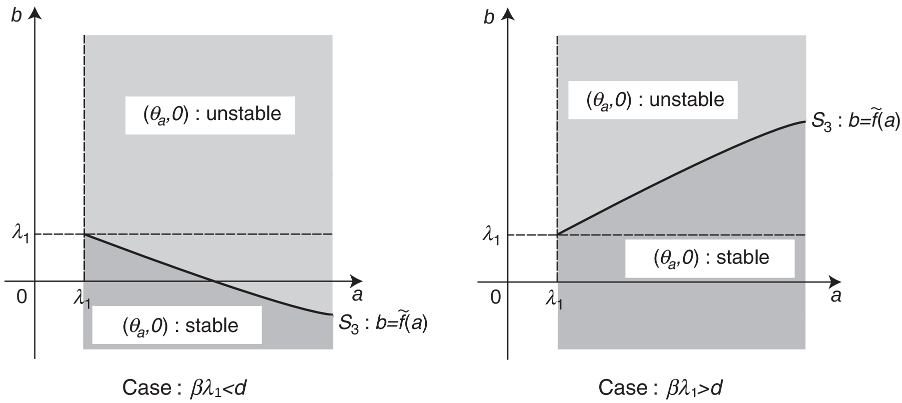

Fig. 1.3. S_3 curve.

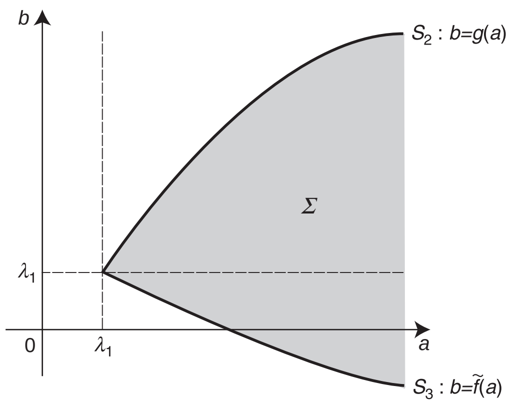

Fig. 1.4. Coexistence region for prey–predator model with small α and β.

such that, if $(\beta, b, d) \in \mathcal{O}_1$, then S contains an unbounded smooth curve expressed as

$$\Gamma_1 = \{(u(s), v(s), a(s));\ s \in (0, \infty)\},$$

where $(u(s), v(s), a(s))$ satisfies the following properties:

(i) $(u(0), v(0)) = (0, \theta_b)$ and $a(0) = a^*$ with $(a^*, b) \in S_2$,

(ii) $a(s) > a(0)$ for $s \in (0, \infty)$ and $\lim_{s\to\infty} a(s) = \infty$,

(iii) $a(s)$ takes a strict local maximum at $s = \overline{s}$ and a strict local minimum at $s = \underline{s}$ with $0 < \overline{s} < \underline{s}$,

(see Theorem 4.1 and Figure 1.5). The above result implies that the multiple existence of positive solutions holds true for $a(\underline{s}) \le a \le a(\overline{s})$; especially (SP-3) admits at least three positive solutions for $(\beta, b, d) \in \mathcal{O}_1$ when a satisfies $a(\underline{s}) < a < a(\overline{s})$. Moreover, one can see the nonexistence of positive solutions of (SP-3) for $a \le a^*$.

For some (β, b, c, d) with $\beta b > \beta\lambda_1 > d$, we will also prove that S contains a bounded S or ⊃-shaped curve, which bifurcates from $(0, \theta_b, a)$ at $a = a^*$ (with $(a^*, b) \in S_2$) and is connected to $(\theta_a, 0, a)$ at $a = a_*$ (with $(a_*, b) \in S_3$). This curve has similar properties as stated in the preceding case $\min\{\beta b, d\} > \beta\lambda_1$. Especially, it will be shown that, in addition to the multiple existence of positive solutions, (SP-3) has a positive solution even if (a, b) lies outside the region surrounded by S_2 and S_3.

The analysis of bifurcating curves of positive solutions is carried out by using the Lyapunov–Schmidt reduction procedure. If β is large and $|d/\beta - \lambda_1|$ are small, then this reduction leads us to a suitable limiting problem. We will use the perturbation technique to derive precise information on positive solutions from the limiting problem. The perturbation technique is also valid to study the stability of positive solutions. We will derive stability properties of positive solutions from those for limiting problem.

Finally we should state some results on the global existence of solutions for reaction diffusion systems with nonlinear diffusion. For example, it is a very important problem to show the existence of global solutions to (1.1) for any initial functions (u_0, v_0) in suitable function spaces without any size restrictions on (u_0, v_0). But this is a very delicate problem and the existing results on the global solutions depend on the positivity of self-diffusion coefficients and space dimension. For details, we refer the reader to [2,5,6,13,19,34–36, 48,51,58–60] and [63].

1.2. *Preliminaries*

1.2.1. *Logistic equation with diffusion*

For each $q \in C(\bar{\Omega})$, let $\lambda_1(q)$ be the principal eigenvalue of

$$\begin{cases} -\Delta u + q(x)u = \lambda u & \text{in } \Omega, \\ \qquad\qquad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.11}$$

As is well known, the principal eigenvalue is given by the following variational characterization

$$\lambda_1(q) = \inf_{u \in H_0^1, \|u\|=1} \left\{ \|\nabla u\|^2 + \int_\Omega q(x)u^2 dx \right\}, \tag{1.12}$$

where $\|\cdot\|$ denotes $L^2(\Omega)$-norm. It should be noted that the infimum in (1.12) is attained by a unique positive function $\varphi \in H_0^1(\Omega)$ satisfying $\|\varphi\| = 1$.

We will collect some useful results on the principal eigenvalue.

PROPOSITION 1.1. (i) *If q_i $(i = 1, 2)$ satisfy $q_1 \geq q_2$ $(q_1 \neq q_2)$, then $\lambda_1(q_1) > \lambda_1(q_2)$.*
(ii) *For q_n $(n = 1, 2, 3, \ldots)$ and $q \in C(\bar{\Omega})$, let φ_n $(n = 1, 2, 3, \ldots)$ and $\varphi \in H_0^1(\Omega)$ be the corresponding eigenfunctions of* (1.11) *satisfying $\|\varphi_n\| = 1$ $(n = 1, 2, 3, \ldots)$ and $\|\varphi\| = 1$. If $\lim_{n\to\infty} \|q_n - q\|_\infty = 0$, then*

$$\lim_{n\to\infty} \lambda_1(q_n) = \lambda_1(q) \quad \textit{and} \quad \lim_{n\to\infty} \varphi_n = \varphi \textit{ strongly in } H_0^1(\Omega).$$

Here $\|\cdot\|_\infty$ denotes a supremum norm defined by $\|q\|_\infty = \sup_{x\in\bar{\Omega}} |q(x)|$.

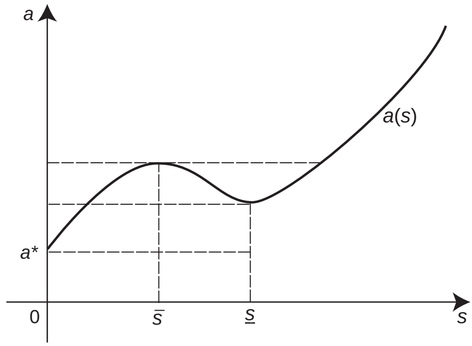

Fig. 1.5. Graph of $a(s)$.

(iii) *Let I be an open interval and assume that a mapping $\alpha \to q_\alpha$ is continuously differentiable from I to $C(\bar{\Omega})$ with respect to supremum norm. If $\varphi_\alpha \in H_0^1(\Omega)$ with $\|\varphi_\alpha\| = 1$ is a unique positive eigenfunction corresponding to $\lambda_1(q_\alpha)$, then $\alpha \longrightarrow \lambda_1(q_\alpha)$ is of class $C^1(I)$ and*

$$\frac{d}{d\alpha}\lambda_1(q_\alpha) = \int_\Omega \frac{\partial q_\alpha}{\partial \alpha}\varphi_\alpha^2\,dx \quad \textit{for all } \alpha \in I.$$

PROOF. It is easy to see the assertion (i) from the variational characterization (1.12). In order to prove (ii), we use the following inequality

$$|\lambda_1(p) - \lambda_1(q)| \le \|p - q\|_\infty \quad \text{for every } p, q \in C(\bar{\Omega}). \tag{1.13}$$

To prove (1.13), let $\varphi_p, \varphi_q \in H_0^1(\Omega)$ be the corresponding positive eigenfunctions to $\lambda_1(p), \lambda_1(q)$, respectively, such that $\|\varphi_p\| = \|\varphi_q\| = 1$. It follows from (1.12) that

$$\begin{aligned}
\lambda_1(p) &= \|\nabla\varphi_p\|^2 + \int_\Omega p\varphi_p^2\,dx \\
&\le \|\nabla\varphi_q\|^2 + \int_\Omega p\varphi_q^2\,dx \\
&= \|\nabla\varphi_q\|^2 + \int_\Omega q\varphi_q^2\,dx + \int_\Omega (p-q)\varphi_q^2\,dx \\
&= \lambda_1(q) + \int_\Omega (p-q)\varphi_p^2\,dx \le \lambda_1(q) + \|p-q\|_\infty.
\end{aligned} \tag{1.14}$$

Similarly, $\lambda_1(q) \le \lambda_1(p) + \|p - q\|_\infty$. Hence one can get (1.13), which implies

$$|\lambda_1(q_n) - \lambda_1(q)| \le \|q_n - q\|_\infty \to 0 \quad \text{as } n \to \infty.$$

To verify the convergence of $\{\varphi_n\}$, we observe that $\{\varphi_n\}$ is bounded in $H_0^1(\Omega)$. Therefore, by Rellich's theorem, there exists a subsequence $\{\varphi_{n'}\}$ of $\{\varphi_n\}$ such that $\lim_{n'\to\infty}\varphi_{n'} = \varphi_*$

strongly in $L^2(\Omega)$ and $\lim_{n'\to\infty} \nabla\varphi_{n'} = \nabla\varphi_*$ weakly in $L^2(\Omega)$ for some $\varphi_* \in H_0^1(\Omega)$. Since $\varphi_{n'}$ satisfies

$$\lambda_1(q_{n'}) = \|\nabla\varphi_{n'}\|^2 + \int_\Omega q_{n'}\varphi_{n'}^2\,dx,$$

letting $n' \to \infty$ in the above identity we see that

$$\begin{aligned}\liminf_{n'\to\infty} \lambda_1(q_{n'}) &\geq \liminf_{n'\to\infty} \|\nabla\varphi_{n'}\|^2 + \lim_{n'\to\infty} \int_\Omega q_{n'}\varphi_{n'}^2\,dx \\ &\geq \|\nabla\varphi_*\|^2 + \int_\Omega q\varphi_*^2\,dx \geq \lambda_1(q).\end{aligned}$$

On the other hand, we have already shown $\lim_{n\to\infty} \lambda_1(q_n) = \lambda_1(q)$; so that

$$\lim_{n'\to\infty} \lambda_1(q_{n'}) = \lambda_1(q) = \|\nabla\varphi\|^2 + \int_\Omega q\varphi^2\,dx = \|\nabla\varphi_*\|^2 + \int_\Omega q\varphi_*^2\,dx.$$

Hence the uniqueness of a normalized positive eigenfunction implies $\varphi = \varphi_*$. Moreover, one can see $\lim_{n'\to\infty} \|\nabla\varphi_{n'}\|^2 = \|\nabla\varphi\|^2$. Since $\varphi_{n'} \to \varphi$ weakly in $H_0^1(\Omega)$ as $n' \to \infty$, it is standard to prove that $\varphi_{n'} \to \varphi$ strongly in $H_0^1(\Omega)$. Thus one can show that $\{\varphi_n\}$ itself converges to φ in $H_0^1(\Omega)$ as $n \to \infty$.

(iii) By (1.13)

$$|\lambda_1(q_\alpha) - \lambda_1(q_\beta)| \leq \|q_\alpha - q_\beta\|_\infty \leq C|\alpha - \beta| \quad \text{for } \alpha, \beta \in I$$

with some $C > 0$. This fact implies that $\alpha \longrightarrow \lambda_1(q_\alpha)$ is Lipschitz-continuous; so that $\lambda_1(q_\alpha)$ is differentiable for almost every $\alpha \in I$. Let $\alpha \in I$ be any differentiable point in I. If $|h|$ is sufficiently small, then

$$\lambda_1(q_{\alpha+h}) \leq \|\nabla\varphi_\alpha\|^2 + \int_\Omega q_{\alpha+h}\varphi_\alpha^2\,dx = \lambda_1(q_\alpha) + \int_\Omega (q_{\alpha+h} - q_\alpha)\varphi_\alpha^2\,dx,$$

as in the proof of (1.14); so that

$$\lambda_1(q_{\alpha+h}) - \lambda_1(q_\alpha) \leq \int_\Omega (q_{\alpha+h} - q_\alpha)\varphi_\alpha^2\,dx. \tag{1.15}$$

Dividing (1.15) by $h > 0$ (resp. $h < 0$) and letting $h \to 0$ we get

$$\frac{d}{d\alpha}\lambda_1(q_\alpha) \leq \int_\Omega \frac{\partial q_\alpha}{\partial\alpha}\varphi_\alpha^2\,dx \quad \left(\text{resp.}\quad \frac{d}{d\alpha}\lambda_1(q_\alpha) \geq \int_\Omega \frac{\partial q_\alpha}{\partial\alpha}\varphi_\alpha^2\,dx\right).$$

Therefore,

$$\frac{d}{d\alpha}\lambda_1(q_\alpha) = \int_\Omega \frac{\partial q_\alpha}{\partial\alpha}\varphi_\alpha^2\,dx \quad \text{for almost all } \alpha \in I.$$

We note that the right-hand side of the above identity is continuous for all $\alpha \in I$; so that this identity is valid for all $\alpha \in I$. □

In what follows, if $q \equiv 0$, we simply write λ_1 in place of $\lambda_1(0)$. We denote by ϕ_1 the corresponding positive eigenfunction satisfying $\|\phi_1\| = 1$; ϕ_1 satisfies

$$\begin{cases} -\Delta\phi_1 = \lambda_1\phi_1 & \text{in } \Omega, \\ \phi_1 = 0 & \text{on } \partial\Omega, \\ \phi_1 > 0 & \text{in } \Omega. \end{cases}$$

We denote by $C_0(\bar{\Omega})$ the space of all continuous functions u in $\bar{\Omega}$ such that u vanishes on $\partial\Omega$ and $C_0(\bar{\Omega})$ is equipped with supremum norm $\|\cdot\|_\infty$.

We now study steady-state problem (1.3) as an auxiliary problem to discuss positive steady-state solutions for a certain class of reaction diffusion systems. Existence, nonexistence and some other properties of positive solutions for (1.3) are given by the following proposition.

PROPOSITION 1.2. (i) *If $a \le \lambda_1$, then* (1.3) *has no nontrivial solutions.*

(ii) *If $a > \lambda_1$, then there exists a unique positive solution θ_a of* (1.3) *such that $\theta_a(x)$ is strictly increasing with respect to a and $\theta_a(x) < a$ for every $x \in \Omega$.*

(iii) $\lim_{a\to\lambda_1} \theta_a = 0$ *uniformly in Ω. More precisely,*

$$\theta_a = \frac{a-\lambda_1}{m^*}\phi_1 + o(a-\lambda_1) \quad \text{as } a \to \lambda_1, \tag{1.16}$$

where $m^* = \int_\Omega \phi_1(x)^3 dx$.

(iv) *For any compact subset K in Ω and for any positive number M, there exists a positive constant $a_{K,M}$ such that $\theta_a(x) > M$ for all $x \in K$ and $a > a_{K,M}$.*

PROOF. All assertions come from [10, Lemma 1] or [18, Propositions 6.1–6.4]. □

We now state some results on large-time behavior of the solution for

$$\begin{cases} w_t = \Delta w + w(a-w) & \text{in } \Omega \times (0,\infty), \\ w = 0 & \text{on } \partial\Omega \times (0,\infty), \\ w(\cdot,0) = w_0 & \text{in } \Omega, \end{cases} \tag{1.17}$$

where $w_0 \in C_0(\bar{\Omega}) (\not\equiv 0)$ is any nonnegative function. It is well known that the solution $w(x,t)$ of (1.17) satisfies

$$\lim_{t\to\infty} w(\cdot,t) = \begin{cases} 0 & \text{if } a \le \lambda_1, \\ \theta_a & \text{if } a > \lambda_1, \end{cases}$$

uniformly in Ω (see, e.g., [56]). In this sense, θ_a is a global attractor for (1.17) whenever it exists. We can also give an important result on the linearized stability of θ_a. The spectral problem for the linearization for (1.3) at θ_a is written as

$$\begin{cases} -\Delta w + (2\theta_a - a)w = \lambda w & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Note that $\lambda_1(2\theta_a - a)$ is the principal eigenvalue for $-\Delta + (2\theta_a - a)I$ with zero Dirichlet boundary condition, where I denotes the identity operator.

LEMMA 1.1. (i) $\lambda_1(2\theta_a - a) > 0$.
(ii) *$a \to \theta_a$ is a C^1-mapping from (λ_1, ∞) to $C_0(\bar{\Omega})$ and $\frac{\partial \theta_a}{\partial a} > 0$ in Ω.*

PROOF. (i) Since θ_a is a positive solution of (1.3), the Krein–Rutman theorem (see [56]) implies that zero is the principal eigenvalue for the following eigenvalue problem

$$\begin{cases} -\Delta w + (\theta_a - a)w = \lambda w & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore, $\lambda_1(\theta_a - a) = 0$; so that Proposition 1.1 yields $\lambda_1(2\theta_a - a) > 0$.

(ii) The continuity of $a \to \theta_a$ can be proved in the same way as the proof of [3, Lemma 3.1]. In order to show the C^1-dependence, it is sufficient to observe

$$\frac{\partial \theta_a}{\partial a} = (-\Delta + (2\theta_a - a)I)^{-1}\theta_a,$$

where $(-\Delta + (2\theta_a - a)I)^{-1}$ denotes the inverse operator of $-\Delta + (2\theta_a - a)I$ with zero Dirichlet boundary condition and is a strictly order-preserving mapping (see, e.g., [49]). Therefore, $\frac{\partial \theta_a}{\partial a} > 0$ in Ω. □

1.2.2. *Degree theory*

In this subsection we will summarize the index theory on a positive cone, which has been developed by Amann [1] and Dancer [9,10] to study positive solutions for nonlinear elliptic equations. See also [28–30,52].

Let E be a real Banach space and let W be a closed convex set in E. We use the notation following the paper of Dancer [9]. Let y be any element in W and define W_y by

$$W_y := \{x \in E;\ y + \gamma x \in W \text{ for some } \gamma > 0\},$$

which is also a convex set.

If $E := C_0(\overline{\Omega}) \times C_0(\overline{\Omega})$ and $W = K \times K$ with $K := \{u \in C_0(\overline{\Omega});\ u \geq 0 \text{ in } \Omega\}$, then it is easy to see that W is a closed convex cone; W is sometimes called a positive cone. For $y = (y_1, y_2) \in W$, note that $\overline{W_y} = W$ for $y = (0,0)$, $\overline{W_y} = C_0(\overline{\Omega}) \times K$ for $y = (y_1, 0)$ with $y_1 > 0$ and $\overline{W_y} = K \times C_0(\overline{\Omega})$ for $y = \{0, y_2\}$ with $y_2 > 0$. Especially, if both y_1 and y_2 are positive, then $\overline{W_y} = E$. For the proof of these results, see Dancer [10, Lemma 3]).

Define

$$S_y = \overline{W_y} \cap (-\overline{W_y}) \quad \text{for } y \in W, \tag{1.18}$$

which is a closed subspace of E. Assume that $T : E \longrightarrow E$ is a compact linear operator such that

$$T(\overline{W_y}) \subseteq \overline{W_y}. \tag{1.19}$$

If $u \in S_y$, then $Tu \in \overline{W_y}$ and $-Tu \in \overline{W_y}$ because of (1.18) and (1.19); so that $Tu \in S_y$. This fact implies that T induces a compact linear mapping $\widetilde{T}$ from E/S_y into itself. We denote by $\widetilde{W_y}$ an image of $\overline{W_y}$ under the quotient mapping $E \longrightarrow E/S_y$. Since $T(\overline{W}_y) \subseteq \overline{W}_y$, it follows that $\widetilde{T}(\widetilde{W_y}) \subseteq \widetilde{W_y}$.

Let $A : W \longrightarrow W$ be a compact and Fréchet differentiable mapping. Denote by $A'(x)$ the Fréchet derivative of A at $x \in W$. Let $y \in W$ be any fixed point of A, i.e., $Ay = y$, and assume that $A'(x)$ is compact. By Lemma 1 in [9, §2], $A'(y)$ maps $\overline{W_y}$ into itself. Then one can define the Leray–Schauder degree $\deg_W(I - A, U, 0)$ for any open subset U in W if A has no fixed points on ∂U. For each isolated fixed point $y \in W$, $\mathrm{index}_W(A, y)$ means $\deg_W(I - A, N(y), 0)$, where $N(y)$ is a suitable neighborhood of y in W. Moreover, it is known that, if $\deg_W(I - A, U, 0) \neq 0$, then A has at least one fixed point in U. For more details, see also [1].

We now give an important index formula which is essentially due to Dancer [9]. See [43, Proposition 2].

PROPOSITION 1.3. *Let $y \in W$ be a fixed point of A. If $(I - A'(y))x \neq 0$ for every $x \in \overline{W_y} \setminus \{0\}$, then*

(i) $\mathrm{index}_W(A, y) = 0$ *if* $r(\widetilde{A'(y)}) > 1$,

(ii) $\mathrm{index}_W(A, y) = 1$ *if* $r(A'(y)) < 1$,

where $r(T)$ denotes the spectral radius of bounded linear operator T.

If we want to apply the degree theory to study the following semi-linear elliptic problem

$$\begin{cases} \Delta u + uf(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u \geq 0 & \text{in } \Omega, \end{cases} \tag{1.20}$$

it is standard to take $E = C_0(\overline{\Omega})$, $W = \{u \in C_0(\overline{\Omega}); u \geq 0 \text{ in } \Omega\}$ and define $A : E \longrightarrow E$ by

$$Au = (-\Delta + pI)^{-1}(pu + uf(u))$$

with a suitable positive constant p, where $(-\Delta + pI)^{-1}g$ for $g \in C(\overline{\Omega})$ denotes a unique solution of

$$\begin{cases} -\Delta u + pu = g & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Note that u is a solution of (1.20) if and only if u is a fixed point of A in W.

For $q \in C(\overline{\Omega})$, define a bounded linear operator $T : E \longrightarrow E$ by

$$Tu = (-\Delta + pI)^{-1}(p - q(x))u \quad \text{for } u \in E$$

with a sufficiently large number p. The relationship between $\lambda_1(q)$ and $r(T)$ can be given by the following proposition.

PROPOSITION 1.4. *Let $q \in C(\overline{\Omega})$ and let p be a sufficiently large number such that $p > q(x)$ for every $x \in \Omega$.*

(i) $\lambda_1(q) > 0$ *if and only if* $r((-\Delta + pI)^{-1}(p - q(x))) < 1$,

(ii) $\lambda_1(q) < 0$ *if and only if* $r((-\Delta + pI)^{-1}(p - q(x))) > 1$,

(iii) $\lambda_1(q) = 0$ *if and only if* $r((-\Delta + pI)^{-1}(p - q(x))) = 1$.

PROOF. For the proof, see Dancer [10, Proposition 1] (see also Li [28, Lemmas 2.1 and 2.3]). □

1.3. *Semi-trivial steady states for competition system*

In this subsection we study the steady-state problem corresponding to (1.4). By Proposition 1.2, (SP-1) has, in addition to the trivial steady state $(0, 0)$, two semi-trivial steady states

$$(\theta_a, 0) \quad \text{if } a > \lambda_1 \quad \text{and} \quad (0, \theta_b) \quad \text{if } b > \lambda_1.$$

When (u, v) satisfies (SP-1), it is called a positive solution if $u > 0$ and $v > 0$ in Ω. We can also show that (SP-1) admits no positive solutions if $a \le \lambda_1$ or $b \le \lambda_1$. Indeed, let (u, v) be any positive solution of (SP-1) and set $U = (1 + \alpha v)u$. Since U satisfies

$$-\Delta U = u(a - u - cv) \quad \text{in } \Omega,$$

taking $L^2(\Omega)$-inner product of this equation with U leads to

$$\|\nabla U\|^2 = \int_\Omega uU(a - u - cv)dx < a \int_\Omega uU\, dx \le a\|U\|^2,$$

where we have used $u \le U$. Note $\|\nabla U\|^2 \ge \lambda_1 \|U\|^2$ for all $U \in H_0^1(\Omega)$. Hence, if (SP-1) has a positive solution, then a must satisfy $a > \lambda_1$; this fact implies that (SP-1) admits no positive solution for $a \le \lambda_1$.

Similarly, one can also prove that (SP-1) admits no positive solutions for $b \le \lambda_1$.

We will discuss stability properties of trivial and semi-trivial steady states by studying the spectrum of the linearized operator around each steady state for (1.4). See the work of Potier-Ferry [47], where the linearization principle for quasi-linear evolution equations is studied.

PROPOSITION 1.5. (i) *Trivial steady state* $(0, 0)$ *is asymptotically stable if* $a < \lambda_1$ *and* $b < \lambda_1$, *while it is unstable if* $a > \lambda_1$ *or* $b > \lambda_1$.
(ii) *Semi-trivial steady state* $(\theta_a, 0)$ *is asymptotically stable if* $\lambda_1\left(\frac{d\theta_a - b}{1+\beta\theta_a}\right) > 0$, *while it is unstable if* $\lambda_1\left(\frac{d\theta_a - b}{1+\beta\theta_a}\right) < 0$.
(iii) *Semi-trivial steady state* $(0, \theta_b)$ *is asymptotically stable if* $\lambda_1\left(\frac{c\theta_b - a}{1+\alpha\theta_b}\right) > 0$, *while it is unstable if* $\lambda_1\left(\frac{c\theta_b - a}{1+\alpha\theta_b}\right) < 0$.

PROOF. We will prove only (ii); the other cases can be studied in a similar manner. The linearized parabolic system of (1.4) at $(\theta_a, 0)$ is given by

$$\begin{cases} u_t = \Delta[u + \alpha\theta_a v] + (a - 2\theta_a)u - c\theta_a v & \text{in } \Omega \times (0, \infty), \\ \rho v_t = \Delta[(1 + \beta\theta_a)v] + (b - d\theta_a)v & \text{in } \Omega \times (0, \infty), \\ u = v = 0 & \text{on } \partial\Omega \times (0, \infty). \end{cases}$$

From the linearization principle, the stability of $(\theta_a, 0)$ is determined by studying the following spectral problem

$$\begin{cases} -\Delta u - \alpha\Delta[\theta_a v] + (2\theta_a - a)u + c\theta_a v = \sigma u & \text{in } \Omega, \\ -\Delta[(1+\beta\theta_a)v] + (d\theta_a - b)v = \sigma\rho v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.21}$$

Here it is convenient to introduce $V = (1+\beta\theta_a)v$ and rewrite (1.21) as

$$\begin{cases} -\Delta u - \alpha\Delta\left[\dfrac{\theta_a}{1+\beta\theta_a}V\right] + (2\theta_a - a)u + \dfrac{c\theta_a}{1+\beta\theta_a}V = \sigma u & \text{in } \Omega, \\ -\Delta V + \dfrac{d\theta_a - b}{1+\beta\theta_a}V = \dfrac{\sigma\rho}{1+\beta\theta_a}V & \text{in } \Omega, \\ u = V = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.22}$$

Let σ be an eigenvalue of (1.21) (resp. (1.22)) and let (u, v) (resp. (u, V)) be the corresponding eigenfunction. Assume $V = 0$; then σ must be an eigenvalue of $-\Delta + (2\theta_a - a)I$ with zero Dirichlet boundary condition. Then Lemma 1.1 assures $\sigma > 0$.

If $V \neq 0$, then σ is an eigenvalue for the second equation of (1.22). The least eigenvalue σ^* among such eigenvalues is given by the following variational characterization

$$\sigma^* = \inf_{V \in H_0^1, V \neq 0} \left\{ \frac{\|\nabla V\|^2 + \displaystyle\int_\Omega \frac{d\theta_a - b}{1+\beta\theta_a} V^2\, dx}{\displaystyle\int_\Omega \frac{\rho}{1+\beta\theta_a} V^2\, dx} \right\}. \tag{1.23}$$

Note $\dfrac{\rho}{1+\beta a} < \dfrac{\rho}{1+\beta\theta_a} < \rho$ in Ω because $0 < \theta_a < a$ by Proposition 1.2. Then it is possible to see

$$\sigma^* \begin{cases} > \dfrac{1}{\rho}\lambda_1\left(\dfrac{d\theta_a - b}{1+\beta\theta_a}\right) > 0 & \text{if } \lambda_1\left(\dfrac{d\theta_a - b}{1+\beta\theta_a}\right) > 0, \\ < \dfrac{1}{\rho}\lambda_1\left(\dfrac{d\theta_a - b}{1+\beta\theta_a}\right) < 0 & \text{if } \lambda_1\left(\dfrac{d\theta_a - b}{1+\beta\theta_a}\right) < 0. \end{cases}$$

Combining the above results one can prove that, if $\lambda_1((d\theta_a - b)/(1+\beta\theta_a)) > 0$, then all eigenvalues of (1.21) are positive; so that $(\theta_a, 0)$ is asymptotically stable. On the other hand, if $\lambda_1((d\theta_a - b)/(1+\beta\theta_a)) < 0$, then (1.21) has a negative eigenvalue, which implies the instability of $(\theta_a, 0)$. □

In order to understand the meaning of Proposition 1.5, we define two sets S_1 and S_2 by (1.9) in ab-plane. Set

$$S(a, b) = \lambda_1\left(\frac{d\theta_a - b}{1+\beta\theta_a}\right) \tag{1.24}$$

to study the structure of S_1. By Proposition 1.2, θ_a is continuous and strictly increasing with respect to a and $z \to \frac{dz-b}{1+\beta z}$ is also strictly increasing; so that $S(a, b)$ is continuous in

(a, b), strictly increasing with respect to a and strictly decreasing with respect to b. Let $w_{a,b} \in H_0^1(\Omega)$ be a positive function satisfying

$$S(a, b) = \|\nabla w_{a,b}\|^2 + \int_\Omega \frac{d\theta_a - b}{1 + \beta\theta_a} w_{a,b}^2 \, dx \tag{1.25}$$

and $\|w_{a,b}\|^2 = 1$.

LEMMA 1.2. *Define S by* (1.24). *Then it has the following properties.*

(i) $\lim_{a\to\lambda_1} S(a, b) = \lambda_1 - b$.

(ii) $\lim_{a\to\infty} S(a, b) = \lambda_1 + \frac{d}{\beta}$.

PROOF. By Proposition 1.2, θ_a is a continuous and strictly increasing function with respect to a such that $\lim_{a\to\lambda_1} \theta_a = 0$ uniformly in Ω and $\lim_{a\to\infty} \theta_a(x) = \infty$ for each $x \in \Omega$.

We begin with the proof of (i). Since

$$\lim_{a\to\lambda_1} \frac{d\theta_a - b}{1 + \beta\theta_a} = -b \qquad \text{uniformly in } \Omega,$$

it follows from Proposition 1.1 that $\lim\limits_{a\to\lambda_1} S_1(a, b) = \lambda_1(-b) = \lambda_1 - b$.

In order to show (ii), we note that

$$\left|\frac{d\theta_a - b}{1 + \beta\theta_a}\right| \le \max\left\{b, \frac{d}{\beta}\right\} \qquad \text{for } x \in \Omega$$

and that

$$\lim_{a\to\infty} \frac{d\theta_a - b}{1 + \beta\theta_a} = \frac{d}{\beta} \qquad \text{for each } x \in \Omega^i,$$

where Ω^i denotes the interior of Ω. Moreover,

$$\lambda_1\left(\frac{d\theta_a - b}{1 + \beta\theta_a}\right) \le \|\nabla w\|^2 + \int_\Omega \frac{d\theta_a - b}{1 + \beta\theta_a} w^2 \, dx \tag{1.26}$$

for all $w \in H_0^1(\Omega)$ such that $\|w\| = 1$. Here Lebesgue's dominated convergence theorem assures

$$\lim_{a\to\infty} \int_\Omega \frac{d\theta_a - b}{1 + \beta\theta_a} w^2 \, dx = \frac{d}{\beta} \int_\Omega w^2 \, dx \tag{1.27}$$

for any $w \in L^2(\Omega)$. Therefore, it follows from (1.26) and (1.27) that

$$\limsup_{a\to\infty} \lambda_1\left(\frac{d\theta_a - b}{1 + \beta\theta_a}\right) \le \lim_{a\to\infty}\left\{\|\nabla w\|^2 + \int_\Omega \frac{d\theta_a - b}{1 + \beta\theta_a} w^2 \, dx\right\} = \|\nabla w\|^2 + \frac{d}{\beta}$$

for every $w \in H_0^1(\Omega)$ satisfying $\|w\| = 1$. Taking the infimum for all $w \in H_0^1(\Omega)$ in the above relations one can see

$$\limsup_{a\to\infty} \lambda_1\left(\frac{d\theta_a - b}{1 + \beta\theta_a}\right) \le \lambda_1 + \frac{d}{\beta}. \tag{1.28}$$

Recalling (1.24) and (1.25) we see from (1.28) that, when b is fixed, $\{w_{a,b}\}$ is bounded in $H_0^1(\Omega)$. Rellich's theorem enables us to choose a sequence $\{w_{a_n,b}\}$ with $a_n \to \infty$ such that

$$\begin{aligned} &w_{a_n,b} \to w_\infty \quad \text{in } L^2(\Omega), \\ &w_{a_n,b} \to \nabla w_\infty \quad \text{weakly in } H_0^2(\Omega), \end{aligned}$$

(note $\|w_\infty\| = 1$ and $\| \nabla w_\infty \| \le \liminf_{n\to\infty} \|\nabla w_{a_n,b}\|$). Consider the following:

$$\begin{aligned} &\left| \int_\Omega \frac{d\theta_{a_n} - b}{1 + \beta\theta_{a_n}} w_{a_n,b}^2 dx - \frac{d}{\beta} \int_\Omega w_\infty^2 dx \right| \\ &\quad \le \left| \int_\Omega \frac{d\theta_{a_n} - b}{1 + \beta\theta_{a_n}} (w_{a_n,b}^2 - w_\infty^2) dx \right| + \left| \int_\Omega \left(\frac{d\theta_{a_n} - b}{1 + \beta\theta_{a_n}} - \frac{d}{\beta} \right) w_\infty^2 dx \right|. \end{aligned} \tag{1.29}$$

The first term on the right-hand side of (1.29) is bounded from above by

$$\begin{aligned} \max\left\{\frac{d}{\beta}, b\right\} \int_\Omega |w_{a_n,b}^2 - w_\infty^2| dx \le \max\left\{\frac{d}{\beta}, b\right\} &\| w_{a_n,b} - w_\infty \| \\ &\| w_{a_n,b} + w_\infty \|, \end{aligned}$$

whose right-hand side converges to zero as $n \to \infty$. Owing to Lebesgue's dominated convergence theorem, the second term in (1.29) also converges to zero as $n \to \infty$. Thus the right-hand side of (1.29) converges to zero as $n \to \infty$. Therefore,

$$\begin{aligned} \liminf_{n\to\infty} \lambda_1 \left(\frac{d\theta_{a_n} - b}{1 + \beta\theta_{a_n}} \right) &\ge \liminf_{n\to\infty} \|\nabla w_{a_n,b}\|^2 + \lim_{n\to\infty} \int_\Omega \frac{d\theta_{a_n} - b}{1 + \beta\theta_{a_n}} w_{a_n,b}^2 \, dx \\ &\ge \|\nabla w_\infty\|^2 + \frac{d}{\beta} \|w_\infty\|^2 \ge \lambda_1 + \frac{d}{\beta}. \end{aligned} \tag{1.30}$$

Combining (1.28) and (1.30) we get

$$\lim_{n\to\infty} \lambda_1 \left(\frac{d\theta_{a_n} - b}{1 + \beta\theta_{a_n}} \right) = \lambda_1 + \frac{d}{\beta},$$

which implies $\lim_{a\to\infty} S(a, b) = \lim_{a\to\infty} \lambda_1((d\theta_a - b)/(1 + \beta\theta_a)) = \lambda_1 + d/\beta$. □

By Lemma 1.2, $S(\lambda_1, b) = \lambda_1 - b < 0$ for $b > \lambda_1$ and $\lim_{a\to\infty} S(a, b) = \lambda_1 + d/\beta > 0$. Since $S(a, b)$ is strictly increasing for $a > \lambda_1$, there exists a unique a_0 satisfying $S(a_0, b) = 0$ for each $b > \lambda_1$. Define a continuous function $f^*(b)$ by

$$a_0 = f^*(b) \quad \text{for each } b \ge \lambda_1. \tag{1.31}$$

Here we should observe by Propositions 1.1 and 1.2 that $S(a,b)$ is of class C^1 for $(a,b) \in (\lambda_1, \infty) \times (\lambda_1, \infty)$,

$$S_a(a,b) := \frac{\partial S}{\partial a}(a,b) = \int_\Omega \frac{\partial}{\partial a}\left\{\frac{d\theta_a - b}{1+\beta\theta_a}\right\} w_{a,b}^2\, dx$$
$$= \int_\Omega \frac{d+\beta b}{(1+\beta\theta_a)^2}\frac{\partial\theta_a}{\partial a} w_{a,b}^2\, dx > 0 \tag{1.32}$$

(use Lemma 1.1) and that

$$S_b(a,b) := \frac{\partial S}{\partial b}(a,b) = \int_\Omega \frac{\partial}{\partial b}\left\{\frac{d\theta_a - b}{1+\beta\theta_a}\right\} w_{a,b}^2\, dx$$
$$= -\int_\Omega \frac{1}{1+\beta\theta_a} w_{a,b}^2\, dx < 0. \tag{1.33}$$

The implicit function theorem assures that $f^*(b)$ is a C^1-function for $b > \lambda_1$ such that

$$(f^*)'(b) = -\frac{S_b(f^*(b), b)}{S_a(f^*(b), b)}, \tag{1.34}$$

which is positive by (1.32) and (1.33). Therefore, we are ready to prove the following result.

LEMMA 1.3. *Define S_1 by (1.9). Then S_1 is expressed as*

$$S_1 = \{(a,b) \in R^2; b = f(a), a \geq \lambda_1\}, \tag{1.35}$$

where f is a strictly increasing function of class C^1 for $a \geq \lambda_1$ with the following properties:

(i) $f(\lambda_1) = \lambda_1$ *and* $\lim_{a\to\infty} f(a) = \infty$.

(ii) $f'(\lambda_1) = d + \beta\lambda_1$.

PROOF. From the preceding arguments, S_1 is a C^1 curve expressed as $a = f^*(b)$ with $b \geq \lambda_1$, where f^* is a strictly increasing function. Define f as the inverse of f^*, i.e., $f(a) = (f^*)^{-1}(a)$. We will prove that f possesses properties (i) and (ii).

Since $S(\lambda_1, b) = \lambda_1 - b$, it is easy to show $f(\lambda_1) = \lambda_1$. In order to show $\lim_{a\to\infty} f(a) = \infty$, we will prove $\lim_{b\to\infty} f^*(b) = \infty$ by contradiction. Assume $\lim_{b\to\infty} f^*(b) = a_\infty < \infty$. Since f^* is strictly increasing

$$0 = S(f^*(b), b) < S(a_\infty, b) = \lambda_1\left(\frac{d\theta_{a_\infty} - b}{1+\beta\theta_{a_\infty}}\right). \tag{1.36}$$

Note

$$\|\nabla w\|^2 + \int_\Omega \frac{d\theta_{a_\infty} - b}{1+\beta\theta_{a_\infty}} w^2\, dx \leq \|\nabla w\|^2 + \int_\Omega \frac{d\theta_{a_\infty}}{1+\beta\theta_{a_\infty}} w^2\, dx - \frac{b}{1+\beta a_\infty}$$

for every $w \in H_0^1(\Omega)$ satisfying $\|w\| = 1$. Hence taking the infimum for all $w \in H_0^1(\Omega)$ in the above inequality we get

$$\lambda_1\left(\frac{d\theta_{a_\infty} - b}{1+\beta\theta_{a_\infty}}\right) \leq \lambda_1\left(\frac{d\theta_{a_\infty}}{1+\beta\theta_{a_\infty}}\right) - \frac{b}{1+\beta a_\infty} \longrightarrow -\infty \quad \text{as } b \to \infty.$$

This is a contradiction to (1.36); thus we have shown (i).

Recall $S(a, f(a)) = 0$; then the implicit function theorem implies

$$f'(a) = -\frac{S_a(a, f(a))}{S_b(a, f(a))}. \tag{1.37}$$

In order to prove (ii), we use Propositions 1.1 and 1.2 to derive

$$\begin{aligned} \theta_a &\longrightarrow 0 && \text{uniformly in } \Omega, \\ w_{a,f(a)} &\longrightarrow \phi_1 && \text{strongly in } H_0^1(\Omega), \\ \frac{\partial\theta_a}{\partial a} &\longrightarrow \frac{\phi_1}{m^*} && \text{uniformly in } \Omega, \end{aligned}$$

as $a \to \lambda_1$, where $m^* = \int_\Omega \phi_1^3\,dx$. Therefore, in view of (1.32) and (1.33),

$$\begin{aligned} S_a(a, f(a)) &\longrightarrow \int_\Omega \frac{d+\beta\lambda_1}{m^*}\phi_1^3\,dx = d+\beta\lambda_1 \\ S_b(a, f(a)) &\longrightarrow -\int_\Omega \phi_1^2\,dx = -1 \end{aligned}$$

as $a \to \lambda_1$. Hence assertion (ii) comes from (1.37). □

Similarly we can also obtain the following result on S_2.

LEMMA 1.4. *Define S_2 by (1.9). Then S_2 is expressed as*

$$S_2 = \{(a, b) \in R^2; b = g(a), a \geq \lambda_1\}, \tag{1.38}$$

where g is a strictly increasing function of class C^1 for $a \geq \lambda_1$ with the following properties:

(i) $g(\lambda_1) = \lambda_1$ *and* $\lim_{a\to\infty} g(a) = \infty$.

(ii) $g'(\lambda_1) = \frac{1}{c+\alpha\lambda_1}$.

By virtue of Lemmas 1.3 and 1.4, stability results for trivial and semi-trivial steady states (Proposition 1.5) read as follows. See Figure 1.1.

THEOREM 1.1. (i) *Trivial steady state $(0, 0)$ is asymptotically stable if $a < \lambda_1$ and $b < \lambda_1$, while it is unstable if $a > \lambda_1$ or $b > \lambda_1$.*

(ii) *Semi-trivial steady state $(\theta_a, 0)$ is asymptotically stable if $b < f(a)$ for $a > \lambda_1$, while $(\theta_a, 0)$ is unstable if $b > f(a)$ for $a > \lambda_1$.*

(iii) *Semi-trivial steady state $(0, \theta_b)$ is asymptotically stable if $b > g(a)$ for $a \geq \lambda_1$ or $b > \lambda_1$ for $0 < a < \lambda_1$, while $(0, \theta_b)$ is unstable if $b < g(a)$ for $(a, b) \in [\lambda_1, \infty) \times [\lambda_1, \infty)$.*

PROOF. It is easy to show (i). Define $S(a, b)$ by (1.24). It follows from the definition of f, $S(a, f(a)) = 0$. Since $S(a, b)$ is strictly decreasing with respect to b, it is easy to see $S(a, b) > 0$ if $b < f(a)$ and $S(a, b) < 0$ if $b > f(a)$ for $a > \lambda_1$. Thus we can derive the assertion of (ii) from Proposition 1.5.

The assertion (iii) can be proved in the same way. □

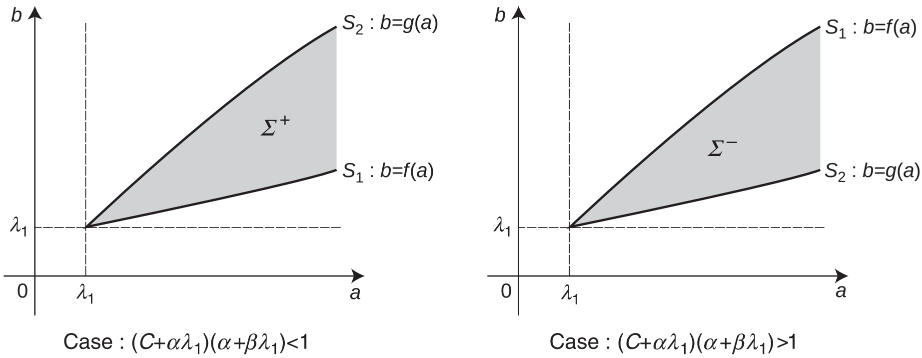

Fig. 1.6. Stability of semi-trivial steady states.

REMARK 1.1. Making use of Theorem 1.1 together with Lemmas 1.3 and 1.4 we can depict each stability region of trivial or semi-trivial steady-state in ab-plane. See Figure 1.6. Especially, if $(c+\alpha\lambda_1)(d+\beta\lambda_1) > 1$, then there exists a region of (a, b) where two semi-trivial steady-states are asymptotically stable.

1.4. *Semi-trivial steady states for prey–predator system*

In this subsection we study the steady-state problem (SP-2) corresponding to (1.6). By Proposition 1.2, (SP-2) has, in addition to the trivial steady state $(0, 0)$, two semi-trivial steady states

$$(\theta_a, 0) \quad \text{if } a > \lambda_1 \quad \text{and} \quad (0, \theta_b) \quad \text{if } b > \lambda_1.$$

As in Section 1.3 it is possible to show that (SP-2) admits no positive solution if $a \leq \lambda_1$. The stability properties of the above trivial and semi-trivial steady states can be given by the following proposition.

PROPOSITION 1.6. (i) *Trivial steady state* $(0, 0)$ *is asymptotically stable if* $a < \lambda_1$ *and* $b < \lambda_1$, *while it is unstable if* $a > \lambda_1$ *or* $b > \lambda_1$.

(ii) *Semi-trivial steady state* $(\theta_a, 0)$ *is asymptotically stable if* $\lambda_1\left(\frac{-d\theta_a - b}{1+\beta\theta_a}\right) > 0$, *while it is unstable if* $\lambda_1\left(\frac{-d\theta_a - b}{1+\beta\theta_a}\right) < 0$.

(iii) *Semi-trivial steady state* $(0, \theta_b)$ *is asymptotically stable if* $\lambda_1\left(\frac{c\theta_b - a}{1+\alpha\theta_b}\right) > 0$, *while it is unstable if* $\lambda_1\left(\frac{c\theta_b - a}{1+\alpha\theta_b}\right) < 0$.

PROOF. It is sufficient to repeat the proof of Proposition 1.5 with d replaced by $-d$. □

Additionally to S_2 defined in (1.9), we define S_3 by (1.10). Set

$$\widetilde{S}(a,b) = \lambda_1\left(\frac{-d\theta_a - b}{1+\beta\theta_a}\right) \tag{1.39}$$

to study the structure of S_3. As in (1.25), it is possible to choose a suitable positive function $w_{a,b} \in H_0^1(\Omega)$ satisfying

$$\widetilde{S}(a,b) = \|\nabla w_{a,b}\|^2 - \int_\Omega \frac{d\theta_a + b}{1+\beta\theta_a} w_{a,b}^2\,dx \tag{1.40}$$

and $\|w_{a.b}\|^2 = 1$.

By Propositions 1.1 and 1.2, $\widetilde{S}(a,b)$ is continuous with respect to (a,b). Moreover, one can show the following result.

LEMMA 1.5. *Define $\widetilde{S}$ by (1.39). Then it has the following properties.*

(i) $\widetilde{S}(a,b)$ *is strictly increasing* (resp. strictly decreasing) with respect to a if $b\beta > d$ (resp. $b\beta < d$) *and* $\widetilde{S}(a,b) = \lambda_1 - b$ *if* $b\beta = d$.

(ii) $\widetilde{S}(a,b)$ *is strictly decreasing with respect to* b.

(iii) $\lim_{a\to\lambda_1} \widetilde{S}(a,b) = \lambda_1 - b$.

(iv) $\lim_{a\to\infty} \widetilde{S}(a,b) = \lambda_1 - \frac{d}{\beta}$.

PROOF. We first note that $z \to \frac{-dz-b}{1+\beta z}$ is strictly increasing (resp. strictly decreasing) if $b\beta > d$ (resp. $b\beta < d$). Since the proofs of (i), (ii) and (iii) are almost the same as that for $S(a,b)$ in Lemma 1.2, we omit them.

In order to show (iv), we use

$$\left|\frac{d\theta_a + b}{1+\beta\theta_a}\right| \le \max\left\{b, \frac{d}{\beta}\right\} \quad \text{for } x \in \Omega$$

and that

$$\lim_{a\to\infty} \frac{d\theta_a + b}{1+\beta\theta_a} = \frac{d}{\beta} \quad \text{for each } x \in \Omega^i.$$

Then the proof of (iv) can be accomplished along the same idea as Lemma 1.2 with obvious modification. □

Making use of Lemma 1.5 we will show that S_3 can be expressed by a monotone function. See Figure 1.3.

LEMMA 1.6. *Define S_3 by (1.10). Then S_3 can be expressed as*

$$S_3 = \{(a,b);\, b = \tilde{f}(a) \text{ for } a \ge \lambda_1\},$$

where $\tilde{f}(\cdot)$ is a C^1 function with respect to $a \in [\lambda_1, \infty)$ with the following properties:

(i) $\tilde{f}$ *is strictly monotone decreasing if* $\beta\lambda_1 < d$, *while* $\tilde{f}$ *is strictly monotone increasing if* $\beta\lambda_1 > d$.

(ii) $\tilde{f}(\lambda_1) = \lambda_1$, $\lim_{a\to\infty} \tilde{f}(a) = -\infty$ *if* $\beta\lambda_1 < d$ *and* $\lim_{a\to\infty} \tilde{f}(a) = \infty$ *if* $\beta\lambda_1 > d$.

(iii) $\tilde{f}'(\lambda_1) = \beta\lambda_1 - d$.

PROOF. We begin with the proof in case $\beta\lambda_1 < d$. By virtue of Lemma 1.5, $\widetilde{S}(\lambda_1, b_0) = \lambda_1 - b_0 \leq 0$ if $b_0 \geq \lambda_1$ (resp. $\widetilde{S}(\lambda_1, b_0) > 0$ if $b_0 < \lambda_1$) and $\lim_{a\to\infty} \widetilde{S}(a, b_0) = \lambda_1 - \frac{d}{\beta} < 0$. Since $\widetilde{S}(a, b_0)$ is a monotone function with respect to a by Lemma 1.5, one can see that, if $b_0 \geq \lambda_1$, then $\widetilde{S}(a, b_0) \leq 0$ for all $a \in (\lambda_1, \infty)$. On the other hand, if $b_0 < \lambda_1$, then the intermediate theorem gives a unique $a_0 \in (\lambda_1, \infty)$ such that $\widetilde{S}(a_0, b_0) = 0$. We set $a_0 = f_*(b_0)$ for $b_0 \leq \lambda_1$.

It follows from Propositions 1.1 and 1.2 that $\widetilde{S}(a, b)$ is of class C^1 for $(a, b) \in (\lambda_1, \infty) \times (-\infty, \infty)$ and that

$$\begin{aligned}\widetilde{S}_a(a, b) := \frac{\partial \widetilde{S}}{\partial a}(a, b) &= -\int_\Omega \frac{\partial}{\partial a}\left\{\frac{d\theta_a + b}{1 + \beta\theta_a}\right\} w_{a,b}^2\, dx \\ &= \int_\Omega \frac{\beta b - d}{(1 + \beta\theta_a)^2} \frac{\partial \theta_a}{\partial a} w_{a,b}^2\, dx\end{aligned} \tag{1.41}$$

(use (1.40)). Moreover,

$$\begin{aligned}\widetilde{S}_b(a, b) := \frac{\partial \widetilde{S}}{\partial b}(a, b) &= -\int_\Omega \frac{\partial}{\partial b}\left\{\frac{d\theta_a + b}{1 + \beta\theta_a}\right\} w_{a,b}^2\, dx \\ &= -\int_\Omega \frac{1}{1 + \beta\theta_a} w_{a,b}^2\, dx < 0.\end{aligned} \tag{1.42}$$

By the implicit function theorem $f_*(b)$ is a C^1-function for $b < \lambda_1$.

We now define $\tilde{f}$ by $\tilde{f}(a) = (f_*)^{-1}(a) < \lambda_1$. Since $\widetilde{S}(a, \tilde{f}(a)) = 0$, one can see from the implicit function theorem that

$$\tilde{f}'(a) = -\frac{\widetilde{S}_a(a, \tilde{f}(a))}{\widetilde{S}_b(a, \tilde{f}(a))}, \tag{1.43}$$

which is negative by (1.41) and (1.42). Thus we have shown that $\tilde{f}$ satisfies (i) in case $\beta\lambda_1 < d$.

We will prove (ii) in case $\beta\lambda_1 < d$. Since $\widetilde{S}(\lambda_1, b) = \lambda_1 - b$, it is easy to see $\lim_{a\to\lambda_1} \tilde{f}(a) = \lambda_1$. In order to show $\lim_{a\to\infty} \tilde{f}(a) = -\infty$, we will show $\lim_{b\to-\infty} f_*(b) = \infty$ by contradiction. Assume $\lim_{b\to-\infty} f_*(b) = a_\infty < \infty$. Since $f_*(b) < a_\infty$ for $b < \lambda_1$,

$$0 = \widetilde{S}(f_*(b), b) > \widetilde{S}(a_\infty, b) = \lambda_1\left(\frac{-d\theta_{a_\infty} - b}{1 + \beta\theta_{a_\infty}}\right). \tag{1.44}$$

Note

$$\|\nabla w\|^2 - \int_\Omega \frac{d\theta_{a_\infty} + b}{1 + \beta\theta_{a_\infty}} w^2\, dx \geq \|\nabla w\|^2 - \int_\Omega \frac{d\theta_{a_\infty}}{1 + \beta\theta_{a_\infty}} w^2\, dx - b$$

for every $w \in H_0^1(\Omega)$ satisfying $\|w\| = 1$. Hence taking the infimum for all $w \in H_0^1(\Omega)$ in the above inequality we get

$$\lambda_1\left(\frac{-d\theta_{a_\infty} - b}{1 + \beta\theta_{a_\infty}}\right) \geq \lambda_1\left(\frac{-d\theta_{a_\infty}}{1 + \beta\theta_{a_\infty}}\right) - b \longrightarrow \infty \quad \text{as} \quad b \to -\infty.$$

This is a contradiction to (1.44); the proof of (ii) is complete.

In case $\beta\lambda_1 > d$, $\widetilde{S}(a, b)$ is strictly increasing with respect to a. Taking account of this fact we can repeat the above argument to derive the assertions of (i) and (ii).

Finally, the proof of (iii) is the same as (ii) of Lemma 1.3. □

In addition to Lemma 1.6, recall that Lemma 1.4 still holds true. Then making use of Proposition 1.6 we can prove the following results on stability properties of trivial or semi-trivial steady states in the same way as Theorem 1.1

THEOREM 1.2. (i) *Trivial steady state* $(0, 0)$ *is asymptotically stable if* $a < \lambda_1$ *and* $b < \lambda_1$, *while it is unstable if* $a > \lambda_1$ *or* $b > \lambda_1$.

(ii) *Semi-trivial steady state* $(\theta_a, 0)$ *is asymptotically stable if* $b < \tilde{f}(a)$ *for* $a > \lambda_1$, *while* $(\theta_a, 0)$ *is unstable if* $b > \tilde{f}(a)$ *for* $a > \lambda_1$.

(iii) *Semi-trivial steady state* $(0, \theta_b)$ *is asymptotically stable if* $b > g(a)$ *for* $a \geq \lambda_1$ *or* $b > \lambda_1$ *for* $0 < a < \lambda_1$, *while* $(0, \theta_b)$ *is unstable if* $b < g(a)$ *for* $(a, b) \in [\lambda_1, \infty) \times [\lambda_1, \infty)$.

2. Positive steady states of competition model with cross-diffusion

In this section we will study the structure of a set of positive steady states for the competition model with cross-diffusion:

$$
\text{(SP-1)} \qquad \begin{cases} \Delta[(1+\alpha v)u] + u(a - u - cv) = 0 & \text{in } \Omega, \\ \Delta[(1+\beta u)v] + v(b - du - v) = 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \\ u \geq 0, \quad v \geq 0 & \text{in } \Omega, \end{cases}
$$

where α, β are nonnegative constants and a, b, c, d are positive constants.

2.1. *A priori estimates*

We are concerned with positive solutions to (SP-1). It is convenient to introduce two unknown functions U and V by (1.8):

$$
U = (1+\alpha v)u \quad \text{and} \quad V = (1+\beta u)v,
$$

which induces a one-to-one correspondence between (u, v) with $u \geq 0$, $v \geq 0$ and (U, V) with $U \geq 0$, $V \geq 0$. One can describe their relations by

$$
\begin{aligned}
u = u(U, V) &= \frac{1}{2\beta}[\{(1 - \beta U + \alpha V)^2 + 4\beta U\}^{1/2} + \beta U - \alpha V - 1], \\
v = v(U, V) &= \frac{1}{2\alpha}[\{(1 - \alpha V + \beta U)^2 + 4\alpha V\}^{1/2} + \alpha V - \beta U - 1].
\end{aligned} \tag{2.1}
$$

Using (2.1) we can rewrite (SP-1) as follows:

$$\text{(RSP-1)}\qquad \begin{cases} \Delta U + U\left(\dfrac{a-u-cv}{1+\alpha v}\right) = 0 & \text{in } \Omega,\\ \Delta V + V\left(\dfrac{b-du-v}{1+\beta u}\right) = 0 & \text{in } \Omega,\\ U = V = 0 & \text{on } \partial\Omega,\\ U \geq 0, \quad V \geq 0 & \text{in } \Omega, \end{cases}$$

where $u = u(U, V)$ and $v = v(U, V)$ are understood to be functions of (U, V) defined by (2.1). By virtue of the one-to-one correspondence between (u, v) and (U, V), we get the equivalence between quasi-linear system (SP-1) and semi-linear system (RSP-1). In what follows, we mainly discuss (RSP-1) in place of (SP-1).

We will derive some a priori estimates of positive solutions of (SP-1) or (RSP-1) in case $a > \lambda_1$ and $b > \lambda_1$,

LEMMA 2.1. *Let* (U, V) *be a positive solution of* (RSP-1). *Then*

$$0 \leq u(x) \leq U(x) \leq M_1(a) := \begin{cases} a & \text{if } a\alpha \leq c,\\ \dfrac{(c+a\alpha)^2}{4c\alpha} & \text{if } a\alpha > c, \end{cases}$$

$$0 \leq v(x) \leq V(x) \leq M_2(b) := \begin{cases} b & \text{if } b\beta \leq d,\\ \dfrac{(d+b\beta)^2}{4d\beta} & \text{if } b\beta > d, \end{cases}$$

for all $x \in \Omega$.

PROOF. Assume $\|U\|_\infty = U(x_0) > 0$ for some $x_0 \in \Omega$. It follows from (RSP-1) that

$$0 \leq -\Delta U(x_0) = u(x_0)(a - u(x_0) - cv(x_0));$$

so that $a - u(x_0) - cv(x_0) \geq 0$ because $u(x_0) > 0$. Therefore,

$$u(x_0) \leq a \quad \text{and} \quad v(x_0) \leq \frac{a - u(x_0)}{c}.$$

These estimates yield

$$\|U\|_\infty = (1 + \alpha v(x_0))u(x_0) \leq \frac{1}{c}u(x_0)(c + a\alpha - \alpha u(x_0)).$$

The right-hand side is regarded as a function of $X := u(x_0)$. Taking its maximum for $0 \leq X \leq a$, we can obtain the desired estimate for $\|U\|_\infty$.

A bound for $\|V\|_\infty$ can be derived in the same way. □

LEMMA 2.2. *Let* (U, V) *be a positive solution of* (RSP-1). *If* $a\alpha \leq c$, *then*

$$u \leq U \leq \theta_a \quad \text{in } \Omega. \tag{2.2}$$

If $b\beta \leq d$, *then*

$$v \leq V \leq \theta_b \quad \text{in } \Omega. \tag{2.3}$$

PROOF. Let (U, V) be a positive solution of (RSP-1). Then Lemma 2.1 implies $U \le a$ if $a\alpha \le c$. So

$$\begin{aligned}\Delta U + U(a-U) &= \frac{U}{1+\alpha v}\{(1+\alpha v)(a-U) - (a-u-cv)\}\\ &= u\left\{(a\alpha + c)v - \frac{2\alpha v + \alpha^2 v^2}{1+\alpha v}U\right\}\\ &\ge uv\left(a\alpha + c - \frac{2\alpha + \alpha^2 v}{1+\alpha v}a\right)\\ &= \frac{uv}{1+\alpha v}(c - a\alpha + c\alpha v) \ge 0.\end{aligned}$$

This fact implies that U is a subsolution of (1.3). Since (1.3) has a unique positive solution θ_a, the standard comparison method yields $\theta_a \ge U$, which gives (2.2). A similar argument is also valid to derive (2.3); so we omit the details. □

2.2. *Existence of positive steady state*

In this subsection we will construct a positive steady-state for (SP-1) (or equivalently (RSP-1)). Our strategy is to employ an approach by the degree theory which has been prepared in Section 1.2.2. Elliptic problem (RSP-1) will be reduced to a fixed point problem for a suitable nonlinear mapping A in a Banach space $E = C_0(\overline{\Omega}) \times C_0(\overline{\Omega})$. We will use the notion of fixed point index to show the existence of a nontrivial element in a positive cone for E.

Choose a sufficiently large number p such that

$$p + \frac{a-u-cv}{1+\alpha v} \ge 0 \quad \text{and} \quad p + \frac{b-du-v}{1+\beta u} \ge 0$$

for $0 \le u \le M_1(a) + 1$ and $0 \le v \le M_2(b) + 1$, where $M_1(a)$, $M_2(b)$ are positive constants in Lemma 2.1. Define a mapping A in $E = C_0(\overline{\Omega}) \times C_0(\overline{\Omega})$ by

$$\begin{aligned}A(U,V) &= (-\Delta + pI)^{-1}\left(\left(p + \frac{a-u-cv}{1+\alpha v}\right)U, \left(p + \frac{b-du-v}{1+\beta u}\right)V\right)\\ &= ((-\Delta+pI)^{-1}(pU + F(u,v)), (-\Delta+pI)^{-1}(pV + G(u,v))),\end{aligned} \tag{2.4}$$

where u, v are functions of U, V (see (2.1)) and

$$F(u,v) = u(a-u-cv), \qquad G(u,v) = v(b-du-v).$$

Define $W = K \times K$ with $K = \{u \in C_0(\overline{\Omega}); u \ge 0 \text{ in } \Omega\}$. Clearly, (U, V) is a solution of (RSP-1) if and only if it is a fixed point of A in W. We set

$$D := \{(U,V) \in W;\ U(x) \le M_1(a) + 1 \text{ and } V(x) \le M_2(b) + 1 \text{ for } x \in \Omega\};$$

then we can see from Lemma 2.1 that all nonnegative solutions of (RSP-1) lie in the interior of D (= int D) with respect to W. (Note that $\theta_a \le a \le M_1(a)$ and $\theta_b \le b \le M_2(b)$). By

(2.4) and the maximum principle for elliptic equations, A maps D into W and, furthermore, the strong maximum principle (see, e.g., [49]) implies that A maps $D\backslash\{(0,0)\}$ into the demi-interior of W (see [9,10]). Moreover, the regularity theory for elliptic equations tells us that A is completely continuous in E. Therefore, one can define the degree, $\deg_W(I-A, \operatorname{int} D)$, for A with respect to W because A has no fixed point on the boundary of D with respect to W by Lemma 2.1.

In order to study $\deg_W(I-A, \operatorname{int} D)$ for A, we will calculate the index of each fixed point of A in W. For this purpose, it is convenient to give an expression of the Fréchet derivative of A at any fixed point (U, V) of A. From (2.4)

$$A'(U,V)\begin{pmatrix}\hat{U}\\ \hat{V}\end{pmatrix} = (-\Delta + pI)^{-1}\left[pI + \begin{pmatrix}F_u & F_v\\ G_u & G_v\end{pmatrix}\begin{pmatrix}u_U & u_V\\ v_U & v_V\end{pmatrix}\right]\begin{pmatrix}\hat{U}\\ \hat{V}\end{pmatrix}, \tag{2.5}$$

where F_u, F_v denote the partial derivatives of F with respect to u, v and u_U, u_V also denote the partial derivatives of u with respect to U, V. Differentiation of (1.8) gives

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1+\alpha v & \alpha u\\ \beta v & 1+\beta u\end{pmatrix}\begin{pmatrix}u_U & u_V\\ v_U & v_V\end{pmatrix}.$$

Since u, v are both nonnegative and above matrices are all invertible, we get

$$\begin{pmatrix}u_U & u_V\\ v_U & v_V\end{pmatrix} = \frac{1}{1+\alpha v+\beta u}\begin{pmatrix}1+\beta u & -\alpha u\\ -\beta v & 1+\alpha v\end{pmatrix}. \tag{2.6}$$

Thus it follows from (2.5) and (2.6) that

$$\begin{aligned} A'(U,V)\begin{pmatrix}\hat{U}\\ \hat{V}\end{pmatrix} &= (-\Delta + pI)^{-1}\Bigg[pI + \frac{1}{1+\alpha v+\beta u}\\ &\quad\times\begin{pmatrix}a-2u-cv & -cu\\ -dv & b-du-2v\end{pmatrix}\begin{pmatrix}1+\beta u & -\alpha u\\ -\beta v & 1+\alpha v\end{pmatrix}\Bigg]\begin{pmatrix}\hat{U}\\ \hat{V}\end{pmatrix}. \end{aligned} \tag{2.7}$$

LEMMA 2.3. $\deg_W(I-A, \operatorname{int} D) = 1$.

PROOF. For $0 \le t \le 1$, we set

$$A_t(U,V) = (-\Delta + pI)^{-1}\left(\left(p + \frac{at-u-cv}{1+\alpha v}\right)U, \left(p + \frac{bt-du-v}{1+\beta u}\right)V\right).$$

As in the study of A, one can prove that A_t is a completely continuous mapping in E and that A_t maps D into W. Let (U^t, V^t) be a fixed point in W: it satisfies

$$\begin{cases} \Delta U^t + U^t\left(\dfrac{at-u^t-cv^t}{1+\alpha v^t}\right) = 0 & \text{in } \Omega,\\ \Delta V^t + V^t\left(\dfrac{bt-du^t-v^t}{1+\beta u^t}\right) = 0 & \text{in } \Omega,\\ U^t = V^t = 0 & \text{on } \partial\Omega,\\ U^t \ge 0, \quad V^t \ge 0 & \text{in } \Omega, \end{cases}$$

where $U^t = (1+\alpha v^t)u^t$ and $V^t = (1+\beta u^t)v^t$. It follows from Lemma 2.1 that (U^t, V^t) satisfies, for each $0 \le t \le 1$,

$$0 \le u^t \le U^t \le M_1(at) \le M_1(a) \quad \text{and} \quad 0 \le v^t \le V^t \le M_2(bt) \le M_2(b).$$

These estimates imply that any fixed point $(U^t, V^t) \in W$ of A_t satisfies $(U^t, V^t) \in \text{int}\, D$ with respect to W; so that A_t has no fixed point on ∂D. Hence it follows from the homotopy invariance that $\deg_W(I - A_t, \text{int}\, D)$ is independent of $t \in [0, 1]$ (see Amann [1, Theorem 11.1]); so that

$$\deg_W(I - A, \text{int}\, D) = \deg_W(I - A_1, \text{int}\, D) = \deg_W(I - A_0, \text{int}\, D). \tag{2.8}$$

Here it should be noted that $(0, 0)$ is a unique fixed point of A_0 in D (see the first paragraph in Section 1.3). So the excision property (see [1, Corollary 11.2]) gives

$$\deg_W(I - A_0, \text{int} D) = \text{index}_W(A_0, (0, 0))$$

and we will calculate $\text{index}_W(A_0, (0, 0))$ with use of Proposition 1.3.

Let $A_0'(0, 0)$ be the Fréchet derivative of A_0, which is given by

$$A_0'(0, 0)(\hat{U}, \hat{V}) = (-\Delta + pI)^{-1}(p\hat{U}, p\hat{V}).$$

Observe that

$$r(A_0'(0, 0)) = \frac{p}{\lambda_1 + p} < 1.$$

Then (ii) of Proposition 1.3 yields $\text{index}_W(A_0, (0, 0)) = 1$. Thus one can show $\deg_W(I - A_0, \text{int} D) = 1$, which together with (2.8), enables us to get the assertion. □

We will study the index of trivial and semi-trivial steady states of (RSP-1) in case $a > \lambda_1$ and $b > \lambda_1$.

LEMMA 2.4. *Let $a > \lambda_1$ and $b > \lambda_1$.*

(i) $\text{index}_W(A, (0, 0)) = 0$.

(ii) *It holds that* $\begin{cases} \text{index}_W(A, (\theta_a, 0)) = 0 & \text{if } \lambda_1\left(\frac{d\theta_a - b}{1+\beta\theta_a}\right) < 0, \\ \text{index}_W(A, (\theta_a, 0)) = 1 & \text{if } \lambda_1\left(\frac{d\theta_a - b}{1+\beta\theta_a}\right) > 0. \end{cases}$

(iii) *It holds that* $\begin{cases} \text{index}_W(A, (0, \theta_b)) = 0 & \text{if } \lambda_1\left(\frac{c\theta_b - a}{1+\alpha\theta_b}\right) < 0, \\ \text{index}_W(A, (0, \theta_b)) = 1 & \text{if } \lambda_1\left(\frac{c\theta_b - a}{1+\alpha\theta_b}\right) > 0. \end{cases}$

PROOF. We will apply Proposition 1.3 to calculate each fixed point index.

(i) We begin with the study of $\operatorname{index}_W(A, (0, 0))$. By (2.7),

$$A'(0, 0)(\hat{U}, \hat{V}) = (-\Delta + pI)^{-1}((p + a)\hat{U}, (p + b)\hat{V}).$$

Clearly, $\overline{W_{(0,0)}} = W$ and, therefore, $S_{(0,0)} = \{(0, 0)\}$; so that $\widetilde{A'(0, 0)}$ is identical with $A'(0, 0)$. Observe that

$$r(A'(0, 0)) = \max\left\{\frac{a + p}{\lambda_1 + p}, \frac{b + p}{\lambda_1 + p}\right\}.$$

For $a > \lambda_1$ and $b > \lambda_1$, it is easy to see that $A'(0, 0)y \neq y$ on $W\backslash\{(0, 0)\}$ and that $r(A'(0, 0)) = r(\widetilde{A')(0, 0})) > 1$. Hence Proposition 1.3 yields $\operatorname{index}_W(A, (0, 0)) = 0$.

(ii) From (2.7),

$$A'(\theta_a, 0)(\hat{U}, \hat{V}) = (-\Delta + pI)^{-1}\left((p + a - 2\theta_a)\hat{U}\right.$$
$$\left. + \left\{\frac{\theta_a((2\theta_a - a)\alpha - c)}{1 + \beta\theta_a}\right\}\hat{V}, \left(p + \frac{b - d\theta_a}{1 + \beta\theta_a}\right)\hat{V}\right).$$

Define T_1 and T_2 by

$$T_1 := (-\Delta + pI)^{-1}(p + a - 2\theta_a),$$
$$T_2 := (-\Delta + pI)^{-1}\left(p + \frac{b - d\theta_a}{1 + \beta\theta_a}\right).$$

Since $\overline{W_{(\theta_a,0)}} = C_0(\overline{\Omega}) \times K$, it is easy to see $S_{(\theta_a,0)} = C_0(\overline{\Omega}) \times \{0\}$; so that one can identify $\widetilde{A'(\phi_a, 0)}$ with T_2.

We will show by contradiction that $I - A'(\theta_a, 0) \neq 0$ on $\overline{W_{(\theta_a,0)}} \setminus \{(0, 0)\}$ if

$$\lambda_1(q^*) \neq 0 \quad \text{with } q^* = -\frac{b - d\theta_a}{1 + \beta\theta_a}.$$

Assume that there exists $(\xi_1, \xi_2) \in C_0(\overline{\Omega}) \times K$ such that $A'(\theta_a, 0)(\xi_1, \xi_2) = (\xi_1, \xi_2)$ with $(\xi_1, \xi_2) \neq (0, 0)$. If $\xi_2 = 0$, then $T_1\xi_1 = \xi_1$, which implies

$$-\Delta\xi_1 + (2\theta_a - a)\xi_1 = 0 \quad \text{in } \Omega, \quad \xi_1 = 0 \quad \text{on } \partial\Omega.$$

Therefore, since $\lambda_1(2\theta_a - a) > 0$ by Lemma 1.1, we see $\xi_1 = 0$, which is a contradiction. So $\xi_2 \in K$ must satisfy $\xi_2 \neq 0$ and $T_2\xi_2 = \xi_2$. Hence it follows from the Krein–Rutman theorem that $r(T_2) = 1$, which, together with Proposition 1.4, implies $\lambda_1(q^*) = 0$. This is also a contradiction. Thus we have shown $I - A'(\theta_a, 0) \neq 0$ on $\overline{W_{(\theta_a,0)}} \setminus \{(0, 0)\}$.

We will apply Proposition 1.3 to calculate the index of $(\theta_a, 0)$. In case $\lambda_1(q^*) < 0$, Proposition 1.4 gives

$$r(T_2) > 1.$$

Since $\widetilde{A'(\phi_a, 0)}$ is identical with T_2, we see $r(\widetilde{A'(\phi_a, 0)}) > 1$. Then it follows from (i) of Proposition 1.3 that $\operatorname{index}_W(A, (\theta_a, 0)) = 0$.

We next consider the case $\lambda_1(q^*) > 0$, which is equivalent to

$$r(T_2) < 1$$

by Proposition 1.4. We will show $r(A'(\theta_a, 0)) < 1$. Let ν be any eigenvalue of $A'(\theta_a, 0)$ and let $(\xi_1, \xi_2) \in E = C_0(\overline{\Omega}) \times C_0(\overline{\Omega})$ be the corresponding eigenfunction. If $\xi_2 \neq 0$, then $T_2\xi_2 = \nu\xi_2$. From $r(T_2) < 1$ it follows that $|\nu| < 1$. Whereas, if $\xi_2 = 0$, then $T_1\xi_1 = \nu\xi_1$ from the definition of $A'(\theta_a, 0)$. Recall $r(T_1) < 1$ by Lemma 1.1 and Proposition 1.4. Therefore, we see that any eigenvalue satisfies $|\nu| < 1$. Thus we have shown $r(A'(\phi_a, 0)) < 1$. Hence it is sufficient to employ (ii) of Proposition 1.3 to get $\mathrm{index}_W(A, (\phi_a, 0)) = 1$.

The proof (iii) is the same as that of (ii); so we omit it. □

REMARK 2.1. According to Proposition 1.5, Lemma 2.4 implies that $\mathrm{index}_W(A, (\theta_a, 0)) = 1$ (resp. $\mathrm{index}_W(A, (\theta_a, 0)) = 0$) if $(\theta_a, 0)$ is asymptotically stable (resp. unstable). The analogous result is valid for $(0, \theta_b)$.

We are now ready to derive an existence result of positive steady states for (SP-1) or (RSP-1).

THEOREM 2.1 (Existence of positive steady-state). *Assume $a > \lambda_1$ and $b > \lambda_1$. Then* (SP-1) (*or equivalently,* (RSP-1)) *admits a positive steady state if one of the following conditions holds true:*

(i) $$\lambda_1\left(\frac{d\theta_a - b}{1 + \beta\theta_a}\right) < 0 \quad \textit{and} \quad \lambda_1\left(\frac{c\theta_b - a}{1 + \alpha\theta_b}\right) < 0,$$

(ii) $$\lambda_1\left(\frac{d\theta_a - b}{1 + \beta\theta_a}\right) > 0 \quad \textit{and} \quad \lambda_1\left(\frac{c\theta_b - a}{1 + \alpha\theta_b}\right) > 0.$$

PROOF. We will prove this theorem with use of the degree theory.

We begin the proof when (i) is satisfied. Assume that (SP-1) (or (RSP-1)) has no positive solution. It follows from the definition of degree that

$$\begin{aligned}&\deg_W(I - A, \mathrm{int}\, D)\\&\quad = \mathrm{index}_W(A, (0, 0)) + \mathrm{index}_W(A, (\theta_a, 0)) + \mathrm{index}_W(A, (0, \theta_b))\end{aligned} \tag{2.9}$$

(see, e.g., [1]). By Lemma 2.3

$$\deg_W(I - A, \mathrm{int}\, D) = 1. \tag{2.10}$$

On the other hand, Lemma 2.4 implies that the right-hand side of (2.9) is equal to zero. This is a contradiction to (2.10); so that (SP-1) must possess at least one positive solution.

When (ii) is satisfied, we also assume that there exists no positive solution of (SP-1) or (RSP-1). Note that both (2.9) and (2.10) are valid. However, Lemma 2.4 together with (2.9) implies $\deg_W(I - A, \ \mathrm{int}\, D) = 2$, which is a contradiction to (2.10). Thus we can conclude that (SP-1) has at least one positive solution in case (ii) is satisfied. □

We recall the results in Section 1.3 and depict two curves S_1 and S_2 defined by (1.9) in the ab-plane. By Lemma 1.3, S_1 is expressed as $b = f(a)$ for $a \geq \lambda_1$ and, by Lemma 1.4, S_2 is expressed as $b = g(a)$ for $a \geq \lambda_1$. Define the following two sets;

$$\begin{aligned} \Sigma^+ &= \{(a,b) \in [\lambda_1, \infty) \times [\lambda_1, \infty);\ f(a) < b < g(a)\}, \\ \Sigma^- &= \{(a,b) \in [\lambda_1, \infty) \times [\lambda_1, \infty);\ g(a) < b < f(a)\}, \end{aligned} \tag{2.11}$$

(see Figure 1.6). By Theorem 1.1, Σ^+ is a region where both $(\theta_a, 0)$ and $(0, \theta_b)$ are unstable, while Σ^- is a region where both $(\theta_a, 0)$ and $(0, \theta_b)$ are asymptotically stable. We note by Lemmas 1.3 and 1.4 that Σ^+ is nonempty if $(c + \alpha\lambda_1)(d + \beta\lambda_1) < 1$, and that Σ^- is nonempty if $(c + \alpha\lambda_1)(d + \beta\lambda_1) > 1$. Lemmas 1.3 and 1.4 also give us useful information about the dependence of Σ^+ and Σ^- upon cross-diffusion coefficients: Σ^+ shrinks to an empty set near $(a,b) = (\lambda_1, \lambda_1)$ as α or β increases, while Σ^- expands near $(a,b) = (\lambda_1, \lambda_1)$ as α or β increases.

Generally, it is possible to show that S_1 curve approaches $a = \lambda_1$ as $\beta \to \infty$ and that S_2 curve approaches $b = \lambda_1$ as $\alpha \to \infty$. Thus Σ^- expands according as $\alpha \to \infty$ or $\beta \to \infty$. It is also a very interesting problem to study limiting behaviors of positive solutions of (SP-1) when α or β tends to ∞. We will discuss these topics elsewhere (see [26]).

The following corollary comes from Theorem 2.1.

COROLLARY 2.1. *If* $(a,b) \in \Sigma^+ \cup \Sigma^-$, *then there exists at least one positive steady-state solution of* (SP-1) (*or equivalently* (RSP-1)).

REMARK 2.2. Theorem 2.1 and Corollary 2.1 imply that (SP-1) possesses a positive solution when two semi-trivial states $(\theta_a, 0)$ and $(0, \theta_b)$ are unstable or asymptotically stable at the same time.

2.3. *Bifurcation theory for competition model*

In the previous subsection, we have shown that (SP-1) has a positive solution when (a,b) satisfies $(a,b) \in \Sigma^+ \cup \Sigma^-$. However, the degree theory gives no information on the existence or nonexistence of positive solutions for $(a,b) \notin \Sigma^+ \cup \Sigma^-$. Our main interest is to know the structure of positive solutions. So we will reconsider (SP-1) or (RSP-1) from the viewpoint of bifurcation theory and study positive steady states as bifurcating positive solutions from semi-trivial states $(\theta_a, 0)$ or $(0, \theta_b)$.

Let $a > \lambda_1$ be fixed. We first study bifurcating positive solutions from $(\theta_a, 0)$ by regarding b as a bifurcation parameter. Define b_* by

$$b_* = f(a), \tag{2.12}$$

where f is an increasing function defined in Lemma 1.3 (see Figure 1.2). From (1.9)

$$\lambda_1\left(\frac{d\theta_a - b_*}{1 + \beta\theta_a}\right) = 0. \tag{2.13}$$

Let Ψ_* be a unique positive solution of

$$\begin{cases} -\Delta\Psi_* + \dfrac{d\theta_a - b_*}{1+\beta\theta_a}\Psi_* = 0 & \text{in } \Omega, \\ \Psi_* = 0 & \text{on } \partial\Omega, \\ \Psi_* > 0 & \text{in } \Omega, \end{cases} \tag{2.14}$$

satisfying $\|\Psi_*\| = 1$. We also define Φ_* as a solution of

$$\begin{cases} -\Delta\Phi_* + (2\theta_a - a)\Phi_* = -\dfrac{(a\alpha + c - 2\alpha\theta_a)\theta_a}{1+\beta\theta_a}\Psi_* & \text{in } \Omega, \\ \Phi_* = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.15}$$

where we have used the invertibility of $-\Delta + (2\theta_a - a)I$ with zero Dirichlet boundary condition (see Lemma 1.1). Furthermore, for $p > N$, set

$$\begin{cases} X := \left[W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)\right] \times \left[W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)\right], \\ Y := L^p(\Omega) \times L^p(\Omega). \end{cases} \tag{2.16}$$

By Sobolev's theorem, X is continuously embedded into $C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega})$. Then one can show the following result by the local bifurcation theory.

PROPOSITION 2.1. *Define b_* by* (2.12). *Then positive solutions of* (RSP-1) *bifurcate from a semi-trivial solution curve $\{(\theta_a, 0, b);\ b > \lambda_1\}$ if and only if $b = b_*$. More precisely, there exists a positive number δ such that all positive solutions of* (RSP-1) *near $(\theta_a, 0, b_*) \in X \times \boldsymbol{R}$ can be expressed as*

$$(U, V, b) = (U(\varepsilon), V(\varepsilon), b(\varepsilon)) \quad \text{for } 0 \le \varepsilon \le \delta$$

with

$$\begin{aligned} U(\varepsilon) &= \theta_a + \varepsilon\Phi_* + \varepsilon\hat{U}(\varepsilon) = \theta_a + \varepsilon\Phi_* + O(\varepsilon^2), \\ V(\varepsilon) &= \varepsilon\Psi_* + \varepsilon\hat{V}(\varepsilon) = \varepsilon\Psi_* + O(\varepsilon^2), \\ b(\varepsilon) &= b_* + b'(0)\varepsilon + O(\varepsilon^2), \end{aligned} \tag{2.17}$$

where Φ_* and Ψ_* are defined by (2.15) *and* (2.14), *respectively, and $\{(\hat{U}(\varepsilon), \hat{V}(\varepsilon), b(\varepsilon))\}$ for $0 < \varepsilon \le \delta$ is a family of smooth functions with respect to ε satisfying $(\hat{U}(0), \hat{V}(0), b(0)) = (0, 0, b_*)$ and $\int_\Omega \hat{V}(s)\Psi_*\,dx = 0$.*

PROOF. We will employ the local bifurcation theorem due to Crandall and Rabinowitz [7]. Set

$$F(u, v) = u(a - u - cv), \qquad G(u, v) = v(b - du - v),$$

where u and v should be regarded as functions of U and V (see (1.8) and (2.1)). By Taylor's expansions of F and G at (U^*, V^*), (RSP-1) can be written in the following form

$$\begin{pmatrix} \Delta U \\ \Delta V \end{pmatrix} + \begin{pmatrix} F(u(U^*, V^*), v(U^*, V^*)) \\ G(u(U^*, V^*), v(U^*, V^*)) \end{pmatrix} + \begin{pmatrix} F_u^* & F_v^* \\ G_u^* & G_v^* \end{pmatrix} \begin{pmatrix} u_U^* & u_V^* \\ v_U^* & v_V^* \end{pmatrix} \begin{pmatrix} U - U^* \\ V - V^* \end{pmatrix} + \begin{pmatrix} \rho^1(U - U^*, V - V^*) \\ \rho^2(U - U^*, V - V^*) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{2.18}$$

where $F_u^* := F_u(u(U^*, V^*), v(U^*, V^*))$, $u_U^* := u_U(U^*, V^*)$ and other notations are used as partial derivatives. Here $\rho^i(U - U^*, V - V^*)$ $(i = 1, 2)$ are smooth functions such that $\rho^i(0, 0) = \rho^i_{(U,V)}(0, 0) = 0$. Recall by (2.6)

$$\begin{pmatrix} u_U & u_V \\ v_U & v_V \end{pmatrix} = \frac{1}{1 + \alpha v + \beta u} \begin{pmatrix} 1 + \beta u & -\alpha u \\ -\beta v & 1 + \alpha v \end{pmatrix}.$$

and note

$$F(\theta_a, 0) = \theta_a(a - \theta_a) = -\Delta\theta_a, \quad G(\theta_a, 0) = 0.$$

Set $(U^*, V^*) = (\theta_a, 0)$ and $\overline{U} := U - \theta_a$ in (2.18); then it can be reduced to the following semi-linear elliptic system

$$\begin{aligned} &\begin{pmatrix} \Delta\overline{U} \\ \Delta V \end{pmatrix} + \frac{1}{1 + \beta\theta_a} \begin{pmatrix} a - 2\theta_a & -c\theta_a \\ 0 & b - d\theta_a \end{pmatrix} \begin{pmatrix} 1 + \beta\theta_a & -\alpha\theta_a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \overline{U} \\ V \end{pmatrix} \\ &\quad + \begin{pmatrix} \rho^1(\overline{U}, V; b) \\ \rho^2(\overline{U}, V; b) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \end{aligned} \tag{2.19}$$

where $\rho^i(\overline{U}, V; b)$ $(i = 1, 2)$ are smooth functions satisfying

$$\rho^1_{(\overline{U},V)}(0, 0; b) = \rho^2_{(\overline{U},V)}(0, 0; b) = 0 \quad \text{for all } a > \lambda_1. \tag{2.20}$$

Define a mapping $\mathcal{F} : X \times \boldsymbol{R} \to Y$ by the left-hand side of (2.19):

$$\begin{aligned} &\mathcal{F}(\overline{U}, V, b) \\ &\quad = \begin{pmatrix} \Delta\overline{U} + (a - 2\theta_a)\overline{U} - \dfrac{(a\alpha + c - 2\alpha\theta_a)\theta_a}{1 + \beta\theta_a} V + \rho^1(\overline{U}, V, b) \\ \Delta V + \dfrac{b - d\theta_a}{1 + \beta\theta_a} V + \rho^2(\overline{U}, V; b) \end{pmatrix}. \end{aligned} \tag{2.21}$$

Since $(U, V) = (\theta_a, 0)$ is a semi-trivial solution of (RSP-1), it turns out $\mathcal{F}(0, 0, b) = 0$ for $b > \lambda_1$.

We will take the Fréchet derivative of $\mathcal{F}(\overline{U}, V, b)$ at $(\overline{U}, V, b) = (0, 0, b)$. From (2.20) and (2.21) it is given by

$$\mathcal{F}_{(\overline{U},V)}(0, 0, b) \begin{pmatrix} \Phi \\ \Psi \end{pmatrix} = \begin{pmatrix} \Delta\Phi + (a - 2\theta_a)\Phi - \dfrac{(a\alpha + c - 2\alpha\theta_a)\theta_a}{1 + \beta\theta_a}\Psi \\ \Delta\Psi + \dfrac{b - d\theta_a}{1 + \beta\theta_a}\Psi \end{pmatrix}. \tag{2.22}$$

Here it should be noted by (2.12) that $\mathcal{F}_{(\overline{U},V)}(0,0,b)$ is singular if and only if $b = b_*$. Moreover, one can see from (2.14) and (2.15) that

$$\operatorname{Ker}\mathcal{F}_{(\overline{U},V)}(0,0,b_*) = \operatorname{span}\{(\Phi_*,\Psi_*)\}.$$

We will show that the codimension of Range $\mathcal{F}_{(\overline{U},V)}(0,0,b_*)$ is one. If $(\tilde{h},\tilde{k}) \in$ Range $\mathcal{F}_{(\overline{U},V)}(0,0,b_*)$, there must exist $(h,k) \in X$ such that

$$\begin{cases} \Delta h + (a-2\theta_a)h - \dfrac{(a\alpha + c - 2\alpha\theta_a)\theta_a}{1+\beta\theta_{a_*}}k = \tilde{h} & \text{in } \Omega, \\ \Delta k + \dfrac{b_* - d\theta_a}{1+\beta\theta_a}k = \tilde{k} & \text{in } \Omega, \\ h = k = 0 & \text{on } \partial\Omega. \end{cases}$$

It is well known that the second equation has a solution k if and only if $\int_\Omega \tilde{k}\Psi_*\,dx = 0$. For such a solution k, the first equation has a unique solution h because of the invertibility of $-\Delta + (2\theta_a - a)I$. Thus we have proved codim Range $\mathcal{F}_{(\overline{U},V)}(0,0,b_*) = 1$.

In order to use the local bifurcation theory by Crandall and Rabinowitz [7] at $(\overline{U},V,b) = (0,0,b_*)$, it remains to verify

$$\mathcal{F}_{(\overline{U},V),b}(0,0,b_*)\begin{pmatrix}\Phi_*\\ \Psi_*\end{pmatrix} \notin \operatorname{Range}\mathcal{F}_{(\overline{U},V)}(0,0,b_*). \tag{2.23}$$

Since $\rho^i_{(\overline{U},V),b}(0,0,b_*) = 0$ by (2.20), it follows from (2.21) that

$$\mathcal{F}_{(\overline{U},V),b}(0,0,b_*)\begin{pmatrix}\Phi_*\\ \Psi_*\end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{1}{1+\beta\theta_a}\Psi_* \end{pmatrix}.$$

Using the argument in the preceding paragraph we can check the range condition (2.23) because $\int_\Omega \frac{\Psi_*^2}{1+\beta\theta_a}dx \neq 0$.

Recall $\overline{U} = U - \theta_a$; we can immediately obtain the assertion of this proposition by the local bifurcation theorem [7]. □

REMARK 2.3. Using Taylor's expansions of u and v with respect to $\overline{U} = U - \theta_a$ and V at $(U,V) = (\theta_a, 0)$ we get

$$u = \theta_a + \overline{U} - \frac{\alpha\theta_a}{1+\beta\theta_a}V + \eta_1(\overline{U},V),$$
$$v = \frac{1}{1+\beta\theta_a}V + \eta_2(\overline{U},V),$$

where $\eta_1 = O(\overline{U}^2 + V^2)$ and $\eta_2 = O(\overline{U}^2 + V^2)$. Hence Proposition 2.1 implies that bifurcating positive solutions of (SP-1) for $b = b(\varepsilon) = b_* + b'(0)\varepsilon + O(\varepsilon^2)$ can be expressed as

$$\begin{aligned} u(\varepsilon) &= \theta_a + \varepsilon\left(\Phi_* - \frac{\alpha\theta_a}{1+\beta\theta_a}\Psi_*\right) + O(\varepsilon^2), \\ v(\varepsilon) &= \frac{\varepsilon}{1+\beta\theta_a}\Psi_* + O(\varepsilon^2), \end{aligned} \tag{2.24}$$

where Φ_* and Ψ_* are defined by (2.15) and (2.14).

In Proposition 2.1, it is very important to know the direction of bifurcation with respect to parameter b or equivalently the sign of $b'(0)$. If $b'(0) > 0$ (resp. $b'(0) < 0$), then the bifurcation of positive solutions of (SP-1) (or (RSP-1)) from $(\theta_a, 0, b)$ at $b = b_*$ is supercritical (resp. subcritical). As to the direction of bifurcation, we can obtain the following result.

LEMMA 2.5. *Let* $\{(U(\varepsilon), V(\varepsilon), b(\varepsilon))\}$ *be a family of positive solutions of* (RSP-1) *as in Proposition* 2.1*. Then it holds that*

$$\begin{aligned} &b'(0)\int_\Omega \frac{\Psi_*^2}{1+\beta\theta_a}dx \\ &\quad = \int_\Omega \frac{\Psi_*^2}{(1+\beta\theta_a)^2}\left\{\Psi_* - (d+b_*\beta)\left(\frac{\alpha\theta_a}{1+\beta\theta_a}\Psi_* - \Phi_*\right)\right\}dx. \end{aligned}$$

PROOF. We will use (2.17), (2.24) and substitute these expressions into

$$\Delta V + v(b - du - v) = 0 \quad \text{in } \Omega. \tag{2.25}$$

By (2.17)

$$\Delta V(\varepsilon) = \varepsilon(\Delta\Psi_* + \Delta\hat{V}(\varepsilon)) \tag{2.26}$$

and by (2.24)

$$\begin{aligned} &b(\varepsilon) - du(\varepsilon) - v(\varepsilon) \\ &\quad = b_* - d\theta_a + b'(0)\varepsilon + \varepsilon\left(\frac{d\alpha\theta_a - 1}{1+\beta\theta_a}\Psi_* - d\Phi_*\right) + O(\varepsilon^2). \end{aligned} \tag{2.27}$$

Moreover, it follows from (2.6) that the second partial derivatives of v are given by

$$\begin{aligned} v_{UU} &= \frac{2\beta^2 v(1+\beta u)}{(1+\alpha v+\beta u)^3}, \\ v_{UV} &= -\frac{\beta(1+\beta u+\alpha v+2\alpha\beta uv)}{(1+\alpha v+\beta u)^3}, \\ v_{VV} &= \frac{2\alpha\beta u(1+\alpha v)}{(1+\alpha v+\beta u)^3}; \end{aligned}$$

so that

$$v_{UU}\Big|_{(U,V)=(\theta_a,0)}=0\,,\quad v_{UV}\Big|_{(U,V)=(\theta_a,0)}=-\frac{\beta}{(1+\beta\theta_a)^2},$$
$$v_{VV}\Big|_{(U,V)=(\theta_a,0)}=\frac{2\alpha\beta\theta_a}{(1+\beta\theta_a)^3}.$$

Taking Taylor's expansion of v with respect to $\overline{U}=U-\theta_a$ and V we get

$$v=\frac{1}{1+\beta\theta_a}V+\frac{1}{2}\left\{\frac{2\alpha\beta\theta_a}{(1+\beta\theta_a)^3}V^2-\frac{2\beta}{(1+\beta\theta_a)^2}\overline{U}V\right\}+\eta_3(\overline{U},V),$$

where $\eta_3(\overline{U},V)=O(\overline{U}^3+V^3)$. Therefore,

$$\begin{aligned} v(\varepsilon)=&\frac{\varepsilon}{1+\beta\theta_a}(\Psi_*+\hat{V}(\varepsilon))\\ &+\frac{\beta\varepsilon^2}{(1+\beta\theta_a)^2}\left(\frac{\alpha\theta_a}{1+\beta\theta_a}\Psi_*-\Phi_*\right)\Psi_*+O(\varepsilon^3). \end{aligned}\tag{2.28}$$

Substitution of (2.26), (2.27) and (2.28) into (2.25) leads to

$$\begin{aligned} 0=&\left(\Delta+\frac{b_*-d\theta_a}{1+\beta\theta_a}\right)(\Psi_*+\hat{V}(\varepsilon))+\frac{\varepsilon b'(0)}{1+\beta\theta_a}\Psi_*\\ &+\varepsilon\left\{\frac{\beta(b_*-d\theta_a)}{(1+\beta\theta_a)^2}\left(\frac{\alpha\theta_a}{1+\beta\theta_a}\Psi_*-\Phi_*\right)\Psi_*\right.\\ &\left.\qquad+\frac{1}{1+\beta\theta_a}\left(\frac{d\alpha\theta_a-1}{1+\beta\theta_a}\Psi_*-d\Phi_*\right)\Psi_*\right\}+O(\varepsilon^2)\\ =&\left(\Delta+\frac{b_*-d\theta_a}{1+\beta\theta_a}\right)\hat{V}(\varepsilon)+\frac{\varepsilon b'(0)}{1+\beta\theta_a}\Psi_*\\ &+\frac{\varepsilon}{(1+\beta\theta_a)^2}\left\{-\Psi_*+(d+b_*\beta)\left(\frac{\alpha\theta_a}{1+\beta\theta_a}\Psi_*-\Phi_*\right)\right\}\Psi_*+O(\varepsilon^2). \end{aligned}$$

Taking $L^2(\Omega)$-inner product of the right-hand side of the above identity with Ψ_* one can derive

$$\begin{aligned} &b'(0)\int_\Omega\frac{\Psi_*^2}{1+\beta\theta_a}dx\\ &\quad=\int_\Omega\frac{\Psi_*^2}{(1+\beta\theta_a)^2}\left\{\Psi_*-(d+b_*\beta)\left(\frac{\alpha\theta_a}{1+\beta\theta_a}\Psi_*-\Phi_*\right)\right\}dx+O(\varepsilon), \end{aligned}$$

which yields the assertion by letting $\varepsilon\to 0$. □

Let $(U(\varepsilon),V(\varepsilon),b(\epsilon))$ be bifurcating positive solutions of (RSP-1) and define $u(\varepsilon)$, $v(\varepsilon)$ and $b(\epsilon)$ by (2.1). Clearly, $\{(u(\varepsilon),v(\varepsilon),b(\epsilon))\}_{0<\varepsilon\le\delta}$ is a family of positive solutions of (SP-1).

We will study stability properties of this family.

LEMMA 2.6. *Let* $\{(u(\varepsilon), v(\varepsilon), b(\epsilon))\}_{0<\varepsilon\le\delta}$ *be a family of positive solutions of* (SP-1) *corresponding to* $\{(U(\varepsilon), V(\varepsilon), b(\epsilon))\}$ *derived in Proposition* 2.1. *Then* $(u(\varepsilon), v(\varepsilon))$ *for sufficiently small* $\varepsilon > 0$ *is asymptotically stable if the bifurcation is supercritical* (*i.e.,* $b'(0) > 0$), *while it is unstable if the bifurcation is subcritical* (*i.e.,* $b'(0) < 0$).

PROOF. We observe that the eigenvalue problem for the linearization associated with $(u(\varepsilon), v(\varepsilon))$ for $b = b_*(\varepsilon)$ is given by

$$\begin{cases} \Delta[(1+\alpha v(\varepsilon))\varphi + \alpha u(\varepsilon)\psi] + (a - 2u(\varepsilon) - cv(\varepsilon))\varphi \\ \qquad - cu(\varepsilon)\psi = -\sigma\varphi & \text{in } \Omega, \\ \Delta[(1+\beta u(\varepsilon))\psi + \beta v(\varepsilon)\varphi] + (b_*(\varepsilon) - du(\varepsilon) - 2v(\varepsilon))\psi \\ \qquad - dv(\varepsilon)\varphi = -\rho\sigma\psi & \text{in } \Omega, \\ \varphi = \psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.29}$$

For $\varepsilon = 0$, (2.29) can be written as

$$\begin{cases} \Delta[\varphi + \alpha\theta_a\psi] + (a - 2\theta_a)\varphi - c\theta_a\psi = -\sigma\varphi & \text{in } \Omega, \\ \Delta[(1+\beta\theta_a)\psi] + (b_* - d\theta_a)\psi = -\rho\sigma\psi & \text{in } \Omega, \\ \varphi = \psi = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.30}$$

which is identical with (1.22) by setting $u = \varphi$ and $V = (1+\beta\theta_a)\psi$. From the arguments in Section 1.3, it is possible to see that zero is a simple eigenvalue of (2.30) and that all other eigenvalues are positive. If we set

$$\Phi = \varphi + \alpha\theta_a\psi \quad \text{and} \quad \Psi = (1+\beta\theta_a)\psi,$$

then (2.30) is equivalent to

$$\begin{cases} -\Delta\Phi + (2\theta_a - a)\Phi + \dfrac{(a\alpha + c - 2\alpha\theta_a)\theta_a}{1+\beta\theta_a}\Psi \\ \qquad = \sigma\left(\Phi - \dfrac{\alpha\theta_a}{1+\beta\theta_a}\Psi\right) & \text{in } \Omega, \\ -\Delta\Psi + \dfrac{d\theta_a - b_*}{1+\beta\theta_a}\Psi = \dfrac{\rho\sigma}{1+\beta\theta_a}\Psi & \text{in } \Omega, \\ \Phi = \Psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.31}$$

An eigenfunction corresponding to zero eigenvalue of (2.31) can be given by (Φ_*, Ψ_*) defined by (2.15) and (2.14). Set

$$\varphi_* = \Phi_* - \frac{\alpha\theta_a}{1+\beta\theta_a}\Psi_* \quad \text{and} \quad \psi_* = \frac{1}{1+\beta\theta_a}\Psi_*; \tag{2.32}$$

then it is easy to see that (φ_*, ψ_*) becomes an eigenfunction corresponding to zero eigenvalue of (2.30).

We are ready to employ the idea and method developed in the work of Crandall and Rabinowitz [8] in order to study the linearized eigenvalue problem (2.29). This problem has a simple eigenvalue $\sigma(\varepsilon)$ satisfying $\sigma(0) = 0$ and real parts of all other eigenvalues are positive and bounded away from zero for sufficiently small $\varepsilon > 0$. Using the implicit function theorem one can express an eigenfunction corresponding to $\sigma(\varepsilon)$ as

$$\begin{cases} \varphi = \varphi_* + \hat{\varphi}(\varepsilon) = \varphi_* + O(\varepsilon), \\ \psi = \psi_* + \hat{\psi}(\varepsilon) = \psi_* + O(\varepsilon). \end{cases} \tag{2.33}$$

Recall

$$u(\varepsilon) = \theta_a + \varepsilon\varphi_* + O(\varepsilon^2) \quad \text{and} \quad v(\varepsilon) = \varepsilon\psi_* + O(\varepsilon^2) \tag{2.34}$$

by (2.24) and (2.32). Substitution of (2.34) into the second equation of (2.29) gives

$$\begin{aligned} &\Delta[(1+\beta\theta_a)\hat{\psi}(\varepsilon)] + (b_* - d\theta_a)\hat{\psi}(\varepsilon) + 2\varepsilon\beta\Delta[\varphi_*\psi_*] \\ &\quad + \varepsilon b'(0)\psi_* - 2\varepsilon\psi_*(d\varphi_* + \psi_*) = -\rho\sigma'(0)\varepsilon\psi_* + O(\varepsilon^2). \end{aligned} \tag{2.35}$$

Note $\Delta\Psi_* + (b_* - d\theta_a)\psi_* = 0$ in Ω. Taking L^2-inner product of (2.34) with Ψ_* we can deduce

$$\begin{aligned} &2\beta(\Delta[\varphi_*\psi_*], \Psi_*)_{L^2} + b'(0)(\psi_*, \Psi_*)_{L^2} - 2(\psi_*(d\varphi_* + \psi_*), \Psi_*)_{L^2} \\ &\quad = -\rho\sigma'(0)(\psi_*, \Psi_*)_{L^2}, \end{aligned}$$

where $(\cdot, \cdot)_{L^2}$ denotes $L^2(\Omega)$-inner product. Note

$$\begin{aligned} (\Delta[\varphi_*\psi_*], \Psi_*)_{L^2} = (\varphi_*\psi_*, \Delta\Psi_*)_{L^2} &= \left(\varphi_*\psi_*, \frac{d\theta_a - b_*}{1+\beta\theta_a}\Psi_*\right)_{L^2} \\ &= \left(\frac{d\theta_a - b_*}{1+\beta\theta_a}\varphi_*\psi_*, \Psi_*\right)_{L^2}. \end{aligned}$$

Moreover, Lemma 2.5 yields

$$\begin{aligned} b_*'(0)(\psi_*, \Psi_*)_{L^2} &= \int_\Omega \psi_*^2\{\Psi_* + (d + b_*\beta)\varphi_*\}dx \\ &= \left(\psi_*^2 + \frac{d + b_*\beta}{1+\beta\theta_a}\varphi_*\psi_*, \Psi_*\right)_{L^2}. \end{aligned}$$

After some arrangements we have

$$\begin{aligned} \rho\sigma'(0)(\psi_*, \Psi_*)_{L^2} &= -b_*'(0)(\psi_*, \Psi_*)_{L^2} + 2(\psi_*^2, \Psi_*)_{L^2} \\ &\quad + 2\left(\frac{d + b_*\beta}{1+\beta\theta_a}\varphi_*\psi_*, \Psi_*\right)_{L^2} \\ &= b_*'(0)(\psi_*, \Psi_*)_{L^2}. \end{aligned}$$

Hence the sign of $\sigma'(0)$ is the same as that of $b_*'(0)$; this fact enables us to prove the assertion. □

We can also discuss the bifurcation from $(0, \theta_b, b)$ almost in the same way as in the preceding arguments by regarding b as a bifurcation parameter. Define b^* by

$$b^* = g(a), \tag{2.36}$$

where g is defined by (1.38) (see Figure 1.2). From (1.9)

$$\lambda_1\left(\frac{c\theta_{b^*} - a}{1 + \alpha\theta_{b^*}}\right) = 0. \tag{2.37}$$

Instead of (2.14) and (2.15), define Φ^* by a unique positive solution satisfying

$$\begin{cases} -\Delta\Phi^* + \dfrac{c\theta_{b^*} - a}{1 + \alpha\theta_{b^*}}\Phi^* = 0 & \text{in } \Omega, \\ \Phi^* = 0 & \text{on } \partial\Omega, \\ \Phi^* > 0 & \text{in } \Omega, \end{cases} \tag{2.38}$$

and $\|\Phi^*\| = 1$. We also define Ψ^* as a solution of

$$\begin{cases} -\Delta\Psi^* + (2\theta_{b^*} - b^*)\Psi^* = -\dfrac{(b^*\beta + d - 2\beta\theta_{b^*})\theta_{b^*}}{1 + \alpha\theta_{b^*}}\Phi^* & \text{in } \Omega, \\ \Psi^* = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.39}$$

Note the invertibility of $-\Delta + (2\theta_{b^*} - b^*)I$ with zero Dirichlet boundary condition (see Lemma 1.1).

Define X and Y by (2.16). We will apply the local bifurcation theory of Crandall and Rabinowitz [7] to get the following result.

PROPOSITION 2.2. *Define b^* by (2.36). Then positive solutions of (RSP-1) bifurcate from a semi-trivial solution curve $\{(0, \theta_b, b);\ b > \lambda_1\}$ if and only if $b = b^*$. More precisely, there exists a positive number δ such that all positive solutions of (RSP-1) near $(0, \theta_{b^*}, b_*) \in X \times \boldsymbol{R}$ can be expressed as*

$$(U, V, b) = (U(\varepsilon), V(\varepsilon), b(\varepsilon)) \quad \textit{for } 0 \le \varepsilon \le \delta$$

with

$$\begin{aligned} U(\varepsilon) &= \varepsilon\Phi^* + \varepsilon\hat{U}(\varepsilon) = \varepsilon\Phi^* + O(\varepsilon^2), \\ V(\varepsilon) &= \theta_{b(\varepsilon)} + \varepsilon\Psi^* + \varepsilon\hat{V}(\varepsilon) = \theta_{b^*} + \varepsilon\Psi^* + \varepsilon b'(0)\frac{\partial\theta_b}{\partial b}\bigg|_{b=b^*} + O(\varepsilon^2), \\ b(\varepsilon) &= b^* + b'(0)\varepsilon + O(\varepsilon^2), \end{aligned} \tag{2.40}$$

where Φ^*, Ψ^* are defined by (2.38), (2.39), respectively, and $\{(\hat{U}(\varepsilon), \hat{V}(\varepsilon), b(\varepsilon))\}$ for $0 < \varepsilon \le \delta$ is a family of smooth functions with respect to ε satisfying $(\hat{U}(0), \hat{V}(0), b(0)) = (0, 0, b^*)$ and $\int_\Omega \hat{U}(s)\Phi^*\,dx = 0$.

PROOF. We will repeat the proof of Proposition 2.1 with slight modification and verify the conditions of the local bifurcation theorem in [7].

We will use (2.18) and take $(0, \theta_{b^*})$. By setting $\overline{V} = V - \theta_b$, semi-linear elliptic system (2.18) is written in the following form

$$\begin{pmatrix} \Delta U \\ \Delta \overline{V} \end{pmatrix} + \frac{1}{1+\alpha\theta_b} \begin{pmatrix} a - c\theta_b & 0 \\ -d\theta_b & b - 2\theta_b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\beta\theta_b & 1+\alpha\theta_b \end{pmatrix} \begin{pmatrix} U \\ \overline{V} \end{pmatrix} + \begin{pmatrix} \rho^3(U, \overline{V}; b) \\ \rho^4(U, \overline{V}; b) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{2.41}$$

where $\rho^i(\overline{U}, V; b)$ $(i = 3,\ 4)$ are smooth functions satisfying

$$\rho^3_{(U,\overline{V})}(0, 0; b) = \rho^4_{(U,\overline{V})}(0, 0; b) = 0 \quad \text{for all } a > \lambda_1.$$

Define a mapping $\mathcal{G} : X \times \boldsymbol{R} \to Y$ by the left-hand side of (2.41):

$$\mathcal{G}(U, \overline{V}, b) = \begin{pmatrix} \Delta U + \dfrac{a - c\theta_b}{1+\alpha\theta_b} V + \rho^3(U, \overline{V}; b) \\ \Delta \overline{V} + (b - 2\theta_b)\overline{V} - \dfrac{(b\beta + d - 2\beta\theta_b)\theta_b}{1+\alpha\theta_b} U + \rho^4(U, \overline{V}, b) \end{pmatrix}, \tag{2.42}$$

which satisfies $\mathcal{G}(0, 0, b) = 0$ for all $b > \lambda_1$ because $(U, V) = (0, \theta_b)$ is a semi-trivial solution of (RSP-1).

The Fréchet derivative of $\mathcal{G}(U, \overline{V}, b)$ at $(U, \overline{V}, b) = (0, 0, b)$ is given by

$$\mathcal{G}_{(U,\overline{V})}(0, 0, b) \begin{pmatrix} \Phi \\ \Psi \end{pmatrix} = \begin{pmatrix} \Delta\Phi + \dfrac{a - c\theta_b}{1+\alpha\theta_b}\Phi \\ \Delta\Psi + (b - 2\theta_b)\Psi - \dfrac{(b\beta + d - 2\beta\theta_b)\theta_b}{1+\alpha\theta_b}\Phi \end{pmatrix}. \tag{2.43}$$

From the definition of b^*, it is easy to see that $\mathcal{G}_{(U,\overline{V})}(0, 0, b)$ is singular if and only if $b = b^*$. Moreover, recalling (2.38) and (2.39) one can find

$$\operatorname{Ker} \mathcal{G}_{(U,\overline{V})}(0, 0, b^*) = \operatorname{span}\{(\Phi^*, \Psi^*)\},$$

which means the dimension of ker $\mathcal{G}_{(U,\overline{V})}(0, 0, b^*)$ is one.

The proof that the codimension of Range $\mathcal{G}_{(U,\overline{V})}(0, 0, b^*)$ is one is essentially the same as that for Range $\mathcal{F}_{(\overline{U},V)}(0, 0, b_*)$ in Proposition 2.1. If $(\tilde{h}, \tilde{k}) \in$ Range $\mathcal{G}_{(\overline{U},V)}(0, 0, b^*)$, then

$$\begin{cases} \Delta h + \dfrac{a - c\theta_{b^*}}{1+\alpha\theta_{b^*}} h = \tilde{h} & \text{in } \Omega, \\ \Delta k + (b^* - 2\theta_{b^*})k - \dfrac{(b^*\beta + d - 2\beta\theta_{b^*})\theta_{b^*}}{1+\alpha\theta_{b^*}} h = \tilde{k} & \text{in } \Omega, \\ h = k = 0 & \text{on } \partial\Omega, \end{cases}$$

for some $(h, k) \in X$. In view of $\lambda_1(c\theta_{b^*}/(1 + \alpha\theta_{b^*})) = 0$, the first equation has a solution h if and only if $\int_\Omega \tilde{h}\Phi^* dx = 0$. Since the second equation always has a solution k, we get the assertion.

Finally we have to verify

$$\mathcal{G}_{(U,\overline{V}),b}(0,0,b^*)\begin{pmatrix}\Phi^*\\ \Psi^*\end{pmatrix} \notin \operatorname{Range}\mathcal{G}_{(U,\overline{V})}(0,0,b^*). \tag{2.44}$$

From (2.43)

$$G_{(U,\overline{V}),b}(0,0,b^*)\begin{pmatrix}\Phi^*\\ \Psi^*\end{pmatrix}$$
$$= \begin{pmatrix} -\dfrac{c+a\alpha}{(1+\alpha\theta_{b^*})^2}\dfrac{\partial\theta_b}{\partial b}\Big|_{b=b^*}\Phi^* \\ \left(1-2\dfrac{\partial\theta_b}{\partial b}\Big|_{b=b^*}\right)\Psi-\dfrac{\partial}{\partial b}\left(\dfrac{(b\beta+d-2\beta\theta_b)\theta_b}{1+\alpha\theta_b}\right)\Big|_{b=b^*}\Phi_* \end{pmatrix}.$$

Suppose that (2.44) is false. Then it follows from the arguments in the preceding paragraph that

$$\int_\Omega \frac{c+a\alpha}{(1+\alpha\theta_{b^*})^2}\frac{\partial\theta_b}{\partial b}\Big|_{b=b^*}(\Phi^*)^2\,dx=0.$$

However, (ii) of Lemma 1.1 implies that the above integral must be positive. Thus we arrive at a contradiction.

Since we have verified all the conditions for the local bifurcation theorem, we can obtain the conclusion. □

REMARK 2.4. By Taylor's expansions of u and v with respect to U and $\overline{V}=V-\theta_a$ at $(U,V)=(0,\theta_b)$ we get

$$u=\frac{1}{1+\alpha\theta_b}U+\eta_3(U,\overline{V}),$$
$$v=\theta_b-\frac{\beta\theta_b}{1+\alpha\theta_b}U+\overline{V}+\eta_4(U,\overline{V}),$$

where $\eta_3=O(U^2+\overline{V}^2)$ and $\eta_4=O(U^2+\overline{V}^2)$. Hence Proposition 2.2 implies that bifurcating positive solutions of (SP-1) for $b=b(\varepsilon)=b^*+b'(0)\varepsilon+O(\varepsilon^2)$ can be expressed as

$$\begin{aligned} u(\varepsilon)&=\frac{\varepsilon}{1+\alpha\theta_{b(\varepsilon)}}\Phi^*+O(\varepsilon^2)=\frac{\varepsilon}{1+\alpha\theta_{b^*}}\Phi^*+O(\varepsilon^2),\\ v(\varepsilon)&=\theta_{b(\varepsilon)}+\varepsilon\left(\Psi^*-\frac{\beta\theta_{b(\varepsilon)}}{1+\alpha\theta_{b(\varepsilon)}}\Phi_*\right)+O(\varepsilon^2)\\ &=\theta_{b^*}+\varepsilon\left(\Psi^*-\frac{\beta\theta_{b^*}}{1+\alpha\theta_{b^*}}\Phi_*\right)+\varepsilon b'(0)\frac{\partial\theta_b}{\partial b}\Big|_{b=b^*}+O(\varepsilon^2), \end{aligned} \tag{2.45}$$

where Φ_* and Ψ_* are defined by (2.38) and (2.39).

We will study the direction of the bifurcation in Proposition 2.2 by the idea used in the proof of Proposition 2.1.

LEMMA 2.7. *Let* $\{(U(\varepsilon), V(\varepsilon), b(\varepsilon))\}$ *be a family of positive solutions of* (RSP-1) *as in Proposition 2.2. Then it holds that*

$$(a\alpha+c)b'(0)\int_\Omega \frac{(\Phi^*)^2}{(1+\alpha\theta_{b^*})^2}\frac{\partial\theta_b}{\partial b}\bigg|_{b=b^*}dx$$
$$= -\int_\Omega \frac{(\Phi^*)^2}{(1+\alpha\theta_{b^*})^2}\left\{\Phi^* - (a\alpha+c)\left(\frac{\beta\theta_{b^*}}{1+\alpha\theta_{b^*}}\Phi^* - \Psi^*\right)\right\}dx.$$

PROOF. We will use (2.40), (2.45) and substitute these expressions into

$$\Delta U + u(a-u-cv) = 0 \quad \text{in } \Omega. \tag{2.46}$$

By (2.40)

$$\Delta U(\varepsilon) = \varepsilon(\Delta\Phi^* + \Delta\hat{U}(\varepsilon)) \tag{2.47}$$

and by (2.45)

$$a - u(\varepsilon) - cv(\varepsilon) = a - c\theta_{b(\varepsilon)} + \varepsilon\left(\frac{c\beta\theta_{b^*}-1}{1+\alpha\theta_{b^*}}\Phi^* - c\Psi^*\right) + O(\varepsilon^2). \tag{2.48}$$

We also note that, as a function of U and V, the second derivatives of u are given by

$$u_{UU}\bigg|_{(U,V)=(0,\theta_b)} = \frac{2\alpha\beta\theta_b}{(1+\alpha\theta_b)^3}, \qquad u_{UV}\bigg|_{(U,V)=(0,\theta_b)} = -\frac{\alpha}{(1+\alpha\theta_b)^2},$$
$$u_{VV}\bigg|_{(U,V)=(0,\theta_b)} = 0.$$

The Taylor expansion of u with respect to U and $\overline{V} = V - \theta_b$ yields

$$u = \frac{1}{1+\alpha\theta_b}U + \left\{\frac{\alpha\beta\theta_b}{(1+\alpha\theta_b)^3}U^2 - \frac{\alpha}{(1+\alpha\theta_b)^2}U\overline{V}\right\} + \eta_4(U,\overline{V}),$$

where $\eta_4(U,\overline{V}) = O(U^3 + \overline{V}^3)$. Hence

$$u(\varepsilon) = \frac{\varepsilon}{1+\alpha\theta_{b(\varepsilon)}}(\Phi^* + \hat{U}(\varepsilon))$$
$$+ \frac{\alpha\varepsilon^2\Phi^*}{(1+\alpha\theta_{b^*})^2}\left(\frac{\beta\theta_{b^*}}{1+\alpha\theta_{b^*}}\Phi^* - \Psi^*\right) + O(\varepsilon^3). \tag{2.49}$$

It follows from (2.48) and (2.49) that

$$u(\varepsilon)(a-u(\varepsilon)-cv(\varepsilon)) = \varepsilon\,\frac{a-c\theta_{b(\varepsilon)}}{1+\alpha\theta_{b(\varepsilon)}}(\Phi^*+\hat{U}(\varepsilon))$$
$$+ \frac{\varepsilon^2\Phi^*}{(1+\alpha\theta_{b^*})^2}\left\{-\Phi^* + (a\alpha+c)\left(\frac{\beta\theta_{b^*}}{1+\alpha\theta_{b^*}}\Phi^* - \Psi^*\right)\right\} + O(\varepsilon^3)$$
$$= \varepsilon\,\frac{a-c\theta_{b^*}}{1+\alpha\theta_{b^*}}(\Phi^*+\hat{U}(\varepsilon)) - \varepsilon^2 b'(0)\frac{(a\alpha+c)\Phi^*}{(1+\alpha\theta_{b^*})^2}\frac{\partial\theta_b}{\partial b}\bigg|_{b=b^*}$$
$$+ \frac{\varepsilon^2\Phi^*}{(1+\alpha\theta_{b^*})^2}\left\{-\Phi^* + (a\alpha+c)\left(\frac{\beta\theta_{b^*}}{1+\alpha\theta_{b^*}}\Phi^* - \Psi^*\right)\right\} + O(\varepsilon^3). \tag{2.50}$$

Substitution of (2.47) and (2.50) into (2.46) gives

$$\left(\Delta + \frac{a - c\theta_{b^*}}{1+\alpha\theta_{b^*}}\right)\hat{U}(\varepsilon) - \varepsilon b'(0)\frac{(a\alpha + c)\Phi^*}{(1+\alpha\theta_{b^*})^2}\frac{\partial\theta_b}{\partial b}\bigg|_{b=b^*}$$
$$+\frac{\varepsilon\Phi^*}{(1+\alpha\theta_{b^*})^2}\left\{-\Phi^* + (a\alpha + c)\left(\frac{\beta\theta_{b^*}}{1+\alpha\theta_{b^*}}\Phi^* - \Psi^*\right)\right\} + O(\varepsilon^3) = 0, \tag{2.51}$$

where (2.38) has been used. Taking $L^2(\Omega)$-inner product of (2.51) with Φ^* and letting $\varepsilon \to 0$ one can derive the assertion. □

Let $\{(u(\varepsilon), v(\varepsilon), b(\varepsilon))\}$ be a family of positive solutions of (SP-1) associated with $\{(U(\varepsilon), V(\varepsilon), b(\varepsilon))\}$ constructed in Proposition 2.2. The eigenvalue problem for the linearization with respect to $(u(\varepsilon), v(\varepsilon), b(\varepsilon))$ is also given by (2.29). For $\varepsilon = 0$ this eigenvalue problem is written as

$$\begin{cases} \Delta[(1+\alpha\theta_{b^*})\varphi] + (a - c\theta_{b^*})\varphi = -\sigma\varphi & \text{in } \Omega, \\ \Delta[\psi + \beta\theta_{b^*}\varphi] + (b^* - 2\theta_{b^*})\psi - d\theta_{b^*}\varphi = -\rho\sigma\psi & \text{in } \Omega, \\ \varphi = \psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.52}$$

As in the proof of Lemma 2.6, it is possible to show that zero is a simple eigenvalue of (2.52) and that all other eigenvalues are positive. If we define

$$\Phi = (1+\alpha\theta_{b^*})\varphi \quad \text{and} \quad \Psi = \psi + \beta\theta_{b^*}\varphi,$$

then (2.52) is rewritten as

$$\begin{cases} -\Delta\Phi + \dfrac{c\theta_{b^*} - a}{1+\alpha\theta_{b^*}}\Phi = \dfrac{\sigma}{1+\alpha\theta_{b^*}}\Phi & \text{in } \Omega, \\ -\Delta\Psi + (2\theta_{b^*} - b^*)\Psi + \dfrac{(b^*\beta + d - 2\beta\theta_{b^*})\theta_{b^*}}{1+\alpha\theta_{b^*}}\Phi & \\ \qquad = \rho\sigma\left(\Psi - \dfrac{\beta\theta_{b^*}}{1+\alpha\theta_{b^*}}\Phi\right) & \text{in } \Omega, \\ \Phi = \Psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.53}$$

An eigenfunction corresponding to zero eigenvalue of (2.53) is given by (Φ^*, Ψ^*) defined by (2.38) and (2.39). If we set

$$\varphi^* = \frac{\Phi^*}{1+\alpha\theta_{b^*}} \quad \text{and} \quad \psi^* = \Psi^* - \frac{\beta\theta_{b^*}}{1+\alpha\theta_{b*}}\Phi^*, \tag{2.54}$$

then we see that (φ^*, ψ^*) is an eigenfunction corresponding to zero eigenvalue of (2.53).

We will study the linearized eigenvalue problem (2.29) as in the proof of Lemma 2.6. This problem has a simple eigenvalue $\sigma(\varepsilon)$ satisfying $\sigma(0) = 0$ and real parts of all other eigenvalues are positive and bounded away from zero for sufficiently small $\varepsilon > 0$. By the implicit function theorem we can prove that the eigenfunction corresponding to $\sigma(\varepsilon)$ is expressed as

$$\begin{cases} \varphi = \varphi^* + \hat{\varphi}(\varepsilon) = \varphi^* + O(\varepsilon), \\ \psi = \psi^* + \hat{\psi}(\varepsilon) = \psi^* + O(\varepsilon), \end{cases} \tag{2.55}$$

where φ^* and ψ^* are defined by (2.54). Note that bifurcating solutions have the following forms

$$u(\varepsilon) = \varepsilon\varphi^* + O(\varepsilon^2) \quad \text{and} \quad v(\varepsilon) = \theta_{b^*} + \varepsilon\psi^* + \varepsilon b'(0) \left.\frac{\partial\theta_b}{\partial b}\right|_{b=b^*} + O(\varepsilon^2) \tag{2.56}$$

(see (2.45) and (2.54)). Substituting (2.56) into the first equation of (2.29) we get after some arrangements

$$\begin{aligned} &\Delta[(1+\alpha\theta_{b^*})\hat{\varphi}] + (a - c\theta_{b^*})\hat{\varphi} + 2\varepsilon\alpha\Delta[\varphi^*\psi^*] + \varepsilon\alpha b'(0)\Delta\left[\left.\frac{\partial\theta_b}{\partial b}\right|_{b=b^*}\varphi^*\right] \\ &\quad - 2\varepsilon(\varphi^* + c\psi^*)\varphi^* - \varepsilon c b'(0)\left.\frac{\partial\theta_b}{\partial b}\right|_{b=b^*}\varphi^* = -\sigma'(0)\varphi^* + O(\varepsilon^2). \end{aligned} \tag{2.57}$$

Observe that by (2.38)

$$\begin{aligned} &(\Delta[(1+\alpha\theta_{b^*})\hat{\varphi}], \Phi^*)_{L^2} + ((a - c\theta_{b^*})\hat{\varphi}, \Phi^*)_{L^2} \\ &\quad = ((1+\alpha\theta_{b^*})\hat{\varphi}, \Delta\Phi^*)_{L^2} + ((a - c\theta_{b^*})\hat{\varphi}, \Phi^*)_{L^2} \\ &\quad = (\hat{\varphi}, (c\theta_{b^*} - a)\Phi^*)_{L^2} + ((a - c\theta_{b^*})\hat{\varphi}, \Phi^*)_{L^2} = 0. \end{aligned}$$

Taking $L^2(\Omega)$-inner product of (2.57) with Φ^* leads to

$$\begin{aligned} -\sigma'(0)(\varphi^*, \Phi^*)_{L^2} &= 2\alpha(\Delta[\varphi^*\psi^*], \Phi^*)_{L^2} - 2((\varphi^* + c\psi^*)\varphi^*, \Phi^*)_{L^2} \\ &\quad + \alpha b'(0)\left(\Delta\left[\left.\frac{\partial\theta_b}{\partial b}\right|_{b=b^*}\varphi^*\right], \Phi^*\right)_{L^2} - c b'(0)\left(\left.\frac{\partial\theta_b}{\partial b}\right|_{b=b^*}\varphi^*. \Phi^*\right)_{L^2}. \end{aligned}$$

Here it should be noted that

$$\begin{aligned} &2\alpha(\Delta[\varphi^*\psi^*], \Phi^*)_{L^2} - 2((\varphi^* + c\psi^*)\varphi^*, \Phi^*)_{L^2} \\ &\quad = 2\alpha(\varphi^*\psi^*, \frac{c\theta_{b^*} - a}{1+\alpha\theta_{b^*}}\Phi^*)_{L^2} - 2((\varphi^* + c\psi^*)\varphi^*, \Phi^*)_{L^2} \\ &\quad = -2\left(\left(\varphi^* + \frac{a\alpha + c}{1+\alpha\theta_{b^*}}\psi^*\right)\varphi^*, \Phi^*\right)_{L^2} \end{aligned}$$

and that

$$\begin{aligned} &\alpha\left(\Delta\left[\left.\frac{\partial\theta_b}{\partial b}\right|_{b=b^*}\varphi^*\right], \Phi^*\right)_{L^2} - c\left(\left.\frac{\partial\theta_b}{\partial b}\right|_{b=b^*}\varphi^*, \Phi^*\right)_{L^2} \\ &\quad = \alpha\left(\left.\frac{\partial\theta_b}{\partial b}\right|_{b=b^*}\varphi^*, \frac{c\theta_b - a}{1+\alpha\theta_{b^*}}\Phi^*\right)_{L^2} - c\left(\left.\frac{\partial\theta_b}{\partial b}\right|_{b=b^*}\varphi^*, \Phi^*\right)_{L^2} \\ &\quad = -(a\alpha + c)\left(\frac{\varphi^*}{1+\alpha\theta_{b^*}}\left.\frac{\partial\theta_b}{\partial b}\right|_{b=b^*}, \Phi^*\right)_{L^2}. \end{aligned}$$

Hence we get

$$\sigma'(0)(\varphi^*, \Phi^*)_{L^2} = 2\left(\left(\varphi^* + \frac{a\alpha + c}{1+\alpha\theta_{b^*}}\psi^*\right)\varphi^*, \Phi^*\right)_{L^2} + (a\alpha + c)b'(0)\left(\frac{\varphi^*}{1+\alpha\theta_{b^*}}\left.\frac{\partial\theta_b}{\partial b}\right|_{b=b^*}, \Phi^*\right)_{L^2}. \tag{2.58}$$

On the other hand, Lemma 2.7 yields

$$(a\alpha + c)b'(0)\left(\frac{\varphi^*}{1+\alpha\theta_{b^*}}\left.\frac{\partial\theta_b}{\partial b}\right|_{b=b^*}, \Phi^*\right)_{L^2} = -((\varphi^*)^2, \Phi^*)_{L^2} - (a\alpha + c)\left(\frac{\varphi^*\psi^*}{1+\alpha\theta_{b^*}}, \Phi^*\right)_{L^2}. \tag{2.59}$$

Finally it follows from (2.58) and (2.59) that

$$\sigma'(0)(\varphi^*, \Phi^*)_{L^2} = -(a\alpha + c)b'(0)\left(\frac{\varphi^*}{1+\alpha\theta_{b^*}}\left.\frac{\partial\theta_b}{\partial b}\right|_{b=b^*}, \Phi^*\right)_{L^2}.$$

Thus one can prove the following result.

LEMMA 2.8. *Let* $\{(u(\varepsilon), v(\varepsilon), b(\varepsilon))\}$ *be a family of positive solutions of* (SP-1) *corresponding to* $\{(U(\varepsilon), V(\varepsilon), b(\varepsilon))\}$ *derived in Proposition 2.2. Then* $(u(\varepsilon), v(\varepsilon))$ *for sufficiently small* $\varepsilon > 0$ *is asymptotically stable if the bifurcation is subcritical* (*i.e.,* $b'(0) < 0$)*, while it is unstable if the bifurcation is supercritical* (*i.e.,* $b'(0) > 0$).

We will reconsider Theorem 2.1 and Corollary 2.1 from the viewpoint of global bifurcation theory.

Let $a > \lambda_1$ be fixed and regard b as a bifurcation parameter. Proposition 2.1 implies that positive solutions of (RSP-1) bifurcate from $(\theta_a, 0, b)$ at $b = b_*$ defined by (2.12) and that all positive solutions (U, V, b) of (RSP-1) lie on a curve in $X \times \boldsymbol{R}$ of the form $\{(U(\varepsilon), V(\varepsilon), b(\varepsilon))\}_{0\le\varepsilon\le\delta}$ with $b(0) = b_*$, where $U(\varepsilon)$, $V(\varepsilon)$ and $b(\varepsilon)$ are expressed by (2.17). Set

$$P = \left\{w \in C_0^1(\Omega);\, w(x) > 0 \text{ for } x \in \Omega \text{ and } \frac{\partial w}{\partial\nu}(x) < 0 \text{ for } x \in \partial\Omega\right\}, \tag{2.60}$$

where $\partial/\partial\nu$ denotes the outward normal derivative on $\partial\Omega$. It should be noted that $(U(\varepsilon), V(\varepsilon)) \in P \times P$ for sufficiently small $\varepsilon > 0$ because θ_a and Ψ_* belong to P by the strong maximum principle (see [49]).

We now apply the global bifurcation theory of Rabinowitz [50]. Then there exists a connected set of nontrivial solutions of (RSP-1), denoted by $\mathcal{C}$, such that $\mathcal{C}$ is unbounded in $X \times \boldsymbol{R}$ or $\mathcal{C}$ meets a nontrivial solution at some $b = \tilde{b}$. Let $\mathcal{C}^+ \subset P \times P \times \boldsymbol{R}$ be a connected subset of $\mathcal{C} - \{(\theta_a, 0, b_*)\}$ such that $\mathcal{C}^+$ contains $\{(U(\varepsilon), V(\varepsilon), b(\varepsilon))\}$ for sufficiently small $\varepsilon > 0$. It can be also seen that $\mathcal{C}^+$ satisfies one of the above alternatives.

We will show the following result.

THEOREM 2.2. *For any fixed* $a > \lambda_1$*, let* $\mathcal{C}^+$ *be a connected set of positive solutions of* (RSP-1) *defined as above. Then the following properties hold true.*

(i) *If* $\mathcal{C}^+$ *is bounded in* $P \times P \times \boldsymbol{R}$, *then the following set*
$$\{b \in \boldsymbol{R};\ (U, V, b) \in \mathcal{C}^+ \cap (P \times P \times \boldsymbol{R})\}$$
contains open interval $(\min\{b^*, b_*\}, \max\{b^*, b_*\})$. *Moreover,* $\mathcal{C}^+$ *connects* $(\theta_a, 0, b_*)$ *with* $(0, \theta_{b^*}, b^*)$.

(ii) *If* $\mathcal{C}^+$ *is unbounded in* $P \times P \times \boldsymbol{R}$ *then open interval* (b_*, ∞) *is contained in*
$$\{b \in \boldsymbol{R};\ (U, V, b) \in \mathcal{C}^+ \cap (P \times P \times \boldsymbol{R})\}.$$

Here b_* *and* b^* *are constants defined by* (2.12) *and* (2.36).

REMARK 2.5. This theorem implies that, if $\mathcal{C}^+$ is bounded in $P \times P \times \boldsymbol{R}$, then this is a branch of positive solutions of (RSP-1) connecting $(\theta_a, 0, b_*)$ and $(0, \theta_{b^*}, b^*)$. However, we do not have satisfactory information whether $\mathcal{C}^+$ is bounded or unbounded in $P \times P \times \boldsymbol{R}$.

PROOF OF THEOREM 2.2. Suppose $\mathcal{C}^+ \cap (\partial(P \times P) \times \boldsymbol{R}) \ni (U_0, V_0, b_0)$ with $(U_0, V_0, b_0) \neq (\theta_a, 0, b_*)$. Then (U_0, V_0, b_0) is a limit of a sequence $\{(U_n, V_n, b_n)\} \subset \mathcal{C}^+ \cap (P \times P \times \boldsymbol{R})$. Since $(U_0, V_0) \in \partial(P \times P)$, U_0 (resp. V_0) has an interior zero in Ω or $\partial U_0/\partial \nu$ (resp. $\partial V_0/\partial \nu$) has a zero on $\partial\Omega$ if $U_0 \in \partial P$ (resp. $V_0 \in \partial P$).

We begin with the case $V_0 \in \partial P$. Since (U_0, V_0) is bounded by Lemma 2.1, one can prove that V_0 satisfies

$$\begin{cases} -\Delta V_0 + M V_0 = \left\{ M + \dfrac{1}{1 + \beta u_0}(b_0 - d u_0 - v_0) \right\} V_0 \geq 0 & \text{in } \Omega, \\ V_0 = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.61}$$

for sufficiently large $M > 0$, where (u_0, v_0) is defined by

$$U_0 = (1 + \alpha v_0) u_0 \quad \text{and} \quad V_0 = (1 + \beta u_0) v_0.$$

Recall that V_0 has an interior zero in Ω or $\partial V_0/\partial \nu$ vanishes at a point on $\partial\Omega$. Hence application of the strong maximum principle to (2.61) yields $V_0 \equiv 0$. This fact implies that U_0 satisfies

$$\Delta U_0 + U_0(a - U_0) = 0 \quad \text{in } \Omega \quad \text{and} \quad U_0 = 0 \quad \text{on } \partial\Omega;$$

so that $U_0 \equiv 0$ or $U_0 = \theta_a$.

If $U_0 \equiv 0$, then positive solutions bifurcate from $(0, 0, b)$ at $b = b_0$. From the linearization of (RSP-1) around $(0, 0)$ we get $(a, b_0) = (\lambda_1, \lambda_1)$. This is impossible. If $U_0 = \theta_a$, then $(\theta_a, 0, b_0)$ is a bifurcation point of positive solutions of (RSP-1). Then it follows from Proposition 2.1 that b_0 must be equal to b_*, which is a contradiction.

The above argument enables us to conclude $U_0 \in \partial P$. Since U_0 satisfies a similar problem to (2.61), the maximum principle implies $U_0 \equiv 0$. Therefore, V_0 satisfies

$$\Delta V_0 + V_0(b_0 - V_0) = 0 \quad \text{in } \Omega \quad \text{and} \quad V_0 = 0 \quad \text{on } \partial\Omega;$$

so that $V_0 \equiv 0$ or $V_0 = \theta_{b_0}$. Since $V_0 \equiv 0$ is excluded in the same way as above, V_0 must be identical with θ_{b_0}. In this case, $(0, \theta_{b_0}, b_0)$ is a bifurcation point of positive solutions of (RSP-1). Hence Proposition 2.2 gives $b_0 = b^*$, where b^* is defined by (2.36). Thus we see that $\mathcal{C} \cap (P \times P \times \boldsymbol{R})$ is a branch of positive solutions of (RSP-1) connecting $(\theta_a, 0, b_*)$ and $(0, \theta_{b^*}, b^*)$ when $\mathcal{C}^+$ is bounded in $P \times P \times \boldsymbol{R}$.

We next consider the case $\mathcal{C}^+ \subset P \times P \times \boldsymbol{R}$. In this case, $\mathcal{C}^+$ never meets trivial solutions of (RSP-1) and $\mathcal{C}^+$ is unbounded in $X \times \boldsymbol{R}$. Since (U, V) is a positive solution, Lemma 2.1 gives

$$0 \le u \le U \le M_1(a) \quad \text{and} \quad 0 \le v \le V \le M_2(b) \quad \text{in } \Omega.$$

We apply the regularity theory of elliptic equations to

$$-\Delta U = u(a - u - cv) \quad \text{and} \quad -\Delta V = v(b - du - v) \quad \text{in } \Omega.$$

Then we see that any $(U, V, b) \in \mathcal{C}^+$ satisfies

$$\|U\|_{W^{2,p}} \le M_1^*(b) \quad \text{and} \quad \|V\|_{W^{2,p}} \le M_2^*(b).$$

These estimates together with the unboundedness of $\mathcal{C}^+$ implies that $\{b > \lambda_1;\ (U, V, b) \in \mathcal{C}^+\}$ is unbounded. Thus we get the assertion (ii). □

3. Positive steady states of prey–predator model with cross-diffusion

In this section we will study the structure of positive steady states for the prey–predator model with cross-diffusion:

$$\text{(SP-2)} \qquad \begin{cases} \Delta[(1 + \alpha v)u] + u(a - u - cv) = 0 & \text{in } \Omega, \\ \Delta[(1 + \beta u)v] + v(b + du - v) = 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \\ u \ge 0, \quad v \ge 0 & \text{in } \Omega, \end{cases}$$

where α, β are nonnegative constants, a, c, d are positive constants and b is a real number.

3.1. *A priori estimates*

We use the following two unknown functions U, V as in Section 2:

$$U = (1 + \alpha v)u \quad \text{and} \quad V = (1 + \beta u)v,$$

which induces a one-to-one correspondence between (u, v) with $u \ge 0, v \ge 0$ and (U, V) with $U \ge 0, V \ge 0$ (see (2.1)). Then (SP-2) is written in the following equivalent system:

$$\text{(RSP-2)} \qquad \begin{cases} \Delta U + U\left(\dfrac{a - u - cv}{1 + \alpha v}\right) = 0 & \text{in } \Omega, \\ \Delta V + V\left(\dfrac{b + du - v}{1 + \beta u}\right) = 0 & \text{in } \Omega, \\ U = V = 0 & \text{on } \partial\Omega, \\ U \ge 0, \quad V \ge 0 & \text{in } \Omega, \end{cases}$$

where $u = u(U, V)$ and $v = v(U, V)$ are understood to be functions of (U, V) defined by (2.1). In what follows, we sometimes discuss (RSP-2) in place of (SP-2).

LEMMA 3.1. *Let* (U, V) *be a positive solution of* (RSP-2). *Then*

$$0 \le u(x) \le U(x) \le M_1(a) := \begin{cases} a & \text{if } a\alpha \le c, \\ \dfrac{(c+a\alpha)^2}{4c\alpha} & \text{if } a\alpha > c, \end{cases}$$
$$0 \le v(x) \le V(x) \le M_3(a,b) := (1+\beta M_1(a))(b+dM_1(a))$$

for all $x \in \Omega$.

PROOF. Estimate for U is the same as in Lemma 2.1. Assume $\|V\|_\infty = V(x_0) > 0$ for some $x_0 \in \Omega$. It follows from (RSP-2) that

$$0 \le -\Delta V(x_0) = v(x_0)(b + du(x_0) - v(x_0));$$

so that $b + du(x_0) - v(x_0) \ge 0$ because $v(x_0) > 0$. Therefore,

$$v(x_0) \le b + du(x_0);$$

so that we see

$$\|V\|_\infty = (1+\beta u(x_0))v(x_0) \le (1+\beta u(x_0))(b+du(x_0))$$
$$\le (1+\beta M_1(a))(b+dM_1(a)). \quad \square$$

LEMMA 3.2. *Let* (U, V) *be a positive solution of* (RSP-2). *If* $a\alpha \le c$*, then*

$$u \le U \le \theta_a \quad \text{in } \Omega. \tag{3.1}$$

If $b\beta \le d$*, then*

$$\theta_b \le V \quad \text{in } \Omega. \tag{3.2}$$

PROOF. The proof of (3.1) is the same as (2.2). To prove (3.2), we use the following inequality for $d \ge b\beta$

$$\Delta V + V(b - V) = uv\{b\beta - d - (2\beta + \beta^2 u)v\} \le 0,$$

which implies that V is a supersolution of (1.3) with a replaced by b. So (3.2) is shown by the comparison method. □

3.2. *Existence and uniqueness of positive steady state*

In this subsection we will construct positive steady state for (SP-2) (or equivalently (RSP-2)) by using the degree theory as in Section 2.2.

Choose a sufficiently large p such that

$$p + \frac{a - u - cv}{1 + \alpha v} \geq 0 \quad \text{and} \quad p + \frac{b + du - v}{1 + \beta u} \geq 0$$

for $0 \leq u \leq M_1(a) + 1$ and $0 \leq v \leq M_3(a, b) + 1$, where $M_1(a)$, $M_3(a, b)$ are positive constants in Lemma 3.1. Define a mapping B in $E = C_0(\overline{\Omega}) \times C_0(\overline{\Omega})$ by

$$\begin{aligned} B(U, V) &= (-\Delta + pI)^{-1} \left\{ \left(p + \frac{a - u - cv}{1 + \alpha v} \right) U, \left(p + \frac{b + du - v}{1 + \beta u} \right) V \right\} \\ &= ((-\Delta + pI)^{-1}(pU + F(u, v)), (-\Delta + pI)^{-1}(pV + \widetilde{G}(u, v))), \end{aligned} \tag{3.3}$$

where u, v are functions of U, V (see (2.1)) and

$$F(u, v) = u(a - u - cv), \qquad \widetilde{G}(u, v) = v(b + du - v).$$

As in Section 2, define $W = K \times K$ with $K = \{u \in C_0(\overline{\Omega}); u \geq 0 \text{ in } \Omega\}$. Clearly, (U, V) is a solution of (RSP-2) if and only if it is a fixed point of B in W. By setting

$$D := \{(U, V) \in W;\ U \leq M_1(a) + 1 \text{ and } V \leq M_3(a, b) + 1 \text{ in } \Omega\},$$

it can be seen from Lemma 3.1 that all nonnegative solutions of (RSP-2) lie in the interior of D ($=$ int D) with respect to W. Note that B has no fixed points on the boundary of D with respect to W and that $\theta_a \leq a \leq M_1(a)$ and $\theta_b \leq b \leq M_3(a, b)$. Moreover, one can show that B is completely continuous and maps $D \backslash \{(0, 0)\}$ into the demi-interior of W. Then it is possible to define $\deg_W(I - B, \text{int } D)$ for B with respect to W.

In order to study $\deg_W(I - B, \text{int } D)$, it is sufficient to calculate the index of each fixed point of B in W. The Fréchet derivative of B at any fixed point (U, V) of B is given by

$$\begin{aligned} B'(U, V) \begin{pmatrix} \hat{U} \\ \hat{V} \end{pmatrix} &= (-\Delta + pI)^{-1} \Bigg[pI + \frac{1}{1 + \alpha v + \beta u} \\ &\times \begin{pmatrix} a - 2u - cv & -cu \\ dv & b + du - 2v \end{pmatrix} \begin{pmatrix} 1 + \beta u & -\alpha u \\ -\beta v & 1 + \alpha v \end{pmatrix} \Bigg] \begin{pmatrix} \hat{U} \\ \hat{V} \end{pmatrix}. \end{aligned} \tag{3.4}$$

(see the derivation of (2.7)).

LEMMA 3.3. $\deg_W(I - B, \text{int } D) = 1$.

PROOF. The proof of this lemma is essentially the same as Lemma 2.3. So we omit the proof. □

We can also derive the fixed point index of each trivial and semi-trivial steady state of (RSP-2) in the same way as Lemma 2.4.

LEMMA 3.4. *Let* $a > \lambda_1$.

(i) $\text{index}_W(B, (0, 0)) = 0$.

(ii) *It holds that* $\begin{cases} \mathrm{index}_W(B,(\theta_a,0)) = 0 & \text{if } \lambda_1\left(\frac{-d\theta_a - b}{1+\beta\theta_a}\right) < 0, \\ \mathrm{index}_W(B,(\theta_a,0)) = 1 & \text{if } \lambda_1\left(\frac{-d\theta_a - b}{1+\beta\theta_a}\right) > 0. \end{cases}$

(iii) *For $b > \lambda_1$, it holds that* $\begin{cases} \mathrm{index}_W(B,(0,\theta_b)) = 0 & \text{if } \lambda_1\left(\frac{c\theta_b - a}{1+\alpha\theta_b}\right) < 0, \\ \mathrm{index}_W(B,(0,\theta_b)) = 1 & \text{if } \lambda_1\left(\frac{c\theta_b - a}{1+\alpha\theta_b}\right) > 0. \end{cases}$

The existence result of positive steady states for (SP-2) or (RSP-2) reads as follows.

THEOREM 3.1 (Existence of positive steady state). *Assume $a > \lambda_1$. Then* (SP-2) (*or equivalently,* (RSP-2)) *admits a positive steady state if one of the following conditions holds true:*

(i)
$$\lambda_1\left(\frac{-d\theta_a - b}{1+\beta\theta_a}\right) < 0 \quad \textit{and} \quad \lambda_1\left(\frac{c\theta_b - a}{1+\alpha\theta_b}\right) < 0,$$

(ii)
$$\lambda_1\left(\frac{-d\theta_a - b}{1+\beta\theta_a}\right) > 0 \quad \textit{and} \quad \lambda_1\left(\frac{c\theta_b - a}{1+\alpha\theta_b}\right) > 0,$$

where $\theta_b \equiv 0$ for $b \le \lambda_1$.

PROOF. The proof is accomplished in the same way as Theorem 2.1 with use of Lemmas 3.3 and 3.4. □

In Section 1.4 we have defined two curves S_2 by (1.9) and S_3 by (1.10) in ab-plane. By Lemma 1.6, S_3 is expressed as $b = \tilde{f}(a)$ for $a \ge \lambda_1$ (see Figure 1.3). Moreover, by Lemma 1.4, S_2 is expressed as $b = g(a)$ for $a \ge \lambda_1$ (see Figure 1.1). Define two sets $\widetilde{\Sigma}^+$ and $\widetilde{\Sigma}^-$ by

$$\begin{aligned} \widetilde{\Sigma}^+ &= \{(a,b) \in [\lambda_1,\infty) \times (-\infty,\infty);\ \tilde{f}(a) < b < g(a)\}, \\ \widetilde{\Sigma}^- &= \{(a,b) \in [\lambda_1,\infty) \times (-\infty,\infty);\ g(a) < b < \tilde{f}(a)\}. \end{aligned} \tag{3.5}$$

It follows from Theorem 1.2 that both $(\theta_a, 0)$ and $(0, \theta_b)$ are unstable if $(a,b) \in \widetilde{\Sigma}^+$, while both $(\theta_a, 0)$ and $(0, \theta_b)$ are asymptotically stable if $(a,b) \in \widetilde{\Sigma}^-$. Therefore, Theorem 3.1 implies that (SP-2) (or (RSP-2)) possesses at least one positive steady state whenever (a,b) lies in a region where $(\theta_a, 0)$ and $(0, \theta_b)$ are asymptotically stable or unstable at the same time.

We can also see from Lemmas 1.3 and 1.4 that $\widetilde{\Sigma}^+$ is nonempty if $(c+\alpha\lambda_1)(\beta\lambda_1 - d) < 1$, and that $\widetilde{\Sigma}^-$ is nonempty if $(c+\alpha\lambda_1)(\beta\lambda_1 - d) > 1$. Moreover, it can be seen that $\widetilde{\Sigma}^+$ shrinks to an empty set near $(a,b) = (\lambda_1, \lambda_1)$ as β becomes large, while $\widetilde{\Sigma}^-$ expands near $(a,b) = (\lambda_1, \lambda_1)$ as β becomes large.

Furthermore, the following corollary comes from Theorem 3.1.

COROLLARY 3.1. (i) *Let $\beta\lambda_1 < d$. If $(a,b) \in (\lambda_1,\infty) \times (-\infty,\infty)$ satisfies $(a,b) \in \widetilde{\Sigma}^+$, then there exists at least one positive steady state of* (SP-2) (*or equivalently* (RSP-2)).

(ii) *Let* $\beta\lambda_1 \geq d$. *If* $(a, b) \in (\lambda_1, \infty) \times [\lambda_1, \infty)$ *satisfies* $(a, b) \in \widetilde{\Sigma}^+ \cup \widetilde{\Sigma}^-$, *then there exists at least one positive steady state of* (SP-2) (*or equivalently* (RSP-2)).

Finally we will discuss the uniqueness of positive steady states for (SP-2) or (RSP-2). For this purpose, the idea of López-Gómez and Pardo[39] is very useful. They have shown the nondegeneracy of a positive solution for prey–predator model with linear diffusion in the special case when the spatial dimension is one. See also the work of Nakashima and the author [41].

THEOREM 3.2. *Let* $N = 1$. *If* $c \geq a\alpha, d \geq b\beta$ *and* $\alpha - a\alpha\beta - c\beta \geq 0$, *then a positive steady state for* (RSP-2) *(or* (SP-2)*) is uniquely determined.*

PROOF. Let $\Omega = (0, 1)$ and let $(U_i, V_i), i = 1, 2$, be two positive steady states for (RSP-2). Set $\hat{U} = U_1 - U_2$ and $\hat{V} = V_1 - V_2$. Since $U_i = (1 + \alpha v_i)u_i$ and $V_i = (1 + \beta u_i)v_i$ $(i = 1, 2)$,

$$\begin{pmatrix} \hat{U} \\ \hat{V} \end{pmatrix} = \begin{pmatrix} 1 + \alpha v_2 & \alpha u_1 \\ \beta v_2 & 1 + \beta u_1 \end{pmatrix} \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix};$$

so that

$$\begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} = \frac{1}{1 + \alpha v_2 + \beta u_1} \begin{pmatrix} 1 + \beta u_1 & -\alpha u_1 \\ -\beta v_2 & 1 + \alpha v_2 \end{pmatrix} \begin{pmatrix} \hat{U} \\ \hat{V} \end{pmatrix}. \tag{3.6}$$

Since (U_1, V_1) and (U_2, V_2) are solutions for (RSP-2), we can get

$$\frac{d^2}{dx^2} \begin{pmatrix} \hat{U} \\ \hat{V} \end{pmatrix} + \begin{pmatrix} a - u_1 - u_2 - cv_2 & -cu_1 \\ dv_2 & b - v_1 - v_2 + du_1 \end{pmatrix} \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} = 0.$$

By (3.6) the above system can be rewritten as

$$\begin{cases} -\hat{U}'' + p_1\hat{U} - q_1\hat{V} = 0, \\ -\hat{V}'' + p_2\hat{V} - q_2\hat{U} = 0, \end{cases} \tag{3.7}$$

where

$$p_1 = \frac{1}{1 + \alpha v_2 + \beta u_1} \left\{ -a + u_1 + u_2 + cv_2 - a\beta u_1 + \beta u_1(u_1 + u_2) \right\},$$

$$p_2 = \frac{1}{1 + \alpha v_2 + \beta u_1} \left\{ -b + v_1 + v_2 - du_1 - b\alpha v_2 + \alpha v_2(v_1 + v_2) \right\},$$

$$q_1 = \frac{u_1}{1 + \alpha v_2 + \beta u_1} \{(u_1 + u_2)\alpha - a\alpha - c\},$$

$$q_2 = \frac{v_2}{1 + \alpha v_2 + \beta u_1} \{(v_1 + v_2)\beta + d - b\beta\}.$$

In view of (3.7) we define the following operators;

$$\begin{cases} L_1 U := -U'' + p_1 U \\ L_2 V := -V'' + p_2 V \end{cases} \tag{3.8}$$

with zero Dirichlet boundary conditions at $x = 0, 1$. Since U_2 and V_1 are positive solutions for (RSP-2), the Krein–Rutman theorem implies that the principal eigenvalues of both operators

$$-\frac{d^2}{dx^2} + \frac{u_2 + cv_2 - a}{1 + \alpha v_2} \quad \text{and} \quad -\frac{d^2}{dx^2} + \frac{v_1 - du_1 - b}{1 + \beta u_1}$$

with zero Dirichlet boundary conditions are zero; that is,

$$\lambda_1\left(\frac{u_2 + cv_2 - a}{1 + \alpha v_2}\right) = \lambda_1\left(\frac{v_1 - du_1 - b}{1 + \beta u_1}\right) = 0.$$

After some calculations it is possible to show

$$p_1 > \frac{u_2 + cv_2 - a}{1 + \alpha v_2} \quad \text{and} \quad p_2 > \frac{v_1 - du_1 - b}{1 + \beta u_1},$$

where we have used $\alpha - a\alpha\beta - \beta c \geq 0$ and $d \geq b\beta$, respectively. So it follows from Proposition 1.1 that

$$\lambda_1(p_1) > \lambda_1\left(\frac{u_2 + cv_2 - a}{1 + \alpha v_2}\right) = 0 \tag{3.9}$$

and

$$\lambda_1(p_2) > \lambda_1\left(\frac{v_1 - dv_1 - b}{1 + \beta u_1}\right) = 0. \tag{3.10}$$

Note that $q_1 < 0$ and $q_2 > 0$ if $a\alpha \leq c$ and $b\beta \leq d$ (use $u_i < a$ for $i = 1, 2$). Using L_1 and L_2 defined by (3.8), we rewrite (3.7) in the following form:

$$L_1\hat{U} = q_1\hat{V} \quad \text{with } q_1 < 0, \tag{3.11}$$

$$L_2\hat{V} = q_2\hat{U} \quad \text{with } q_2 > 0. \tag{3.12}$$

To complete the proof, we need the following lemma which is, in a sense, the generalization of the maximum principle (see Protter and Weinberger [49]).

LEMMA 3.5 (cf. [39]). *Let $q \in C[\gamma, \delta]$ and define L by*

$$Lw = -w'' + q(x)w, \qquad \gamma < x < \delta$$

with zero Dirichlet boundary conditions at $x = \gamma, \delta$. Assume that the principal eigenvalue of L is strictly positive. If $w \in C[\gamma, \delta] \cap C^2(\gamma, \delta)$ satisfies $Lw > 0$ in (γ, δ), $w(\gamma) \geq 0$ and $w(\delta) \geq 0$, then $w > 0$ in (γ, δ).

We will continue the proof of Theorem 3.2. By (3.9) the principal eigenvalue of L_1 in (0, 1) is positive. Then the variational characterization implies that the principal eigenvalue of L_1 in any subinterval of (0, 1) is also positive. It follows from (3.10) that L_2 has the same property as L_1. These results imply that Lemma 3.5 is applicable to the restrictions of L_1 and L_2 on any subinterval of (0, 1).

In the rest of the proof we follow the argument used by López-Gómez and Pardo [39]. Assume that $\hat{U} \neq 0$. By the uniqueness of solutions for the Cauchy problem to second-order differential equations, $\hat{U}$ has at most a finite number of zeros. If $\hat{U} > 0$ in $(0, 1)$, application of Lemma 3.5 to (3.12) implies $\hat{V} > 0$ in $(0, 1)$. Applying Lemma 3.5 again to (3.11), we see $\hat{U} < 0$ in $(0, 1)$. This is a contradiction. Hence, $\hat{U}$ and $\hat{V}$ must change their signs in $(0, 1)$.

Assume that $\hat{U}$ vanishes at $x = x_0, x_1, x_2, \ldots, x_n, x_{n+1}$ with $x_0 = 0, x_{n+1} = 1$:

$$\hat{U}(x) > 0 \quad x \in (x_{2j}, x_{2j+1}), \quad j \geq 0, 2j+1 \leq n,$$
$$\hat{U}(x) < 0 \quad x \in (x_{2j-1}, x_{2j}), \quad j \geq 1, 2j \leq n.$$

By hypothesis $\hat{U}(x) > 0$ for $x \in (x_0, x_1)$ and $\hat{U}(x_0) = \hat{U}(x_1) = 0$. We will show $\hat{V}(x_1) < 0$ by contradiction. Suppose $\hat{V}(x_1) \geq 0$. Since $\hat{V}(x_0) = 0$, Lemma 3.5 with $L = L_2$ assures $\hat{V} > 0$ in (x_0, x_1) from (3.12). Applying Lemma 3.5 again with $L = L_1$ to (3.11) we are led to $\hat{U} < 0$ in (x_0, x_1). Since this is a contradiction, we see $\hat{V}(x_1) < 0$. Again using the assumption that $\hat{U}(x) < 0$ for $x \in (x_1, x_2)$ and $\hat{U}(x_1) = \hat{U}(x_2) = 0$, we can apply Lemma 3.5 to get $\hat{V}(x_2) > 0$. A recursive argument yields

$$\hat{V}(x_{2j}) > 0 \quad \text{and} \quad \hat{V}(x_{2j+1}) < 0 \quad \text{for } x_{2j}, x_{2j+1} \in \{x_1, x_2, \ldots, x_n, x_{n+1}\}.$$

This contradicts the fact that $\hat{V}(x_{n+1}) = 0$. Thus $\hat{U}$ (and, therefore, $\hat{V}$) must be identically zero. These results assure the uniqueness of positive solutions of (RSP-2). □

3.3. *Nonexistence of positive steady state*

We will study nonexistence of positive steady states for (SP-2) or (RSP-2) in order to derive an optimal coexistence region in *ab*-plain. Generally this problem is difficult because we cannot get suitable a priori estimates of U, V or u, v. Our nonexistence results are restricted to the case when cross-diffusion coefficients are small in the following sense:

$$a\alpha \leq c \quad \text{and} \quad b\beta \leq d. \tag{3.13}$$

When (3.13) is satisfied, Lemma 3.2 gives

$$\theta_a \geq U \geq u \quad \text{and} \quad V \geq \theta_b \quad \text{in } \Omega.$$

These a priori estimates lead us to the following result.

THEOREM 3.3. *Let α and β satisfy* (3.13). *Suppose that $1 + \{\alpha - \beta(c + a\alpha)\}\theta_b \geq 0$ in Ω. If*

$$\lambda_1\left(\frac{c\theta_b - a}{1 + \alpha\theta_b}\right) \geq 0 \quad \text{for } b \geq \lambda_1, \tag{3.14}$$

or

$$\lambda_1\left(\frac{-d\theta_a - b}{1 + \beta\theta_a}\right) \geq 0 \quad \text{for } b < \lambda_1, \tag{3.15}$$

then (RSP-2) *or equivalently* (SP-2) *does not admit a positive steady state.*

PROOF. Assume that (RSP-2) has a positive steady-state solution (U, V) and that (3.14) is satisfied in case $b \geq \lambda_1$. Since $1 + \{\alpha - \beta(c + a\alpha)\}\theta_b \geq 0$ in Ω, we can show that

$$\frac{a - c\theta_b}{1 + \alpha\theta_b} > \frac{a - u - cv}{1 + \alpha v} \quad \text{in } \Omega$$

(see also [43, Corollary 1]). Hence

$$\Delta U + \frac{a - c\theta_b}{1 + \alpha\theta_b} U > \Delta U + \frac{a - u - cv}{1 + \alpha v} U = 0 \quad \text{in } \Omega.$$

Therefore, for every $p > 0$

$$\left\{ p + \frac{a - c\theta_b}{1 + \alpha\theta_b} \right\} U > (-\Delta + pI)U.$$

Let p be a sufficiently large number and denote by $(-\Delta + pI)^{-1}$ the inverse operator in $C_0(\overline{\Omega})$ of $-\Delta + pI$ with zero Dirichlet boundary condition. By the monotonicity of $(-\Delta + pI)^{-1}$ and the strong maximum principle (see [49]) we have

$$(-\Delta + pI)^{-1} \left\{ p + \frac{a(1 - c\theta_b)}{1 + \alpha\theta_b} \right\} U > U. \tag{3.16}$$

Define

$$T_3 w := (-\Delta + pI)^{-1} \left\{ p + \frac{a - c\theta_b}{1 + \alpha\theta_b} \right\} w.$$

As is discussed by Li [28, Lemma 2.3], it follows from (3.16) that $r(T_3) > 1$. This fact, together with Proposition 1.4, implies $\lambda_1(c\theta_b - a/(1 + \alpha\theta_b)) < 0$, which is a contradiction to (3.14). Therefore, (RSP-2) has no positive solutions when (3.14) is satisfied.

Next we consider the case $b < \lambda_1$. Let (U, V) be a positive solution for (RSP-2). From the second equation it follows that

$$\Delta V + \frac{b + du}{1 + \beta u} V > 0 \quad \text{in } \Omega. \tag{3.17}$$

For $b\beta \leq d$, $z \mapsto (1 + dz)/(1 + \beta z)$ is strictly increasing. Since $u \leq \theta_a$, (3.17) gives

$$\Delta V + \frac{b + d\theta_a}{1 + \beta\theta_a} V > 0 \quad \text{in } \Omega.$$

Therefore, making use of the same arguments as in the case $b \geq \lambda_1$ we can conclude

$$r\left((-\Delta + pI)^{-1} \left\{ p + \frac{b + d\theta_a}{1 + \beta\theta_a} \right\} \right) > 1.$$

This fact, together with Proposition 1.4, yields $\lambda_1(-(b + d\theta_a)/(1 + \beta\theta_a)) < 0$. Therefore, it is possible to prove that (3.15) is also a sufficient condition for the nonexistence. □

3.4. *Bifurcation theory for prey–predator model*

We will reconsider (SP-2) or (RSP-2) from the viewpoint of bifurcation theory as in Section 2.3. Let a be a bifurcation parameter and fix b in this subsection. We discuss bifurcations of positive steady states from semi-trivial states $(\theta_a, 0)$ or $(0, \theta_b)$.

Assume that b satisfies $b\beta > \beta\lambda_1 > d$ or $d > \beta\lambda_1 > b$. Define a_* by

$$b = \tilde{f}(a_*), \tag{3.18}$$

where $\tilde{f}$ is a function appearing in Lemma 1.6. Recall that $\tilde{f}(a)$ is strictly increasing (resp. decreasing) for $a \geq \lambda_1$ if $\beta\lambda_1 > d$ (resp. $\beta\lambda_1 < d$). From (1.10)

$$\lambda_1\left(\frac{-d\theta_{a_*} - b}{1 + \beta\theta_{a_*}}\right) = 0. \tag{3.19}$$

Let Ψ_* be a unique positive solution of

$$\begin{cases} -\Delta\Psi_* - \dfrac{d\theta_{a_*} + b}{1 + \beta\theta_{a_*}}\Psi_* = 0 & \text{in } \Omega, \\ \Psi_* = 0 & \text{on } \partial\Omega, \\ \Psi_* > 0 & \text{in } \Omega, \end{cases} \tag{3.20}$$

satisfying $\|\Psi_*\| = 1$. In view of Lemma 1.1, define Φ_* as a unique solution of

$$\begin{cases} -\Delta\Phi_* + (2\theta_{a_*} - a_*)\Phi_* = -\dfrac{(a_*\alpha + c - 2\alpha\theta_{a_*})\theta_{a_*}}{1 + \beta\theta_{a_*}}\Psi_* & \text{in } \Omega, \\ \Phi_* = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.21}$$

Furthermore, for $p > N$, define X and Y by (2.16) as in Section 2.3.

The local bifurcation theorem for (RSP-2) from $(\theta_a, 0, a)$ reads as follows.

PROPOSITION 3.1. *Let $b\beta > \beta\lambda_1 > d$ or $d > \beta\lambda_1 > b\beta$. Define a_* by* (3.18). *Then positive solutions of* (RSP-2) *bifurcate from a semi-trivial solution curve $\{(\theta_a, 0, a);\ a > \lambda_1\}$ if and only if $a = a_*$. More precisely, there exists a positive number δ such that all positive solutions of* (RSP-2) *near $(\theta_{a_*}, 0, a_*) \in X \times \boldsymbol{R}$ can be expressed as*

$$(U, V, b) = (U(\varepsilon), V(\varepsilon), a(\varepsilon)) \quad \textit{for } 0 \leq \varepsilon \leq \delta$$

with

$$\begin{aligned} U(\varepsilon) &= \theta_{a(\varepsilon)} + \varepsilon\Phi_* + \varepsilon\hat{U}(\varepsilon) = \theta_{a_*} + \varepsilon\Phi_* + \varepsilon a'(0)\frac{\partial\theta_a}{\partial a}\bigg|_{a=a_*} + O(\varepsilon^2), \\ V(\varepsilon) &= \varepsilon\Psi_* + \varepsilon\hat{V}(\varepsilon) = \varepsilon\Psi_* + O(\varepsilon^2), \\ a(\varepsilon) &= a_* + a'(0)\varepsilon + O(\varepsilon^2), \end{aligned} \tag{3.22}$$

where Φ_*, Ψ_* are defined by (3.21), (3.20) *and $\{(\hat{U}(\varepsilon), \hat{V}(\varepsilon), a(\varepsilon))\}$ for $0 \leq \varepsilon \leq \delta$ is a family of smooth functions with respect to ε satisfying $(\hat{U}(0), \hat{V}(0), a(0)) = (0, 0, a_*)$ and $\int_\Omega \hat{V}(s)\Psi_*\,dx = 0$.*

PROOF. The method of the proof is essentially the same as those of Propositions 2.1 and 2.2. So we omit it. □

REMARK 3.1. Set $\overline{U} = U - \theta_a$. Around $(U, V) = (\theta_a, 0)$, we see from Remark 2.3 that

$$u = \theta_a + \overline{U} - \frac{\alpha\theta_a}{1+\beta\theta_a} V + \eta_1(\overline{U}, V) \quad \text{and} \quad v = \frac{1}{1+\beta\theta_a} V + \eta_2(\overline{U}, V),$$

where $\eta_1 = O(\overline{U}^2 + V^2)$ and $\eta_2 = O(\overline{U}^2 + V^2)$. Hence Proposition 3.1 implies that bifurcating positive solutions of (SP-2) for $a = a(\varepsilon) = a_* + a'(0)\varepsilon + O(\varepsilon^2)$ can be expressed as

$$\begin{aligned} u(\varepsilon) &= \theta_{a(\varepsilon)} + \varepsilon\left(\Phi_* - \frac{\alpha\theta_{a_*}}{1+\beta\theta_{a_*}}\Psi_*\right) + O(\varepsilon^2) \\ &= \theta_{a_*} + \varepsilon\left(\Phi_* - \frac{\alpha\theta_{a_*}}{1+\beta\theta_{a_*}}\Psi_*\right) + \varepsilon a'(0)\frac{\partial\theta_a}{\partial a}\bigg|_{a=a_*} + O(\varepsilon^2), \\ v(\varepsilon) &= \frac{\varepsilon}{1+\beta\theta_{a_*}}\Psi_* + O(\varepsilon^2). \end{aligned} \tag{3.23}$$

Similarly to Proposition 3.1, one can also study the bifurcation from $(0, \theta_b)$. Define a^* by

$$b = g(a^*), \tag{3.24}$$

where g is a strictly increasing function defined by (1.38). From (1.9)

$$\lambda_1\left(\frac{c\theta_b - a^*}{1+\alpha\theta_b}\right) = 0.$$

Instead of (3.20) and (3.21), define Φ^* by a unique positive solution of

$$\begin{cases} -\Delta\Phi^* + \dfrac{c\theta_b - a^*}{1+\alpha\theta_b}\Phi^* = 0 & \text{in } \Omega, \\ \Phi^* = 0 & \text{on } \partial\Omega, \\ \Phi^* > 0 & \text{in } \Omega, \end{cases} \tag{3.25}$$

satisfying $\|\Phi^*\| = 1$ and define Ψ^* as a solution of

$$\begin{cases} -\Delta\Psi^* + (2\theta_b - b)\Psi^* = -\dfrac{(b\beta - d - 2\beta\theta_b)\theta_b}{1+\alpha\theta_b}\Phi^* & \text{in } \Omega, \\ \Psi^* = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.26}$$

PROPOSITION 3.2. *For $b > \lambda_1$, define a^* by (3.24). Then positive solutions of* (RSP-2) *bifurcate from a semi-trivial solution curve $\{(0, \theta_b, a);\ a > \lambda_1\}$ if and only if $a = a^*$. More precisely, there exists a positive number δ such that all positive solutions of* (RSP-2) *near $(0, \theta_b, a^*) \in X \times \boldsymbol{R}$ can be expressed as*

$$(U, V, a) = (U(\varepsilon), V(\varepsilon), a(\varepsilon)) \quad \text{for } 0 \le \varepsilon \le \delta$$

with

$$\begin{aligned} U(\varepsilon) &= \varepsilon\Phi^* + \varepsilon\hat{U}(\varepsilon) = \varepsilon\Phi^* + O(\varepsilon^2), \\ V(\varepsilon) &= \theta_b + \varepsilon\Psi^* + \varepsilon\hat{V}(\varepsilon) = \theta_b + \varepsilon\Psi^* + O(\varepsilon^2), \\ a(\varepsilon) &= a^* + a'(0)\varepsilon + O(\varepsilon^2), \end{aligned} \tag{3.27}$$

where Φ^*, Ψ^* are defined by (3.25), (3.26), respectively, and $\{(\hat{U}(\varepsilon), \hat{V}(\varepsilon), a(\varepsilon))\}$ for $0 \le \varepsilon \le \delta$ is a family of smooth functions with respect to ε satisfying $(\hat{U}(0), \hat{V}(0), a(0)) = (0, 0, a^*)$ and $\int_\Omega \hat{U}(s)\Phi^*\,dx = 0$.

PROOF. The proof is almost the same as that of Proposition 3.1. So we omit it. □

We will investigate the direction of bifurcating solutions given in Proposition 3.1.

LEMMA 3.6. *Let* $\{(U(\varepsilon), V(\varepsilon), a(\varepsilon))\}$ *be a family of positive solutions of* (RSP-2) *as in Proposition* 3.1. *Then it holds that*

$$a'(0)(b\beta - d)\int_\Omega \frac{\Psi_*^2}{(1+\beta\theta_{a_*})^2}\frac{\partial\theta_a}{\partial a}\bigg|_{a=a_*} dx = -\int_\Omega \frac{\Psi_*^2}{(1+\beta\theta_{a_*})^2}\left\{\Psi_* - (b\beta - d)\left(\frac{\alpha\theta_{a_*}}{1+\beta\theta_{a_*}}\Psi_* - \Phi_*\right)\right\}dx,$$

where Φ_* *and* Ψ_* *are defined by* (3.21) *and* (3.20).

PROOF. We will use (3.22), (3.23) and substitute these expressions into

$$\Delta V + v(b + du - v) = 0 \quad \text{in } \Omega. \tag{3.28}$$

By (3.22)

$$\Delta V(\varepsilon) = \varepsilon(\Delta\Psi_* + \Delta\hat{V}(\varepsilon)) \tag{3.29}$$

and by (3.23)

$$b + du(\varepsilon) - v(\varepsilon) = b + d\theta_{a(\varepsilon)} + \varepsilon\left(d\Phi_* - \frac{1+d\alpha\theta_{a_*}}{1+\beta\theta_{a_*}}\Psi_*\right) + O(\varepsilon^2). \tag{3.30}$$

Repeating the arguments used in the derivation of (2.28) one can get

$$v(\varepsilon) = \frac{\varepsilon}{1+\beta\theta_{a(\varepsilon)}}(\Psi_* + \hat{V}(\varepsilon)) + \frac{\beta\varepsilon^2}{(1+\beta\theta_{a_*})^2}\left(\frac{\alpha\theta_{a_*}}{1+\beta\theta_{a_*}}\Psi_* - \Phi_*\right)\Psi_* + O(\varepsilon^3). \tag{3.31}$$

Substitution of (3.29), (3.30) and (3.31) into (3.28) leads to

$$\begin{aligned} 0 &= \left(\Delta + \frac{b + d\theta_{a(\varepsilon)}}{1+\beta\theta_{a(\varepsilon)}}\right)(\Psi_* + \hat{V}(\varepsilon)) + \varepsilon\left\{\frac{\beta(b+d\theta_{a_*})}{(1+\beta\theta_{a_*})^2}\left(\frac{\alpha\theta_{a_*}}{1+\beta\theta_{a_*}}\Psi_* - \Phi_*\right)\Psi_*\right. \\ &\quad \left. + \frac{1}{1+\beta\theta_{a_*}}\left(d\Phi_* - \frac{1+d\alpha\theta_{a_*}}{1+\beta\theta_{a_*}}\Psi_*\right)\Psi_*\right\} + O(\varepsilon^2) \\ &= \left(\Delta + \frac{b+d\theta_{a_*}}{1+\beta\theta_{a_*}}\right)\hat{V}(\varepsilon) + \varepsilon a'(0)\frac{d - b\beta}{(1+\beta\theta_{a_*})^2}\frac{\partial\theta_a}{\partial a}\bigg|_{a=a_*}\Psi_* \\ &\quad + \frac{\varepsilon}{(1+\beta\theta_{a_*})^2}\left\{-\Psi_* + (b\beta - d)\left(\frac{\alpha\theta_{a_*}}{1+\beta\theta_{a_*}}\Psi_* - \Phi_*\right)\right\}\Psi_* + O(\varepsilon^2). \end{aligned}$$

Taking $L^2(\Omega)$-inner product of the right-hand side of the above identity with Ψ_* we have

$$
\begin{aligned}
a'(0)(d-b\beta)&\int_\Omega \frac{\Psi_*^2}{(1+\beta\theta_{a_*})^2}\left.\frac{\partial\theta_a}{\partial a}\right|_{a=a_*} dx \\
&= \int_\Omega \frac{\Psi_*^2}{(1+\beta\theta_{a_*})^2}\left\{\Psi_* - (b\beta-d)\left(\frac{\alpha\theta_{a_*}}{1+\beta\theta_{a_*}}\Psi_* - \Phi_*\right)\right\} dx + O(\varepsilon).
\end{aligned}
$$

Hence the assertion comes from the above identity by letting $\varepsilon \to 0$. □

REMARK 3.2. It follows from Lemma 3.6 that the direction of bifurcation becomes opposite according as the sign of $b\beta - d$ changes.

We can also study stability of bifurcating positive solutions in Proposition 3.1.

LEMMA 3.7. *Let* $\{(u(\varepsilon), v(\varepsilon), a(\varepsilon))\}$ *be a family of positive solutions of* (SP-2) *corresponding to* $\{(U(\varepsilon), V(\varepsilon), a(\varepsilon))\}$ *in Proposition* 3.1.

(i) *If* $\beta\lambda_1 < d$, *then* $(u(\varepsilon), v(\varepsilon))$ *for sufficiently small* $\varepsilon > 0$ *is asymptotically stable if the bifurcation is supercritical* (*i.e.*, $a'(0) > 0$), *while it is unstable if the bifurcation is subcritical* (*i.e.*, $a'(0) < 0$).

(ii) *If* $\beta\lambda_1 > d$, *then* $(u(\varepsilon), v(\varepsilon))$ *for sufficiently small* $\varepsilon > 0$ *is asymptotically stable if the bifurcation is subcritical, while it is unstable if the bifurcation is supercritical.*

PROOF. The eigenvalue problem for the linearization associated with $(u(\varepsilon), v(\varepsilon))$ for $a = a(\varepsilon)$ is given by

$$
\begin{cases}
\Delta[(1+\alpha v(\varepsilon))\varphi + \alpha u(\varepsilon)\psi] + (a(\varepsilon) - 2u(\varepsilon) - cv(\varepsilon))\varphi & \\
\qquad - cu(\varepsilon)\psi = -\sigma\varphi & \text{in } \Omega, \\
\Delta[(1+\beta u(\varepsilon))\psi + \beta v(\varepsilon)\varphi] + (b + du(\varepsilon) - 2v(\varepsilon))\psi & \\
\qquad + dv(\varepsilon)\varphi = -\rho\sigma\psi & \text{in } \Omega. \\
\varphi = \psi = 0 & \text{on } \partial\Omega.
\end{cases}
\tag{3.32}
$$

If $\varepsilon = 0$ in (3.32), then

$$
\begin{cases}
\Delta[\varphi + \alpha\theta_{a_*}\psi] + (a_* - 2\theta_{a_*})\varphi - c\theta_{a_*}\psi = -\sigma\varphi & \text{in } \Omega, \\
\Delta[(1+\beta\theta_{a_*})\psi] + (b + d\theta_{a_*})\psi = -\rho\sigma\psi & \text{in } \Omega, \\
\varphi = \psi = 0 & \text{on } \partial\Omega.
\end{cases}
\tag{3.33}
$$

Recall that zero is a simple eigenvalue of (3.33) and that all other eigenvalues are positive. If we set

$$
\Phi = \varphi + \alpha\theta_{a_*}\psi \quad \text{and} \quad \Psi = (1+\beta\theta_{a_*})\psi,
$$

then (3.33) is equivalent to

$$\begin{cases} -\Delta\Phi + (2\theta_{a_*} - a_*)\Phi + \dfrac{(a\alpha + c - 2\alpha\theta_{a_*})\theta_{a_*}}{1+\beta\theta_{a_*}}\Psi \\ \qquad = \sigma\left(\Phi - \dfrac{\alpha\theta_{a_*}}{1+\beta\theta_{a_*}}\Psi\right) & \text{in } \Omega, \\ -\Delta\Psi - \dfrac{d\theta_{a_*}+b}{1+\beta\theta_a}\Psi = \dfrac{\rho\sigma}{1+\beta\theta_a}\Psi & \text{in } \Omega, \\ \Phi = \Psi = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.34}$$

(see (2.31)). An eigenfunction corresponding to zero eigenvalue of (3.34) can be given by (Φ_*, Ψ_*) defined in (3.21) and (3.20). Setting

$$\varphi_* = \Phi_* - \frac{\alpha\theta_{a_*}}{1+\beta\theta_{a_*}}\Psi_* \quad \text{and} \quad \psi_* = \frac{1}{1+\beta\theta_{a_*}}\Psi_*, \tag{3.35}$$

we see that (φ_*, ψ_*) becomes an eigenfunction corresponding to zero eigenvalue of (3.33).

Let $\sigma(\varepsilon)$ be an eigenvalue of (3.32) satisfying $\sigma(0) = 0$. Using the implicit function theorem as in the proof of Lemma 2.8 one can express an eigenfunction corresponding to $\sigma(\varepsilon)$ in the following form with use of (φ_*, ψ_*) defined by (3.35):

$$\begin{cases} \varphi = \varphi_* + \hat{\varphi}(\varepsilon) = \varphi_* + O(\varepsilon), \\ \psi = \psi_* + \hat{\psi}(\varepsilon) = \psi_* + O(\varepsilon). \end{cases} \tag{3.36}$$

Recall

$$u(\varepsilon) = \theta_{a_*} + \varepsilon\varphi_* + \varepsilon a'(0)\left.\frac{\partial\theta_a}{\partial a}\right|_{a=a_*} + O(\varepsilon^2) \quad \text{and} \quad v(\varepsilon) = \varepsilon\psi_* + O(\varepsilon^2) \tag{3.37}$$

by (3.23) and (3.35). Substituting (3.37) into the second equation of (3.32) and arranging the resulting expressions we have

$$\begin{aligned} &\Delta[(1+\beta\theta_{a_*})\hat{\psi}(\varepsilon)] + (b + d\theta_{a_*})\hat{\psi}(\varepsilon) + 2\varepsilon\beta\Delta[\varphi_*\psi_*] + 2\varepsilon\psi_*(d\varphi_* - \psi_*) \\ &\quad + \varepsilon\beta a'(0)\Delta\left[\left.\frac{\partial\theta_a}{\partial a}\right|_{a=a_*}\psi_*\right] + \varepsilon d a'(0)\left.\frac{\partial\theta_a}{\partial a}\right|_{a=a_*}\psi_* = -\rho\sigma'(0)\varepsilon\psi_* + O(\varepsilon^2) \end{aligned} \tag{3.38}$$

(cf. (2.57)).

Taking L^2-inner product of (3.38) with Ψ_* one can derive

$$\begin{aligned} &2\beta(\Delta[\varphi_*\psi_*], \Psi_*)_{L^2} + 2(\psi_*(d\varphi_* - \psi_*), \Psi_*)_{L^2} \\ &\quad + \beta a'(0)\left(\Delta\left[\left.\frac{\partial\theta_a}{\partial a}\right|_{a=a_*}\psi_*\right], \Psi_*\right)_{L^2} + d a'(0)\left(\left.\frac{\partial\theta_a}{\partial a}\right|_{a=a_*}\psi_*, \Psi_*\right)_{L^2} \\ &\qquad = -\rho\sigma'(0)(\psi_*, \Psi_*)_{L^2}. \end{aligned}$$

Then the rest of the proof is the same as Lemma 2.8. □

As to the bifurcating solutions constructed in Proposition 3.2 we can obtain the following results. Their proofs are almost the same as those of Lemmas 2.5 and 2.6.

LEMMA 3.8. (i) *Let* $\{(U(\varepsilon), V(\varepsilon), a(\varepsilon))\}$ *be a family of positive solutions of* (RSP-2) *constructed in Proposition* 3.2. *Then it holds that*

$$a'(0)\int_\Omega \frac{(\Phi^*)^2}{1+\alpha\theta_b}dx = \int_\Omega \frac{(\phi^*)^2}{(1+\alpha\theta_b)}\left\{\Phi^* - (c+a^*\alpha)\left(\frac{\beta\theta_b}{1+\alpha\theta_b}\Phi^* - \Psi^*\right)\right\}dx.$$

(ii) *Let* $\{(u(\varepsilon), v(\varepsilon), a(\varepsilon))\}$ *be a family of positive solutions of* (SP-2) *corresponding to* $\{(U(\varepsilon), V(\varepsilon), a(\varepsilon))\}$ *in* (i). *Then* $(u(\varepsilon), v(\varepsilon)$ *for sufficiently small* $\varepsilon > 0$ *is asymptotically stable if the bifurcation is supercritical, while it is unstable if the bifurcation is subcritical.*

We will apply the global bifurcation theory to (RSP-2) as in Section 2.3. Define positive cone P by (2.60). Let $\mathcal{C}_1 \subset X \times \boldsymbol{R}$ denote a connected subset of nontrivial solutions of (RSP-2) such that $\mathcal{C}_1 \ni (\theta_{a_*}, 0, a_*)$, where a_* is a positive number defined by (3.18) and satisfies $(a_*, b) \in S_3$. We denote by $\mathcal{C}_1^+$ a maximal component of $\mathcal{C}_1 \setminus \{(\theta_{a_*}, 0, a_*)\}$ such that $\mathcal{C}_1^+ \subset P \times P \times \boldsymbol{R}$. By Proposition 3.1, $(U(\varepsilon), V(\varepsilon), a(\varepsilon)) \in \mathcal{C}_1^+$ for sufficiently small $\varepsilon > 0$.

Similarly, we denote by $\mathcal{C}_2^+$ a maximal component of positive solutions of (RSP-2) which contains bifurcating positive solutions from $(0, \theta_b, a)$ at $a = a^*$. Here a^* is a positive number defined by (3.24) and satisfies $(a^*, b) \in S_2$.

We can show the following result by using the global bifurcation theory of Rabinowitz [50] in the same way as Theorem 2.2.

THEOREM 3.4. *For any fixed b, let* $\mathcal{C}_1^+$ *and* $\mathcal{C}_2^+$ *be a connected subset of positive solutions of* (RSP-2) *defined as above.*

(i) *If* $\beta\lambda_1 < d$, *then*

$$\{a \in \boldsymbol{R};\ (U, V, a) \in \mathcal{C}_1^+\} = (a_*, \infty) \quad \text{for } b < \lambda_1,$$
$$\{a \in \boldsymbol{R};\ (U, V, a) \in \mathcal{C}_2^+\} \supset (a^*, \infty) \quad \text{for } b > \lambda_1,$$

where a_* *and* a^* *are defined by* (3.18) *and* (3.24), *respectively.*

(ii) *If* $\beta\lambda_1 > d$, *then for each* $i = 1, 2$, *the set* $\{a \in \boldsymbol{R};\ (U, V, a) \in \mathcal{C}_2^+\}$ *contains an open interval* $(\min\{a_*, a^*\}, \max\{a_*, a^*\})$ *for* $b > \lambda_1$. *Moreover, if either* $\mathcal{C}_1^+$ *or* $\mathcal{C}_2^+$ *is bounded, then* $\mathcal{C}_1^+ = \mathcal{C}_2^+$ *and* $\mathcal{C}_1^+$ *connects* $(\theta_{a_*}, 0, a_*)$ *with* $(0, \theta_b, a^*)$.

PROOF. In the proof of the first assertion in (i), we have to use Theorem 3.3 and, especially, (3.15). □

3.5. *Other prey–predator models with cross-diffusion*

We will study other prey–predator models with cross-diffusion effect, which is different from the model discussed in the preceding subsections. In [20], Kadota and Kuto have proposed the following reaction-diffusion system:

$$\begin{cases} u_t = \Delta[(1+\alpha v)u] + u(a-u-cv) & \text{in } \Omega\times(0,\infty), \\ v_t = \Delta\left[\left(\mu + \dfrac{1}{1+\beta u}\right)v\right] + v(b+du-v) & \text{in } \Omega\times(0,\infty), \\ u = v = 0 & \text{on } \partial\Omega\times(0,\infty), \\ u(\cdot,0) = u_0, \quad v(\cdot,0) = v_0 & \text{in } \Omega, \end{cases} \tag{3.39}$$

where a, c, d, μ are positive constants, b is a real number and α, β are nonnegative constants. In the second equation of (3.39), nonlinear diffusion $\Delta[(\mu + 1/(1+\beta u)v]$ means that the dispersive force of the predator species is weakened in the high population-density area of prey species. For this model, one can also consider the corresponding stationary problem

$$\text{(SP-4)} \qquad \begin{cases} \Delta[(1+\alpha v)u] + u(a-u-cv) = 0 & \text{in } \Omega, \\ \Delta\left[\left(\mu + \dfrac{1}{1+\beta u}\right)v\right] + v(b+du-v) = 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

It is easy to see that (SP-4) has two semi-trivial solutions; $(\theta_a, 0)$ and $(0, \theta_{b,\mu})$ with $\theta_{b,\mu} = (\mu+1)\theta_{b/(\mu+1)}$. Our main interest is to look for positive solutions for (SP-4). For this purpose, both the degree theory and the bifurcation theory developed in the preceding subsections are valid to show the existence of positive solutions. The existence theorem reads as follows (see [20, Theorem 2.1]).

THEOREM 3.5. *Assume $a > \lambda_1$. Then* (SP-4) *has a positive solution if*

$$\lambda_1\left(-\frac{(b+d\theta_a)(1+\beta\theta_a)}{1+\mu(1+\beta\theta_a)}\right) < 0 \quad \textit{and} \quad \lambda_1\left(\frac{c\theta_{b,\mu}-a}{1+\alpha\theta_{b,\mu}}\right) < 0,$$

where $\theta_{b,\mu} = (\mu+1)\theta_{b/(\mu+1)}$ and $\theta_{b,\mu} \equiv 0$ if $b \le (\mu+1)\lambda_1$.

Let a and b be positive parameters and define two curves S_4 and S_5 by

$$\begin{aligned} S_4 &= \left\{(a,b) \in R^2;\ \lambda_1\left(-\frac{(b+d\theta_a)(1+\beta\theta_a)}{1+\mu(1+\beta\theta_a)}\right) = 0 \text{ for } a \ge \lambda_1\right\}, \\ S_5 &= \left\{(a,b) \in R^2;\ \lambda_1\left(\frac{c\theta_{b,\mu}-a}{1+\alpha\theta_{b,\mu}}\right) = 0 \text{ for } b \ge (\mu+1)\lambda_1\right\}. \end{aligned} \tag{3.40}$$

Information on profiles of S_4 and S_5 can be given by the following result (see [20, Lemmas 2.2 and 2.3]).

LEMMA 3.9. *Define S_4 and S_5 by* (3.40).

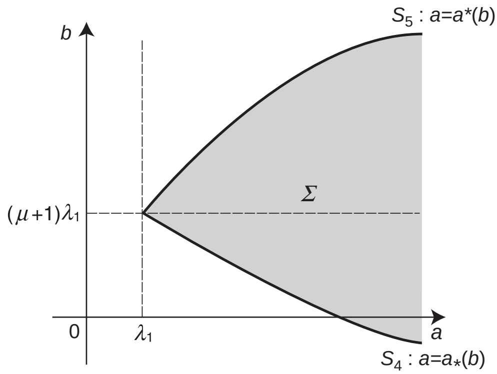

Fig. 3.1.

(i) *The curve S_4 is expressed as*

$$S_4 = \{(a, b) \in R^2;\ a = a_*(b) \text{ for } b \le (\mu + 1)\lambda_1\},$$

where $a_*(b)$ is a strictly decreasing function of class C^1 for $b \le (\mu+1)\lambda_1$ such that $a_*((\mu + 1)\lambda_1) = \lambda_1$ and $\lim_{b\to-\infty} a_*(b) = \infty$.

(ii) *The curve S_5 is expressed as*

$$S_5 = \{(a, b) \in R^2;\ a = a^*(b) \text{ for } b \ge (\mu + 1)\lambda_1\},$$

where $a^*(b)$ is a strictly increasing function of class C^1 for $b \ge (\mu+1)\lambda_1$ such that $a^*((\mu + 1)\lambda_1) = \lambda_1$ and $\lim_{b\to\infty} a^*(b) = \infty$.

By virtue of Lemma 3.9, one can draw two curves S_4 and S_5 in ab-plane as Figure 3.1. Let Σ be a domain surrounded by S_4 and S_5 curves:

$$\Sigma = \{(a, b) \in R^2;\ a > a_*(b) \text{ for } b \le (\mu + 1)\lambda_1 \text{ and } a > a^*(b) \\ \text{ for } b > (\mu + 1)\lambda_1\}.$$

Then the following corollary easily follows from Theorem 3.5.

COROLLARY 3.2. *If* $(a, b) \in \Sigma$, *then* (SP-4) *possesses a positive solution.*

In [32] and [33], Lou and Ni have studied the steady-state problem associated with (1.4) under homogeneous Neumann boundary conditions and discussed the dependence of its positive solutions upon cross-diffusion coefficients. In particular, letting one of the cross-diffusion coefficients to ∞, they have established the limiting characterization of positive solutions for competition systems with large cross-diffusion.

We can also study the dependence of positive solutions for (SP-4) upon nonlinear diffusion coefficients α and β. Now it should be noted that S_4 and S_5 depend on α and β and, therefore, Σ also depends on α and β. So it is a very interesting problem to study the effects of α and β on the structure of positive solutions of (SP-4). For details, see Kadota-Kuto [20], Kuto [24] and Kuto-Yamada [25,26].

4. Multiple existence of positive steady states for prey–predator models with cross-diffusion

In this section we will study sufficient conditions under which (SP-2) admits multiple positive steady states. In particular, we are interested in the case where cross-diffusion coefficient β is sufficiently large. For the sake of simplicity, we set $\alpha = 0$ in (SP-2);

$$\text{(SP-3)} \qquad \begin{cases} \Delta u + u(a - u - cv) = 0 & \text{in } \Omega, \\ \Delta[(1+\beta u)v] + v(b + du - v) = 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \\ u \geq 0, \quad v \geq 0 & \text{in } \Omega. \end{cases}$$

We will discuss the case when β is large, b $(> \lambda_1)$ is close to λ_1 and $|d/\beta - \lambda_1|$ is small.

4.1. *Lyapunov–Schmidt reduction*

If we use two unknown functions

$$U = u \quad \text{and} \quad V = (1 + \beta u)v \tag{4.1}$$

as in the previous sections, we can rewrite (SP-3) in the following equivalent form

$$\text{(RSP-3)} \qquad \begin{cases} \Delta U + U\left(a - U - \dfrac{cV}{1+\beta U}\right) = 0 & \text{in } \Omega, \\ \Delta V + \dfrac{V}{1+\beta U}\left(b + dU - \dfrac{V}{1+\beta U}\right) = 0 & \text{in } \Omega, \\ U = V = 0 & \text{on } \partial\Omega. \\ U \geq 0, \quad V \geq 0 & \text{in } \Omega. \end{cases}$$

It is now convenient to introduce the following change of variables in (RSP-3);

$$\begin{aligned} &a = \lambda_1 + \varepsilon a_1, \quad b = \lambda_1 + \varepsilon b_1, \quad \frac{d}{\beta} = \lambda_1 + \varepsilon\tau, \quad \beta = \frac{\gamma}{\varepsilon}, \\ &\quad U = \varepsilon w, \quad V = \varepsilon z, \end{aligned} \tag{4.2}$$

where ε is a small positive parameter and τ is a constant which is allowed to be negative. With use of (4.2), (RSP-3) is rewritten in the following form

$$\text{(PP)} \qquad \begin{cases} \Delta w + \lambda_1 w + \varepsilon w\left(a_1 - w - \dfrac{cz}{1+\gamma w}\right) = 0 & \text{in } \Omega, \\ \Delta z + \lambda_1 z + \dfrac{\varepsilon z}{1+\gamma w}\left(b_1 + \tau\gamma w - \dfrac{z}{1+\gamma w}\right) = 0 & \text{in } \Omega, \\ w = z = 0 & \text{on } \partial\Omega. \\ w \geq 0, \quad z \geq 0 & \text{in } \Omega. \end{cases}$$

By virtue of (4.2), note that two semi-trivial solutions of (RSP-3)

$$(U, V) = (\theta_a, 0)\ (a > \lambda_1) \quad \text{and} \quad (U, V) = (0, \theta_b)\ (b > \lambda_1)$$

correspond to the following semi-trivial ones of (PP)

$$(w, z) = \left(\frac{1}{\varepsilon}\theta_{\lambda_1+\varepsilon a_1}, 0\right) \quad \text{and} \quad (w, z) = \left(0, \frac{1}{\varepsilon}\theta_{\lambda_1+\varepsilon b_1}\right),$$

respectively.

In what follows, we fix b_1 and regard a_1 as a bifurcation parameter. Moreover, Proposition 3.1 implies that, if $\tau \neq 0$, then positive solutions of (PP) bifurcate from semi-trivial solution curve $\{(\varepsilon^{-1}\theta_{\lambda_1+\varepsilon a_1}, 0, a_1);\ a_1 > 0\}$ if and only if

$$a_1 = a_{1*}(\varepsilon) := \frac{1}{\varepsilon}(a_* - \lambda_1), \tag{4.3}$$

where a_* is a positive number satisfying (3.18) and (3.19). Similarly, recall a^* is a positive number defined by (3.24); so that it satisfies $a^* = \lambda_1(c\theta_b)$. Then Proposition 3.2 assures that positive solutions of (PP) bifurcate from other semi-trivial solution curve $\{(0, \varepsilon^{-1}\theta_{\lambda_1+\varepsilon b_1}, a_1\}; a_1 > 0)$ if and only if

$$a_1 = a_1^*(\varepsilon) := \frac{1}{\varepsilon}(\lambda_1(c\theta_{\lambda_1+\varepsilon b_1}) - \lambda_1). \tag{4.4}$$

We will apply the Lyapunov–Schmidt reduction procedure to (PP) as in the work of Du and Lou [15]. Let X and Y be Banach spaces defined by (2.16) with $p > N$. We study (PP) along the idea developed in the work of Kuto and Yamada [25]. Define two mappings $H : X \to Y$ and $B : X \times \boldsymbol{R} \to Y$ by

$$\begin{aligned} H(w, z) &= (\Delta w + \lambda_1 w, \Delta z + \lambda_1 z), \\ B(w, z, a_1) &= \left(w\left(a_1 - w - \frac{cz}{1+\gamma w}\right),\right. \\ &\qquad \left.\frac{z}{1+\gamma w}\left(b_1 + \tau\gamma w - \frac{z}{1+\gamma w}\right)\right). \end{aligned} \tag{4.5}$$

Clearly, (PP) is equivalent to

$$H(w, z) + \varepsilon B(w, z, a_1) = 0. \tag{4.6}$$

Let X_1 (resp. Y_1) denote the L^2-orthogonal complement of span$\{(\Phi, 0), (0, \Phi)\}$ in X (resp. Y). Then any $(w, z) \in X$ can be expressed as

$$\begin{aligned} w &= s\phi_1 + w_1 \quad \text{with } s = (w, \phi_1)_{L^2} \quad \text{and} \quad (w_1, \phi_1)_{L^2} = 0, \\ z &= t\phi_1 + z_1 \quad \text{with } t = (z, \phi_1)_{L^2} \quad \text{and} \quad (z_1, \phi_1)_{L^2} = 0; \end{aligned}$$

then $\boldsymbol{u} := (w_1, z_1) \in X_1$. Similarly, any element in Y can also be expressed as above. Let $P : X \to X_1$ and $Q : Y \to Y_1$ be the L^2-orthogonal projections. Then any $(w, z) \in X$ satisfies

$$\begin{aligned} (I - P)(w, z) &= (s, t)\phi_1 \quad \text{with } s = (w, \phi_1)_{L^2}, t = (z, \phi_1)_{L^2} \\ P(w, z) &= \boldsymbol{u} = (w_1, z_1) \quad \text{with } (w_1, \phi_1)_{L^2} = 0, (z_1, \phi_1)_{L^2} = 0. \end{aligned} \tag{4.7}$$

Note that Q and $I - Q$ have similar expressions for $(w, z) \in Y$.

Since $H(s\phi_1, t\phi_1) = 0$ and $(I - Q)H(\boldsymbol{u}) = 0$ for $\boldsymbol{u} \in X_1$, it is easy to see that (4.6) is equivalent to

$$QH(\boldsymbol{u}) + \varepsilon QB((s,t)\Phi + \boldsymbol{u}, a_1) = 0 \tag{4.8}$$

and

$$(I - Q)B((s,t)\Phi + \boldsymbol{u}, a_1) = 0. \tag{4.9}$$

If we denote the left-hand side of (4.8) by $G(s, t, a_1, \varepsilon, \boldsymbol{u})$, it is a C^1-mapping from $\boldsymbol{R}^4 \times X_1$ to Y_1. Clearly,

$$G(s, t, a_1, 0, 0) = 0 \quad \text{for all } (s, t, a_1) \in \boldsymbol{R}^3.$$

Furthermore, it is possible to show

$$G_{\boldsymbol{u}}(s, t, a_1, 0, 0) = QH \quad \text{for any } (s, t, a_1) \in \boldsymbol{R}^3,$$

where $G_{\boldsymbol{u}}$ denotes the Fréchet derivative of G with respect to $\boldsymbol{u}$. Since QH is an isomorphism from X_1 onto Y_1, we can apply the implicit function theorem to (4.8). For any $(\hat{s}, \hat{t}, \hat{a}_1) \in \boldsymbol{R}^3$ there exist a positive constant $\hat{\varepsilon} = \hat{\varepsilon}(\hat{s}, \hat{t}, \hat{a}_1)$ and a neighborhood $\mathcal{N}$ of $(w, z, a_1, \varepsilon) = (\hat{s}\phi_1, \hat{t}\phi_1, \hat{a}_1, 0)$ in $X \times \boldsymbol{R}^2$ such that all solutions of (4.8) in $\mathcal{N}$ can be expressed as

$$\{((s,t)\phi_1 + \boldsymbol{u}(s, t, a_1, \varepsilon), a_1, \varepsilon);\ |s - \hat{s}| < \hat{\varepsilon},$$
$$|t - \hat{t}| < \hat{\varepsilon},\ |a_1 - \hat{a}_1| < \hat{\varepsilon},\ |\varepsilon| \le \hat{\varepsilon}\},$$

where $\boldsymbol{u}(s, t, a_1, \varepsilon) : \boldsymbol{R}^4 \to X_1$ is a smooth function satisfying $\boldsymbol{u}(s, t, a_1, 0) = \boldsymbol{0}$. Let C be any fixed positive number. Making use of the compactness of $\{(s, t, a_1);\ |s| \le C,\ |t| \le C,\ |a_1| \le C\}$, one can find a positive $\varepsilon_0 = \varepsilon_0(C)$ and a neighborhood $\mathcal{N}_0$ of set $\{(s\phi_1, t\phi_1, a_1, 0);\ |s| \le C,\ |t| \le C,\ |a_1| \le C\}$ such that all solutions of (4.8) in $\mathcal{N}_0$ are given by

$$\{((s,t)\phi_1 + \boldsymbol{u}(s, t, a_1, \varepsilon), a_1, \varepsilon);\ |s| \le C + \varepsilon_0,\ |t| \le C + \varepsilon_0,\ |a_1|$$
$$\le C + \varepsilon_0,\ |\varepsilon| \le \varepsilon_0\}, \tag{4.10}$$

where $\boldsymbol{u}(s, t, a_1, \varepsilon)$ is an X_1-valued smooth function such that $\boldsymbol{u}(s, t, a_1, 0) = 0$. If we put

$$\varepsilon \boldsymbol{U}(s, t, a_1, \varepsilon) = \boldsymbol{u}(s, t, a_1, \varepsilon), \tag{4.11}$$

then $\boldsymbol{U}(s, t, a_1, \varepsilon)$ is also a smooth function for $|s| \le C + \varepsilon_0$, $|t| \le C + \varepsilon_0$, $|a_1| \le C + \varepsilon_0$ and $|\varepsilon| \le \varepsilon_0$.

The above considerations give the following lemma.

LEMMA 4.1. *For any fixed positive C, assume $|s| \le C + \varepsilon_0$, $|t| \le C + \varepsilon_0$, $|a_1| \le C + \varepsilon_0$ and $|\varepsilon| \le \varepsilon_0$. Let $\boldsymbol{U}(s, t, a_1, \varepsilon)$ be an X_1-valued smooth function defined by* (4.11)*. Then $(w, z, a_1, \varepsilon) = ((s,t)\phi_1 + \varepsilon \boldsymbol{U}(s, t, a_1, \varepsilon), a_1, \varepsilon)$ becomes a solution of* (4.6) (*or equivalently* (PP)) *in $\mathcal{N}_0$ if and only if*

$$(I - Q)B((s,t)\phi_1 + \varepsilon \boldsymbol{U}(s, t, a_1, \varepsilon), a_1) = 0.$$

Set

$$M = \{(s, t, a_1); |s| \le C + \varepsilon_0, |t| \le C + \varepsilon_0, |a_1| \le C + \varepsilon_0\}.$$

For each $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, define a mapping $F^\varepsilon : M \to \boldsymbol{R}^2$ by

$$F^\varepsilon(s, t, a_1)\phi_1 = (I - Q)B((s, t)\phi_1 + \varepsilon \boldsymbol{U}(s, t, a_1, \varepsilon), a_1)$$

and put

$$\boldsymbol{U}(s, t, a_1, \varepsilon) = (W(s, t, a_1, \varepsilon), Z(s, t, a_1, \varepsilon)).$$

Since $I - Q$ also satisfies the same relation as (4.7) for $I - P$, we see

$$F^\varepsilon(s, t, a_1) = \begin{pmatrix} \displaystyle\int_\Omega (s\phi_1 + \varepsilon W)\left[a_1 - (s\phi_1 + \varepsilon W) - \frac{c(t\phi_1 + \varepsilon Z)}{1 + \gamma(s\phi_1 + \varepsilon W)}\right]\phi_1\,dx \\ \displaystyle\int_\Omega \frac{t\phi_1 + \varepsilon Z}{1 + \gamma(s\phi_1 + \varepsilon W)}\left[b_1 + \tau\gamma(s\phi_1 + \varepsilon W) - \frac{t\phi_1 + \varepsilon Z}{1 + \gamma(s\phi_1 + \varepsilon W)}\right]\phi_1\,dx \end{pmatrix}. \tag{4.12}$$

Lemma 4.1 asserts that solving (4.6) in $\mathcal{N}_0$ is equivalent to solving Ker F^ε. We will concentrate ourselves on getting information on Ker F^ε in the subsequent sections.

4.2. *Analysis of limiting problem*

In this subsection we investigate the structure of Ker $F^0(s, t, a_1)$. It will help us to get useful information on a set of positive solutions of (PP) when $\varepsilon > 0$ is very small. It follows from (4.12) that

$$F^0(s, t, a_1) = \begin{pmatrix} \displaystyle s\left(a_1 - m^* s - ct\int_\Omega \frac{\phi_1^3}{1 + \gamma s\phi_1}\,dx\right) \\ \displaystyle t\left[b_1 - (b_1 - \tau)\gamma s\int_\Omega \frac{\phi_1^3}{1 + \gamma s\phi_1}\,dx - t\int_\Omega \frac{\phi_1^3}{(1 + \gamma s\phi_1)^2}\,dx\right] \end{pmatrix}, \tag{4.13}$$

where $m^* = \int_\Omega \phi_1^3 dx$. Therefore, Ker $F^0(s, t, a_1)$ is a union of the following four sets:

$$\mathcal{L}_0 = \{(0, 0, a_1); a_1 \in \boldsymbol{R}\},$$
$$\mathcal{L}_1 = \{(a_1/m^*, 0, a_1); a_1 \in \boldsymbol{R}\},$$
$$\mathcal{L}_2 = \{(0, b_1/m^*, a_1); a_1 \in \boldsymbol{R}\},$$
$$\mathcal{L}_p = \{(s, h(\gamma s), k(s)); s \in \boldsymbol{R}\},$$

where

$$\begin{cases} h(s) = \left[b_1 - (b_1 - \tau)s \int_\Omega \dfrac{\phi_1^3}{1 + s\phi_1} dx \right] \left(\int_\Omega \dfrac{\phi_1^3}{(1 + s\phi_1)^2} dx \right)^{-1}, \\ k(s) = m^* s + ch(\gamma s) \int_\Omega \dfrac{\phi_1^3}{1 + \gamma s\phi_1} dx. \end{cases} \tag{4.14}$$

By the identification $(s, t)\phi_1$ with (s, t), it is possible to regard $\mathcal{L}_1 \cap \overline{\boldsymbol{R}_+}^3$ and $\mathcal{L}_2 \cap \overline{\boldsymbol{R}_+}^3$ as the limiting sets of semi-trivial solution curves $\{(\varepsilon^{-1}\theta_{\lambda_1+\varepsilon a_1}, 0, a_1); a_1 > 0\}$ and $\{(0, \varepsilon^{-1}\theta_{\lambda_1+\varepsilon b_1}, a_1); a_1 > 0\}$ as $\varepsilon \to 0$, respectively. Indeed, it follows from (iii) of Proposition 1.2 that

$$\theta_{\lambda_1+\varepsilon a_1} = \frac{\varepsilon a_1 \phi_1}{m^*} + o(\varepsilon) \quad \text{and} \quad \theta_{\lambda_1+\varepsilon b_1} = \frac{\varepsilon b_1 \phi_1}{m^*} + o(\varepsilon) \quad \text{as } \varepsilon \to 0;$$

so that one can easily show the above results. By virtue of (4.14),

$$(0, h(0), k(0)) = (0, b_1/m^*, b_1 c) \in \mathcal{L}_2. \tag{4.15}$$

Moreover, since

$$\begin{aligned} b_1 - b_1 s \int_\Omega \frac{\phi_1^3}{1 + s\phi_1} dx &= b_1 \int_\Omega \phi_1^2 \left(1 - \frac{s\phi_1}{1 + s\phi_1} \right) dx \\ &= b_1 \int_\Omega \frac{\phi_1^2}{1 + s\phi_1} dx, \end{aligned} \tag{4.16}$$

it is easy to show that, if $\tau \geq 0$, then

$$h(s) > 0 \quad \text{for all } s \in [0, \infty).$$

On the other hand, if $\tau < 0$, we can find a positive number s_0 such that

$$\begin{cases} h(s) > 0 & \text{for } s \in [0, s_0), \\ h(s) < 0 & \text{for } s \in (s_0, \infty). \end{cases} \tag{4.17}$$

Hence

$$(s_0/\gamma, h(s_0), k(s_0/\gamma)) = (s_0/\gamma, 0, s_0 \|\phi\|_3^3/\gamma) \in \mathcal{L}_1 \tag{4.18}$$

if $\tau < 0$.

LEMMA 4.2. *If k is defined by (4.14), then it possesses the following properties.*

(i) *If $\tau \geq 0$, then $k(s) > k(0) = b_1 c$ for all $s \in (0, \infty)$ and* $\lim_{s\to\infty} k(s) = \infty$.

(ii) *There exist positive constants $\tilde{\tau} = \tilde{\tau}(c, b_1)$ and $\tilde{\gamma} = \tilde{\gamma}(c, b_1)$ with the following properties:*

  (a) *if $(\tau, \gamma) \in [0, \tilde{\tau}] \times [\tilde{\gamma}, \infty)$, then $k(s)$ attains a strict local maximum at $s = \overline{s}$ and a strict local minimum at $s = \underline{s}$, where $\overline{s}$ and $\underline{s}$ are positive numbers satisfying $\overline{s} < \underline{s}$ and $k(\overline{s}) > k(\underline{s})$;*

(b) *if* $(\tau,\gamma)\in[-\tilde{\tau},0)\times[\tilde{\gamma},\infty)$, *then* $k(s)$ *achieves a strict local maximum in* $(0,s_0/\gamma)$. *Furthermore, there exists a continuous function* $\hat{\gamma}(\tau)$ *in* $[-\tilde{\tau},0)$ *with*

$$\tilde{\gamma}<\hat{\gamma}(\tau)\quad \text{for all } \tau\in[-\tilde{\tau},0)\quad \text{and}\quad \lim_{\tau\uparrow 0}\hat{\gamma}(\tau)=\infty \tag{4.19}$$

such that, if $\gamma\in[\tilde{\gamma},\hat{\gamma}(\tau)]$ *for* $\tau\in[-\tilde{\tau},0)$, *then* $k(s)$ *attains a strict local minimum in* $(0,s_0/\gamma)$ *and, moreover, if* $\gamma\in[\hat{\gamma}(\tau),\infty)$ *for* $\tau\in[-\tilde{\tau},0)$, *then* $\max_{s\in[0,s_0/\gamma]}k(s)=k(\hat{s})$ *for some* $\hat{s}\in(0,s_0/\gamma)$.

PROOF. In view of (4.14), if we define

$$\begin{aligned} h^*(s;\tau) &:= h(s)\int_\Omega \frac{\phi_1^3}{1+s\phi_1}dx \\ &= \left[b_1-(b_1-\tau)s\int_\Omega \frac{\phi_1^3}{1+s\phi_1}dx\right]\int_\Omega \frac{\phi_1^3}{1+s\phi_1}dx\left(\int_\Omega \frac{\phi_1^3}{(1+s\phi_1)^2}dx\right)^{-1}, \end{aligned} \tag{4.20}$$

then

$$k(s)=m^*s+ch^*(\gamma s;\tau). \tag{4.21}$$

Note

$$\lim_{s\to\infty} s\int_\Omega \frac{\phi_1^3}{1+s\phi_1}dx=\lim_{s\to\infty}\int_\Omega \frac{\phi_1^3}{\phi_1+1/s}dx=1$$

and

$$\begin{aligned} &\lim_{s\to\infty}\left(\int_\Omega \frac{\phi_1^3}{1+s\phi_1}dx\Big/ s\int_\Omega \frac{\phi_1^3}{(1+s\phi_1)^2}dx\right) \\ &\quad=\lim_{s\to\infty}\left(\int_\Omega \frac{s\phi_1^3}{1+s\phi_1}dx\Big/\int_\Omega \frac{s^2\phi_1^3}{(1+s\phi_1)^2}dx\right)=\frac{1}{\|\phi_1\|_1}. \end{aligned}$$

Therefore,

$$\lim_{s\to\infty}\frac{h^*(s;\tau)}{s}=\frac{\tau}{\|\phi_1\|_1},$$

which immediately yields

$$\lim_{s\to\infty}\frac{k(s)}{s}=m^*+\frac{c\gamma\tau}{\|\phi_1\|_1}\quad \text{for any } \tau\in\boldsymbol{R}. \tag{4.22}$$

In particular, from the Schwarz inequality

$$\left(\int_\Omega \frac{\phi_1^3}{1+s\phi_1}dx\right)^2\le\int_\Omega \frac{\phi_1^4}{(1+s\phi_1)^2}dx\int_\Omega \phi_1^2dx=\int_\Omega \frac{\phi_1^4}{(1+s\phi_1)^2}dx;$$

so that

$$\begin{aligned} h^*(s;0) &= b_1\left(1 - s\int_\Omega \frac{\phi_1^3}{1+s\phi_1}dx\right)\int_\Omega \frac{\phi_1^3}{1+s\phi_1}dx\left(\int_\Omega \frac{\phi_1^3}{(1+s\phi_1)^2}dx\right)^{-1} \\ &\geq b_1\left(\int_\Omega \frac{\phi_1^3}{1+s\phi_1}dx - s\int_\Omega \frac{\phi_1^4}{(1+s\phi_1)^2}dx\right)\left(\int_\Omega \frac{\phi_1^3}{(1+s\phi_1)^2}dx\right)^{-1} \\ &= b_1 = h^*(0;0) \quad \text{for all } s \in [0,\infty). \end{aligned} \tag{4.23}$$

It follows from (4.20), (4.21) and (4.23) that, if $\tau \geq 0$, then

$$k(s) \geq m^* s + ch^*(\gamma s, 0) \geq b_1 c = k(0) \quad \text{for all } s \in (0,\infty).$$

Thus (i) is proved. Furthermore, in view of (4.16), we see

$$\begin{aligned} \lim_{s\to\infty} h^*(s;0) &= b_1 \lim_{s\to\infty}\int_\Omega \frac{s\phi_1^2}{1+s\phi_1}dx\int_\Omega \frac{s\phi_1^3}{1+s\phi_1}dx\left(\int_\Omega \frac{s^2\phi_1^3}{(1+s\phi_1)^2}dx\right)^{-1} \\ &= b_1 = h^*(0;0). \end{aligned} \tag{4.24}$$

Hence it follows from (4.23) and (4.24) that $h^*(s;0)$ attains its global maximum at a point in $(0,\infty)$.

Note

$$k'(s) = m^* + c\gamma\frac{\partial h^*}{\partial s}(\gamma s, \tau).$$

Hence

$$k'(s^*) < 0 \quad \text{for some } s^* \in (0,\infty) \tag{4.25}$$

provided that γ is sufficiently large. After some calculations one can show

$$\frac{\partial h^*}{\partial s}(0,0) = b_1\{\|\phi_1\|_4^4 - \|\phi_1\|_3^6\}/\|\phi_1\|_3^3 > 0,$$

$$\lim_{s\to\infty} s\frac{\partial h^*}{\partial s}(s,0) = 0,$$

where $\|\cdot\|_p$ denotes $L^p(\Omega)$-norm. Therefore,

$$k'(0) > m^* \quad \text{and} \quad \lim_{s\to\infty} k'(s) = m^*. \tag{4.26}$$

By virtue of (4.25) and (4.26) there exist two positive numbers s_1 and s_2 $(s_1 < s_2)$ such that

$$\begin{aligned} &k'(s_1) = k'(s_2) = 0, k'(s) < 0 \quad \text{for } s \in (s_1, s_2) \\ &\text{and} \quad k'(s) > 0 \quad \text{for } s \in (s_1-\delta, s_1) \cup (s_2, s_2+\delta) \end{aligned}$$

with some $\delta > 0$. Therefore, one can see that, if $\tau = 0$ and γ is sufficiently large, then $k(s)$ forms a '$\sim$'-shaped curve in the sense of the assertion (a) of (ii). This property of $k(s)$ is invariant for small $\tau > 0$ and the proof of (a) is complete.

In case $\tau < 0$, we can show from (4.17) and (4.20) that

$$\begin{cases} h^*(s;\tau) > 0 & \text{for } s \in [0, s_0), \\ h^*(s;\tau) < 0 & \text{for } s \in (s_0, \infty). \end{cases}$$

Hence, if $|\tau|$ is sufficiently small, then $h^*(s;\tau)$ attains its global maximum at some point in $(0, s_0)$ because of (4.23)). Thus by (4.21), we may assume that, if $(\tau, \gamma) \in [-\tilde{\tau}, 0) \times [\tilde{\gamma}, \infty)$, then $k(s)$ possesses at least one strict local maximum in $(0, s_0/\gamma)$. Observe that k depends continuously on (τ, γ). Hence there exists a continuous function $\hat{\gamma}(\tau)$ in $[-\tilde{\tau}, 0)$ with (4.19) such that, if $\gamma \in [\tilde{\gamma}, \hat{\gamma}(\tau))$ for $\tau \in [-\tilde{\tau}, 0)$, then $k(s)$ forms a '∼'-shaped curve in $(0, s_0/\gamma)$ and, if $\gamma \in [\hat{\gamma}(\tau), \infty)$ for $\tau \in [-\tilde{\tau}, 0)$, then $\max_{s\in[0,s_0/\gamma]} k(s) = k(\hat{s})$ for some $\hat{s} \in (0, s_0/\gamma)$. Thus we accomplish the proof of (b) in (ii). □

4.3. *Analysis of perturbed problem (PP)*

We will study (PP) in case $\tau \geq 0$. The other case $\tau < 0$ can be discussed in the same manner.

By Lemma 4.2, there exist sufficient large numbers A_1 and C satisfying

$$A_1 = k(C) = \max_{s\in[0,C]} k(s). \tag{4.27}$$

The purpose in this section is to show that, if $\varepsilon > 0$ is small enough, then all positive solutions of (PP) for $a_1 \in [0, A_1]$ form a one-dimensional submanifold near a curve

$$\{(w, z, a_1) = (s\phi_1, h(\gamma s)\phi_1, k(s)); 0 < s \leq C\}.$$

Our main result is stated as follows:

PROPOSITION 4.1. *For $\tau \geq 0$, there exist a positive constant $\varepsilon_0 = \varepsilon_0(A_1)$ and a family of bounded smooth curves*

$$\{S(\xi, \varepsilon) = (s(\xi, \varepsilon), t(\xi, \varepsilon), a_1(\xi, \varepsilon)) \in \boldsymbol{R}^3; (\xi, \varepsilon) \in [0, C(\varepsilon)] \times [0, \varepsilon_0]\}$$

such that for each $\varepsilon \in (0, \varepsilon_0]$, all positive solutions of (PP) *with $a_1 \in (0, A_1]$ can be parameterized as*

$$\begin{aligned} \Gamma^\varepsilon = \{&(w(\xi, \varepsilon), z(\xi, \varepsilon), a_1(\xi, \varepsilon)) = ((s, t)\phi_1 + \varepsilon \boldsymbol{U}(s, t, a_1, \varepsilon), a_1); \\ &(s, t, a_1) = (s(\xi, \varepsilon), t(\xi, \varepsilon), a_1(\xi, \varepsilon)) \text{ for } \xi \in (0, C(\varepsilon)]\}, \end{aligned} \tag{4.28}$$

and

$$S(\xi, 0) = (\xi, h(\gamma\xi), k(\xi)), \qquad S(0, \varepsilon) = (0, t(\varepsilon), a_1^*(\varepsilon)).$$

Here $C(\varepsilon)$ is a smooth positive function in $[0, \varepsilon_0]$ *satisfying $C(0) = C$ and $a_1(C(\varepsilon), \varepsilon) = A_1$, $t(\varepsilon) := \varepsilon^{-1} \int_\Omega \theta_{\lambda_1+\varepsilon b_1}\phi_1\, dx$, $a_1^*(\varepsilon)$ is a positive number defined by* (4.4) *and $\boldsymbol{U}$ is an X_1-valued function defined by* (4.11).

Moreover, Γ^ε can be extended for all $a_1 \in [0, \infty)$ as a curve of positive solutions (PP).

We will prove Proposition 4.1 with use of a series of lemmas. As the first step, we will study the structure of the set of nonnegative solutions of (4.6) (or equivalently (PP)) near the intersection point of $\mathcal{L}_2$ and $\mathcal{L}_p$;

$$(s, t, a_1) = (0, b_1/m^*, b_1 c)$$

(see (4.15)).

LEMMA 4.3. *Let* F^ε *be a mapping defined by* (4.12). *Then there exist a neighborhood* $\mathcal{U}_0$ *of* $(0, b_1/m^*, b_1 c)$ *and a positive constant* δ_0 *such that for any* $\varepsilon \in [0, \delta_0]$,

$$\begin{aligned}&\operatorname{Ker} F^\varepsilon \cap \mathcal{U}_0 \cap \overline{\boldsymbol{R}_+}^3\\ &\quad = \{(s(\xi,\varepsilon), t(\xi,\varepsilon), a_1(\xi,\varepsilon));\ \xi \in [0,\delta_0)\} \cup \{(0, t(\varepsilon), a_1) \in \mathcal{U}_0\}\end{aligned} \tag{4.29}$$

with some smooth functions $s(\xi,\varepsilon), t(\xi,\varepsilon), a_1(\xi,\varepsilon)$ *with respect to* $(\xi,\varepsilon) \in [0,\delta_0] \times \boldsymbol{R}$ *satisfying*

$$\begin{aligned}&(s(\xi,0), t(\xi,0), a_1(\xi,0)) = (\xi, h(\gamma\xi), k(\xi)),\\ &(s(0,\varepsilon), t(0,\varepsilon), a_1(0,\varepsilon)) = (0, t(\varepsilon), a_1^*(\varepsilon)).\end{aligned}$$

PROOF. Proposition 3.1, together with (4.4), enables us to prove that, for any $\varepsilon > 0$, there exist a positive number $\delta = \delta(\varepsilon)$ and a neighborhood $\mathcal{V}_\varepsilon$ of the bifurcation point $(w, z, a_1) = (0, \varepsilon^{-1}\theta_{\lambda_1+\varepsilon b_1}, a_1^*(\varepsilon))$ such that all positive solutions of (PP) in $\mathcal{V}_\varepsilon$ are expressed as

$$\begin{aligned}(w, z, a_1) &= (w(\xi,\varepsilon), z(\xi,\varepsilon), a_1(\xi,\varepsilon))\\ &= (\xi\Phi^* + \xi W(\xi,\varepsilon), \varepsilon^{-1}\theta_{\lambda_1+\varepsilon b_1} + \xi\Psi^* + \xi Z(\xi,\varepsilon), a_1(\xi,\varepsilon))\end{aligned}$$

for $\xi \in (0, \delta]$ with use of Φ^* and Ψ^* defined by (3.25) and (3.26), respectively. Here $W(\xi,\varepsilon), Z(\xi,\varepsilon), a_1(\xi,\varepsilon)$ are appropriate smooth functions satisfying $a_1(0,\varepsilon) = a_1^*(\varepsilon)$ and $\int_\Omega W(\xi,\varepsilon)\Phi^* dx = 0$. Define an open set $\mathcal{U}_\varepsilon$ of $\boldsymbol{R}^3$ by

$$\mathcal{U}_\varepsilon := \left\{(s, t, a_1);\ s = \int_\Omega w\phi_1 dx, t = \int_\Omega z\phi_1 dx, (w, z, a_1) \in \mathcal{V}_\varepsilon\right\}$$

and put

$$s(\xi,\varepsilon) := \int_\Omega w(\xi,\varepsilon)\phi_1 dx, \qquad t(\xi,\varepsilon) := \int_\Omega z(\xi,\varepsilon)\phi_1 dx.$$

Recalling the equivalence of (PP) and (4.6), we see that, if $\varepsilon \in [0, \varepsilon_0]$, then

$$\begin{aligned}&\operatorname{Ker} F^\varepsilon \cap U_\varepsilon \cap \overline{\boldsymbol{R}_+}^3\\ &\quad = \{(s(\xi,\varepsilon), t(\xi,\varepsilon), a_1(\xi,\varepsilon)); \xi \in [0,\delta]\} \cup \{(0, t(\varepsilon), a_1) \in \mathcal{U}_\varepsilon\}.\end{aligned}$$

Since $(0, t(\varepsilon), a_1^*(\varepsilon))$ is a bifurcation point for any $\varepsilon \in [0, \varepsilon_0]$, it is possible to show that $\mathcal{U}_\varepsilon$ contains a neighborhood $\mathcal{U}_0$ of $(0, b_1/m^*, b_1 c)$ if $\varepsilon > 0$ is sufficiently small. Thus the proof is complete. □

LEMMA 4.4. *Let A_1, C be positive numbers defined in (4.27). Then there exist a positive number $\varepsilon_0 = \varepsilon(A_1) > 0$ and a neighborhood $\mathcal{U}$ of $\{(s\phi_1, h(\gamma s)\phi_1, k(s)); 0 \le s \le C\}$ such that, for each $\varepsilon \in (0, \varepsilon_0]$, all positive solutions of* (PP) *in $\mathcal{U} \cap (X \times (0, A_1])$ can be expressed as* (4.28).

PROOF. We will employ the perturbation technique used by Du and Lou [15, Appendix]. Define $\mathcal{L}_p([\delta_0/2, C]) = \{(s, h(\gamma s), k(s));\ s \in [\delta_0/2, C]\}$ for a positive constant δ_0 in Lemma 4.3. By (4.13) and (4.14), one can derive

$$\det F^0_{(s,t)}(s, h(\gamma s), k(s)) = sh(\gamma s)k'(s) \int_\Omega \frac{\phi_1^3}{(1+\gamma s\phi_1)^2} dx, \tag{4.30}$$

where $F^0_{(s,t)}$ denotes a matrix of the first-order derivatives with respect to s and t;

$$\begin{pmatrix} \dfrac{\partial F^0}{\partial s} & \dfrac{\partial F^0}{\partial t} \end{pmatrix}.$$

Let $(\overline{s}, h(\gamma\overline{s}), k(\overline{s})) \in \mathcal{L}_p([\delta_0/2, C])$ be any fixed point. Since $h(\gamma\overline{s}) > 0$ for $\tau \ge 0$, it follows from (4.30) that, if $k'(\overline{s}) \ne 0$, then $F^0_{(s,t)}(\overline{s}, h(\gamma\overline{s}), k(\overline{s}))$ is invertible. In this case, the implicit function theorem assures the existence of a positive number $\delta = \delta(\overline{s})$ and a neighborhood $\mathcal{W}_{\overline{s}}$ of $(\overline{s}, h(\gamma\overline{s}))$ such that for all $\varepsilon \in [0, \delta]$,

$$\operatorname{Ker} F^\varepsilon \cap \mathcal{U}_{\overline{s}} = \{(s(a_1, \varepsilon), t(a_1, \varepsilon), a_1); a_1 \in (k(\overline{s}) - \delta, k(\overline{s}) + \delta)\}, \tag{4.31}$$

with $\mathcal{U}_{\overline{s}} = \mathcal{W}_{\overline{s}} \times (k(\overline{s}) - \delta, k(\overline{s}) + \delta)$. Here $\{(s(a_1, \varepsilon), t(a_1, \varepsilon));\ \varepsilon \in (0, \varepsilon_0]\}$ is a family of smooth functions satisfying $(s(k(\overline{s}), 0), t(k(\overline{s}), 0)) = (\overline{s}, h(\gamma\overline{s}))$.

On the other hand, if $k'(\overline{s}) = 0$, then (4.30) implies

$$\dim \operatorname{Ker} F^0_{(s,t)}(\overline{s}, h(\gamma\overline{s}), k(\overline{s})) = \operatorname{codim} \operatorname{Range} F^0_{(s,t)}(\overline{s}, h(\gamma\overline{s}), k(\overline{s})) = 1. \tag{4.32}$$

After some calculations, one can see

$$\frac{\partial F^0}{\partial a_1}(\overline{s}, h(\gamma\overline{s}), k(\overline{s})) = \begin{pmatrix} \overline{s} \\ 0 \end{pmatrix} \notin \operatorname{Range} F^0_{(s,t)}(\overline{s}, h(\gamma\overline{s}), k(\overline{s})). \tag{4.33}$$

According to the spontaneous bifurcation theory due to Crandall and Rabinowitz [8, Theorem 3.2 and Remark 3.3],we see from (4.32) and (4.33) that there exist a positive number $\delta = \delta(\overline{s})$ and a neighborhood $\mathcal{U}_{\overline{s}}$ of $(\overline{s}, h(\gamma\overline{s}), k(\overline{s}))$ such that for each $\varepsilon \in [0, \delta]$,

$$\operatorname{Ker} F^\varepsilon \cap \mathcal{U}_{\overline{s}} = \{(s(\xi, \varepsilon), t(\xi, \varepsilon), a_1(\xi, \varepsilon)); \xi \in (-\delta, \delta)\}, \tag{4.34}$$

where ξ is a parameter, $s(\xi, \varepsilon), t(\xi, \varepsilon)$ and $a_1(\xi, \varepsilon)$ are smooth functions for $(\xi, \varepsilon) \in [-\delta, \delta] \times [0, \delta]$ such that

$$(s(0, 0), t(0, 0), a_1(0, 0)) = (\overline{s}, h(\gamma\overline{s}), k(\overline{s})).$$

Note that

$$\mathcal{L}_p([\delta_0/2, C]) \subset \bigcup \{\mathcal{U}_{\overline{s}}; \overline{s} \in [\delta_0/2, C]\},$$

where each $\mathcal{U}_{\bar{s}}$ is an open set satisfying (4.31) or (4.34). Since $\mathcal{L}_p([\delta_0/2, C])$ is compact, one can find a finite number of points $(\{s_j\}_{j=1}^k$ such that $s_j, h(\gamma s_j), k(s_j)) \in \mathcal{L}_p([\delta_0/2, C])$ for $1 \le j \le k$ and

$$\mathcal{L}_p([\delta_0/2, C]) \subset \bigcup_{j=1}^k \mathcal{U}_j, \quad \text{with}\, \mathcal{U}_j := \mathcal{U}_{s_j}.$$

Here we may assume that $\mathcal{U}_j \cap \mathcal{U}_{j+1}$ are not empty for all $0 \le j \le k-1$, where $\mathcal{U}_0$ is an open set in Lemma 4.3. Put $\delta_j = \delta(s_j)$ for $1 \le j \le k$. By virtue of (4.31) and (4.34), it can be proved that, for any $\varepsilon \in [0, \delta_j]$ $(1 \le j \le k)$, each $\operatorname{Ker} F^\varepsilon \cap \mathcal{U}_j$ is expressed in the following form

$$\operatorname{Ker} F^\varepsilon \cap \mathcal{U}_j = \{(s^j(\xi,\varepsilon), t^j(\xi,\varepsilon), a_1^j(\xi,\varepsilon)); \xi \in (-\delta_j, \delta_j)\} =: J_j^\varepsilon$$

with some smooth functions $s^j(\xi,\varepsilon)$, $t^j(\xi,\varepsilon)$ and $a_1^j(\xi,\varepsilon)$ satisfying

$$(s^j(0,0), t^j(0,0), a_1^j(0,0)) = (s_j, h(\gamma s_j), k(s_j)).$$

Recall Lemma 4.3 and define

$$J_0^\varepsilon = \{(s(\xi,\varepsilon), t(\xi,\varepsilon), a_1(\xi,\varepsilon)); \xi \in (0, \delta_0]\}$$

and $\mathcal{U} = \bigcup_{j=0}^k \mathcal{U}_j$. Then for any $\varepsilon \in [0, \min_{0\le j\le k} \delta_j]$,

$$\operatorname{Ker} F^\varepsilon \cap \mathcal{U} \cap \boldsymbol{R}_+^3 = \bigcup_{j=0}^k J_j^\varepsilon. \tag{4.35}$$

This fact implies that $\operatorname{Ker} F^\varepsilon \cap \mathcal{U} \cap \boldsymbol{R}_+^3$ forms a one-dimensional submanifold. Indeed, using the procedure by Du and Lou [15, Proposition A3], we can construct a smooth curve $S(\xi,\varepsilon) = (s(\xi,\varepsilon), t(\xi,\varepsilon), a_1(\xi,\varepsilon))$ such that

$$\begin{cases} \bigcup_{j=0}^k J_j^\varepsilon = \{S(\xi,\varepsilon); \xi \in (0, C(\varepsilon)]\}, \\ (s(\xi,0), t(\xi,0), a_1(\xi,0)) = (\xi, h(\gamma\xi), k(\xi)), \\ (s(0,\varepsilon), t(0,\varepsilon), a_1(0,\varepsilon)) = (0, t(\varepsilon), a_1^*(\varepsilon)) \end{cases} \tag{4.36}$$

for sufficiently small $\varepsilon > 0$ and $\xi \in [0, C(\varepsilon)]$ with some smooth function $C(\varepsilon)$.

Finally, the conclusion follows from Lemma 4.1 and (4.36). □

The following lemma allows us to conclude that, if $a_1 \in (0, A_1]$ and $\varepsilon > 0$ is sufficiently small, then (PP) has no positive solution outside $\mathcal{U}$.

LEMMA 4.5. *Let $\mathcal{V}$ be any neighborhood of $\{(s\phi_1, h(\gamma s)\phi_1, k(s)); 0 \le s \le C\}$. Then there exists a positive constant ε_1 such that, for each $\varepsilon \in (0, \varepsilon_1]$, any solution (w, z) of* (PP) *with $a_1 \in (0, A_1]$ satisfies*

$$(w, z, a_1) = ((s,t)\phi_1 + \varepsilon \boldsymbol{U}(s,t,a_1,\varepsilon), a_1) \in \mathcal{V} \quad \text{with some } (s,t) \in \boldsymbol{R}^2,$$

where $\boldsymbol{U}(s,t,a_1,\varepsilon)$ is a function defined by (4.11).

PROOF. We will prove this lemma by contradiction. Suppose that there exists a sequence $\{(a_1^n, \varepsilon_n)\}$ with the following properties: $a_1^n \in (0, A_1]$, $\lim_{n\to 0} \varepsilon_n = 0$ and (PP) with $(a_1, \varepsilon) = (a_1^n, \varepsilon_n)$ admits a positive solution (w_n, z_n) such that $(w_n, z_n, a_1^n) \notin \mathcal{V}$ for $n \in \boldsymbol{N}$. To derive a contradiction, it is sufficient to find a subsequence $\{(w_{n(j)}, z_{n(j)}, a_1^{n(j)}, \varepsilon_{n(j)})\}$ and a sequence $\{(s_j, t_j)\}$ such that

$$\begin{cases} (w_{n(j)}, z_{n(j)}) = (s_j, t_j)\phi_1 + \varepsilon_{n(j)}\boldsymbol{U}(s_j, t_j, a_1^{n(j)}, \varepsilon_{n(j)}) \quad \text{for all } k \in \boldsymbol{N}, \\ \lim\limits_{j\to\infty} (s_j, t_j, a_1^{n(j)}) = (s, h(\gamma s), k(s)) \quad \text{for some } s \in [0, C]. \end{cases} \tag{4.37}$$

We begin with a priori bounds for $\{w_n\}$ and $\{z_n\}$. It follows from (3.1) and (4.2) that

$$w_n \le \frac{1}{\varepsilon_n}\theta_{\lambda_1+\varepsilon_n a_1^n} \le \frac{1}{\varepsilon_n}\theta_{\lambda_1+\varepsilon_n A_1} \quad \text{in } \Omega$$

for all $n \in \boldsymbol{N}$. Making use of (1.16) we see

$$\lim_{n\to\infty} \frac{1}{\varepsilon_n}\theta_{\lambda_1+\varepsilon_n A_1} = \frac{A_1}{m^*}\phi_1 \quad \text{uniformly in } \Omega, \tag{4.38}$$

with $m^* = \int_\Omega \phi_1(x)^3 dx$. Therefore, for sufficiently large n

$$w_n \le 1 + \frac{A_1\|\phi_1\|_\infty}{m^*} =: M \quad \text{in } \Omega, \tag{4.39}$$

which implies that $\{w_n\}$ is a bounded sequence in $C(\overline{\Omega})$.

We next use the second equation of (PP) to get

$$\begin{aligned} -\Delta z_n &= \lambda_1 z_n + \frac{\varepsilon_n z_n}{1+\gamma w_n}\left(b_1 + \tau\gamma w_n - \frac{z_n}{1+\gamma w_n}\right) \\ &\le \lambda_1 z_n + \varepsilon_n z_n\left[b_1 + \tau - \frac{z_n}{(1+\gamma w_n)^2}\right] \\ &\le z_n\left[\lambda_1 + \varepsilon_n(b_1+\tau) - \frac{\varepsilon_n z_n}{(1+\gamma M)^2}\right] \quad \text{in } \Omega \end{aligned}$$

for sufficiently large n. The above result assures that $\varepsilon_n z_n/(1+\gamma M)^2$ is a subsolution of a logistic equation (1.3) with a replaced by $\lambda_1 + \varepsilon_n(b_1+\tau)$. The comparison principle yields $\varepsilon_n z_n/(1+\gamma M)^2 \le \theta_{\lambda_1+\varepsilon_n(b_1+\tau)}$ in Ω; so that

$$z_n \le (1+\gamma M)^2 \frac{\theta_{\lambda_1+\varepsilon_n(b_1+\tau)}}{\varepsilon_n} \quad \text{in } \Omega.$$

Hence it follows from (4.38) that

$$z_n \le 1 + \frac{(1+\gamma M)^2(b_1+\tau)\|\phi_1\|_\infty}{m^*} \quad \text{in } \Omega, \tag{4.40}$$

for sufficiently large n. Thus we see from (4.39) and (4.40) that both $\{w_n\}$ and $\{z_n\}$ are uniformly bounded in $C(\overline{\Omega})$.

Set $\overline{w}_n = w_n/\|w_n\|_\infty$ and $\overline{z}_n = z_n/\|z_n\|_\infty$. Since (w_n, z_n) satisfies (PP), $(\overline{w}_n, \overline{z}_n)$ satisfies

$$\begin{cases} -\Delta\overline{w}_n = \lambda_1\overline{w}_n + \varepsilon_n\overline{w}_n\left(a_1^n - w_n - \dfrac{cz_n}{1+\gamma w_n}\right) & \text{in } \Omega, \\ -\Delta\overline{z}_n = \lambda_1\overline{z}_n + \dfrac{\varepsilon_n\overline{z}_n}{1+\gamma w_n}\left(b_1 + \tau\gamma w_n - \dfrac{z_n}{1+\gamma w_n}\right) & \text{in } \Omega, \\ \overline{w}_n = \overline{z}_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.41}$$

Observe that both $\{a_1^n - w_n - cz_n/(1+\gamma w_n)\}$ and $\{b_1 + \tau\gamma w_n - z_n/(1+\gamma w_n)\}$ are bounded in $C(\overline{\Omega})$ because of the boundedness of $\{(w_n, z_n, a_1^n)\}$. Applying the standard regularity theory for elliptic differential equations to (4.41) we can show that both $\{\overline{w}_n\}$ and $\{\overline{z}_n\}$ are uniformly bounded in $C^2(\overline{\Omega})$. So it is possible to choose a subsequence $\{(\overline{w}_{n(k)}, \overline{z}_{n(k)}, a_1^{n(k)})\}$ such that

$$\lim_{k\to\infty}(\overline{w}_{n(k)}, \overline{z}_{n(k)}, a_1^{n(k)}) = (\overline{w}, \overline{z}, a_1^\infty) \quad \text{in } C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \times \boldsymbol{R}$$

with some $(\overline{w}, \overline{z}, a_1^\infty)$. For the sake of simplicity, this subsequence is still denoted by $\{(w_n, z_n, a_1^n)\}$. Since $\lim_{n\to\infty}\varepsilon_n = 0$, letting $n \to \infty$ in (4.41) leads us to get

$$-\Delta\overline{w} = \lambda_1\overline{w}, \quad -\Delta\overline{z} = \lambda_1\overline{z} \quad \text{in } \Omega, \qquad \overline{w} = \overline{z} = 0 \quad \text{on } \partial\Omega.$$

In view of $\|\overline{w}\|_\infty = \|\overline{z}\|_\infty = 1$, we have $\overline{w} = \overline{z} = \phi_1/\|\phi_1\|_\infty$. So the boundedness of $\{(w_n, z_n)\}$ in $C^2(\overline{\Omega})^2$ yields

$$\lim_{n\to\infty}(w_n, z_n) = (s\phi_1, t\phi_1) \quad \text{in } C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \tag{4.42}$$

for some $s \geq 0$ and $t \geq 0$. Recall the arguments in Section 4.1; by virtue of (4.10) and (4.42), (w_n, z_n) must be expressed as

$$(w_n, z_n) = (s_n, t_n)\phi_1 + \varepsilon_n\boldsymbol{U}(s_n, t_n, a_1^n, \varepsilon_n)$$

for sufficiently large n, where (s_n, t_n) satisfies $\lim_{n\to\infty}(s_n, t_n) = (s, t)$.

To prove $t = h(\gamma s)$, we multiply by ϕ_1 the second equation of (4.41) and integrate the resulting identity; then

$$\int_\Omega \frac{\overline{z}_n\phi_1}{1+\gamma w_n}\left(b_1 + \tau\gamma w_n - \frac{z_n}{1+\gamma w_n}\right)dx = 0.$$

Letting $n \to \infty$ in the above equality we see from (4.42)

$$b_1 - (b_1 - \tau)\gamma s\int_\Omega \frac{\phi_1^3}{1+\gamma s\phi_1}dx = t\int_\Omega \frac{\phi_1^3}{(1+\gamma s\phi_1)^2}dx,$$

which, together with (4.14), implies $t = h(\gamma s)$.

Finally we will prove $a_1^\infty = k(s)$. Multiply the first equation of (4.41) by ϕ_1 and integrate the resulting expression; then

$$\int_\Omega \overline{w}_n\phi_1\left(a_1^n - w_n - \frac{cz_n}{1+\gamma w_n}\right)dx = 0.$$

Letting $n \to \infty$ in the above equality, we have

$$a_1^\infty - s\|\phi_1\|_3^3 - ct\int_\Omega \frac{\phi_1^3}{1+\gamma s\phi_1}dx = 0,$$

which immediately leads to $a_1^\infty = k(s)$ by (4.14)). Thus we obtain (4.37) and we complete the proof. □

We are ready to give a proof of Proposition 4.1.

PROOF OF PROPOSITION 4.1. It is easy to show (4.28) by using Lemmas 4.4 and 4.5. To accomplish the proof, it remains to show that a curve Γ^ε of positive solutions can be extended from $[0, A_1]$ to the whole interval $[0, \infty)$ with respect to a_1. Let $\hat{\Gamma}^\varepsilon$ be a maximal extension of Γ^ε in the direction $a_1 \geq A_1$ as a solution curve of (PP). According to the global bifurcation theorem of Rabinowitz [50], one of the following two properties (i) and (ii) must hold true;

(i) $\hat{\Gamma}^\varepsilon$ is unbounded in $X \times \boldsymbol{R}$,

(ii) $\hat{\Gamma}^\varepsilon$ meets the trivial solution set $(0, 0, a_1)$ or a semi-trivial solution curve at a point except for $(0, \varepsilon^{-1}\theta_{\lambda_1+\varepsilon b_1}, a_1^*)$.

We introduce a positive cone

$$P = \left\{(w, z);\ w > 0,\ z > 0 \text{ in } \Omega \text{ and } \frac{\partial w}{\partial \nu} < 0,\ \frac{\partial z}{\partial \nu} < 0 \text{ on } \partial\Omega\right\}.$$

Suppose that $(\hat{w}, \hat{z}, \hat{a}_1) \in \hat{\Gamma}^\varepsilon$ satisfies $(\hat{w}, \hat{z}) \in \partial P$ at $\hat{a}_1 \in (A_1, \infty)$. Then $(\hat{w}, \hat{z})$ satisfies $\hat{w} \geq 0$, $\hat{z} \geq 0$ in Ω and

$$\hat{w}(x_0)\hat{z}(x_0) = 0 \quad \text{at some } x_0 \in \Omega \tag{4.43}$$

or

$$\frac{\partial \hat{w}}{\partial \nu}(x_1)\frac{\partial \hat{z}}{\partial \nu}(x_1) = 0 \quad \text{at some } x_1 \in \partial\Omega. \tag{4.44}$$

By applying the strong maximum principle [49] to (PP), it can be seen from (4.43) and (4.44) that $\hat{w}$ or $\hat{z}$ must be identically zero.

We now recall Proposition 3.2, which asserts that positive solutions of (PP) bifurcate from the semi-trivial solution curve $\{(0, \varepsilon^{-1}\theta_{\lambda_1+\varepsilon b_1}, a_1);\ a_1 > 0\}$ if and only if $a_1 = a_1^*$. For $\tau \geq 0$, Proposition 3.1 implies that no positive solution bifurcates from other semi-trivial solution curve $\{(\varepsilon^{-1}\theta_{\lambda_1+\varepsilon a_1}, 0, a_1);\ a_1 > 0\}$. Moreover, it is easily verified that the trivial solution is nondegenerate. Therefore, we can deduce that $(\hat{w}, \hat{z}, \hat{a}_1) = (0, \varepsilon^{-1}\theta_{\lambda_1+\varepsilon b_1}, a_1^*)$, which contradicts $\hat{a}_1 > A_1 > a_1^*$.

These considerations imply that $\hat{\Gamma}^{\varepsilon} \setminus \{(0, \varepsilon^{-1}\theta_{\lambda_1+\varepsilon b_1}, a_1^*)\}$ is contained in P and, therefore, (ii) is excluded. Thus (i) must hold true. Lemma 3.1, together with (4.2) implies the boundedness of w and z

$$\begin{cases} w(x) \le \dfrac{1}{\varepsilon}(\lambda_1 + \varepsilon a_1), \\ z(x) \le \dfrac{1}{\varepsilon}\{1 + \beta(\lambda_1 + \varepsilon a_1)\}\{\lambda_1 + \varepsilon b_1 + \beta(\lambda_1 + \varepsilon\tau)(\lambda_1 + \varepsilon a_1)\} \end{cases}$$

for all $x \in \Omega$. Hence Γ^{ε} can be extended over $[0, \infty)$ with respect to a_1 as a curve of positive solutions of (PP). Thus we complete the proof. □

Proposition 4.1, together with Lemma 4.2, implies that Γ^{ε} forms an unbounded S-shaped curve with respect to a_1 provided that $0 \le \tau \le \tilde{\tau}, \gamma \ge \tilde{\gamma}, \infty)$ and $\varepsilon > 0$ is sufficiently small. Therefore, we have the following result.

PROPOSITION 4.2. *Suppose that* $(\tau, \gamma) \in [0, \tilde{\tau}] \times [\tilde{\gamma}, \infty)$ *and* $\varepsilon > 0$ *is sufficiently small. Then the set of positive solutions of* (PP) *contains an unbounded S-shaped curve* Γ^{ε} *which bifurcates from the semi-trivial solution curve* $\{(0, \varepsilon^{-1}\theta_{\lambda_1+\varepsilon b_1}, a_1); a_1 > 0\}$ *at* $a_1 = a_1^*(\varepsilon)$. *Furthermore, there exist two positive numbers* $\overline{a}_1(\varepsilon) > \underline{a}_1(\varepsilon)$ $(> a_1^*(\varepsilon))$ *with the following properties:*

(i) *if* $a_1 \in (0, a_1^*(\varepsilon)]$, *then* (PP) *has no positive solution;*
(ii) *if* $a_1 \in (a_1^*(\varepsilon), \underline{a}_1(\varepsilon)) \cup (\overline{a}_1(\varepsilon), \infty)$, *then* (PP) *has at least one positive solution;*
(iii) *if* $a_1 = \underline{a}_1(\varepsilon)$ *or* $a_1 = \overline{a}_1(\varepsilon)$, *then* (PP) *has at least two positive solutions;*
(iv) *if* $a_1 \in (\underline{a}_1(\varepsilon), \overline{a}_1(\varepsilon))$, *then* (PP) *has at least three positive solutions.*

PROOF. Let $S(\xi, \varepsilon) = (s(\xi, \varepsilon), t(\xi, \varepsilon), a_1(\xi, \varepsilon))$ be a smooth curve obtained in Proposition 4.1. We recall that $S(\xi, 0) = (\xi, h(\gamma\xi), k(\xi))$ and

$$\lim_{\varepsilon \to 0}(t(\xi, \varepsilon), a_1(\xi, \varepsilon)) = (h(\gamma\xi), k(\xi)) \quad \text{in } C^1([0, C]) \times C^1([0, C]),$$

where C is a positive constant defined in (4.27). After some calculations,

$$\begin{aligned} k'(0) &= m^* \left\{1 + c\gamma\tau + c\gamma b_1 \left(\frac{\|\phi_1\|_4^4}{\|\phi_1\|_6^6} - 1\right)\right\} \\ &\ge m^*(1 + c\gamma\tau) > 0. \end{aligned}$$

Hence we see from Lemma 4.2 that, if $(\tau, \gamma) \in [0, \tilde{\tau}] \times [\tilde{\gamma}, \infty)$ and $\varepsilon > 0$ is sufficiently small, then $k_{\varepsilon}(\xi) := a_1(\xi, \varepsilon)$ $(0 \le \xi \le C(\varepsilon))$ satisfies $k_{\varepsilon}'(0) > 0$, $k_{\varepsilon}(\xi) > k_{\varepsilon}(0) = a_1^*(\varepsilon)$ for all $\xi \in (0, C(\varepsilon)]$ and k_{ε} achieves its local maximum and local minimum at $\overline{\xi}(\varepsilon)$ and $\underline{\xi}(\varepsilon)$, respectively. Here $\overline{\xi}(\varepsilon)$ and $\underline{\xi}(\varepsilon)$ satisfy $\lim_{\varepsilon \to 0} \overline{\xi}(\varepsilon) = \overline{s}$ and $\lim_{\varepsilon \to 0} \underline{\xi}(\varepsilon) = \underline{s}$, where $\overline{s}$ and $\underline{s}$ are critical points of k appearing in Lemma 4.2.

Define $\overline{a}_1(\varepsilon) := k_{\varepsilon}(\overline{\xi}(\varepsilon))$, $\underline{a}_1(\varepsilon) := k_{\varepsilon}(\underline{\xi}(\varepsilon))$ and

$$K_{\varepsilon}(a_1) := \{\xi \in (0, \infty); k_{\varepsilon}(\xi) = a_1\}.$$

Obviously, if $\varepsilon > 0$ is small enough, then $K_{\varepsilon}(a_1)$ has no element for $a_1 \in (0, a_1^*(\varepsilon)]$; at least one element for $a_1 \in (a_1^*(\varepsilon), \underline{a}_1(\varepsilon)) \cup (\overline{a}_1(\varepsilon), A_1]$; at least two elements for

$a_1 = \underline{a}_1(\varepsilon)$ or $\overline{a}_1(\varepsilon)$; at least three elements for $a_1 \in (\underline{a}_1(\varepsilon), \overline{a}_1(\varepsilon))$. From (4.28), it should be noted that the number of elements of $K_\varepsilon(a_1)$ is equal to the number of positive solutions of (PP) provided that $\varepsilon \in (0, \varepsilon_0]$ and $a_1 \in (0, A_1]$. Moreover, the extension of Γ^ε implies that (PP) has at least one positive solution for $a_1 \in [A_1, \infty)$. Thus we get the assertion. □

4.4. *Multiple existence for (SP-3)*

We discuss the multiple existence of positive solutions for (SP-3) by making use of Proposition 4.2. Regard a as a parameter and set

$$S = \{(u, v, a);\ (u, v) \text{ is a positive solution of (SP-3) and } a > \lambda_1\}.$$

Our multiple existence result is stated as follows:

THEOREM 4.1. *Assume* $\min\{\beta b, d\} > \beta\lambda_1$. *For any* $c > 0$, *there exist a large number* M *and an open set*

$$\mathcal{O}_1 = \mathcal{O}_1(c) \subset \{(\beta, b, d);\ \beta \geq M, 0 < d/\beta - \lambda_1, b - \lambda_1 \leq M^{-1}\}$$

such that, if $(\beta, b, d) \in \mathcal{O}_1$, *then* S *contains an unbounded smooth curve*

$$\Gamma_1 = \{(u(s), v(s), a(s)) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \times (\lambda_1, \infty);\ s \in (0, \infty)\},$$

which possesses the following properties:

(i) $(u(0), v(0)) = (0, \theta_b), a(0) = a^* > \lambda_1, a'(0) > 0$, *where* a^* *is defined by* (3.24) *and satisfies* $(a^*, b) \in S_2$.

(ii) $a(s) > a(0)$ *for all* $s \in (0, \infty)$ *and* $\lim_{s\to\infty} a(s) = \infty$.

(iii) $a(s)$ *attains a strict local maximum at* $s = \overline{s}$ *and a strict local minimum at* $s = \underline{s}$ *with* $0 < \overline{s} < \underline{s}$.

REMARK 4.1. Let $\overline{a} := a(\overline{s})$ and $\underline{a} := a(\underline{s})$. Theorem 4.1 implies that (SP-3) has at least one positive solution if $a \in (a^*, \underline{a}) \cup (\overline{a}, \infty)$; at least two positive solutions if $a = \underline{a}$ or $a = \overline{a}$ and at least three positive solutions if $a \in (\underline{a}, \overline{a})$. See Figure 1.5.

PROOF OF THEOREM 4.2. By (4.2) it should be noted that the bifurcation point of Γ^ε in Proposition 4.1, $(w, z, a_1) = (0, \varepsilon^{-1}\theta_{\lambda_1+\varepsilon b_1}, a_1^*(\varepsilon))$, is mapped to $(U, V, a) = (0, \theta_b, \lambda_1(c\theta_b))$ on the semi-trivial solution curve $\{(0, \theta_b, a);\ a > 0\}$ of (RSP-3). Define

$$\mathcal{O}_1^0 := \{(\beta, b, d) = (\gamma/\varepsilon, \lambda_1 + \varepsilon b_1, (\lambda_1 + \varepsilon\tau)\gamma/\varepsilon);\ (\tau, \gamma) \in [0, \tilde{\tau}] \times [\tilde{\gamma}, \infty)\}.$$

It follows from Proposition 4.2 that, if $(\beta, b, d) \in \mathcal{O}_1^0$ and $\varepsilon > 0$ is sufficiently small, then the set of positive solutions for (RSP-3) contains an unbounded S-shaped curve Γ^* which bifurcates from the semi-trivial solution curve $\{(0, \theta_b, a);\ a > 0\}$ at $a = \lambda_1(c\theta_b)$. More precisely, if we define

$$\overline{a} = \lambda_1 + \varepsilon\overline{a}_1(\varepsilon) \quad \text{and} \quad \underline{a} = \lambda_1 + \varepsilon\underline{a}_1(\varepsilon),$$

then (RSP-3) has no positive solution for $a \in (0, \lambda_1(c\theta_b)]$; at least one positive solution for $a \in (\lambda_1(c\theta_b), \underline{a}) \cup (\overline{a}, \infty)$; at least two positive solutions for $a = \underline{a}$ or $\overline{a}$; at least three positive solutions for $a \in (\underline{a}, \overline{a})$. In view of the one-to-one correspondence between $(u, v) \geq 0$ and $(U, V) \geq 0$ through (4.1), define $\Gamma_1 = \{(u, v, a); (U, V, a) \in \Gamma^*\}$ for $(\beta, b, d) \in O_1^0$. Clearly, Γ_1 is contained in the set of positive solutions for (SP-3) and possesses at least two turning points with respect to a. Thus the proof is complete. □

REMARK 4.2. The above multiple existence for (SP-3) can be generalized to those for (SP-2) if α is sufficiently small, $\alpha > 0$. For details see [25].

In case $\tau < 0$ for (PP), we can obtain the following result which corresponds to Proposition 4.1 in case $\tau \geq 0$.

PROPOSITION 4.3. *Let $\tau < 0$. Then there exists a positive constant $\varepsilon_0 = \varepsilon_0(A_1)$ such that for each $\varepsilon \in (0, \varepsilon_0]$, all positive solutions of* (PP) *with $a_1 \in (0, A_1]$ are given by*

$$\Gamma^\varepsilon = ((s, t)\Phi + \varepsilon \boldsymbol{U}(s, t, a_1, \varepsilon), a_1); (s, t, a_1) \in \{S(\xi, \varepsilon); 0 < \xi < C(\varepsilon)\}\},$$

where $S(\xi, \varepsilon) \in \boldsymbol{R}^3$ is a suitable smooth curve for $(\xi, \varepsilon) \in [0, C(\varepsilon)] \times [0, \varepsilon_0]$ satisfying

$$S(\xi, 0) = (\xi, h(\gamma\xi), k(\xi)), \qquad S(0, \varepsilon) = (0, t(\varepsilon), a_1^*(\varepsilon)),$$
$$\text{and} \quad S(C(\varepsilon), \varepsilon) = (s(\varepsilon), 0, a_{1*}(\varepsilon)),$$

where $a_{1*}(\varepsilon)$ and $a_1^*(\varepsilon)$ are defined by (4.3) *and* (4.4)*, respectively. Moreover,*

$$t(\varepsilon) := \varepsilon^{-1} \int_\Omega \theta_{\lambda_1 + \varepsilon b_1} \Phi, \qquad s(\varepsilon) := \varepsilon^{-1} \int_\Omega \theta_{\lambda_1 + \varepsilon a_{1*}(\varepsilon)} \Phi$$

and $C(\varepsilon)$ is a certain smooth function in $[0, \varepsilon_0]$ satisfying $C(0) = s_0/\gamma$.

Proposition 4.3 gives an important information on the solution curve connecting S_2 and S_3 discussed in Section 3. Making use of Lemma 4.2 one can prove the following result corresponding to Proposition 4.2 for $\tau \geq 0$.

PROPOSITION 4.4. *Assume that $(\tau, \gamma) \in [-\tilde{\tau}, 0) \times [\tilde{\gamma}, \infty)$ and that $\varepsilon > 0$ is sufficiently small. Then the set of positive solutions of* (PP) *contains a bounded smooth curve*

$$\Gamma^\varepsilon = \{(w(\xi, \varepsilon), z(\xi, \varepsilon), a_1(\xi, \varepsilon)); \xi \in (0, C(\varepsilon))\},$$

which possesses the following properties:

(i) $(w(0, \varepsilon), z(0, \varepsilon), a_1(0, \varepsilon)) = (0, \varepsilon^{-1}\theta_{\lambda_1 + \varepsilon b_1}, a_1^*(\varepsilon))$, $a_1'(0, \varepsilon) > 0$.

(ii) $(w(C(\varepsilon), \varepsilon), z(C(\varepsilon), \varepsilon), a_1(C(\varepsilon), \varepsilon)) = (\varepsilon^{-1}\theta_{\lambda_1 + \varepsilon a_{1*}(\varepsilon)}, 0, a_{1*}(\varepsilon))$.

(iii) *$a_1(\xi, \varepsilon)$ attains a strict local maximum in $(0, C(\varepsilon))$. In particular, if $(\tau, \gamma) \in [-\tilde{\tau}, 0) \times [\tilde{\gamma}, \hat{\gamma}(\tau))$, then $a_1(\xi, \varepsilon)$ attains a strict local minimum in $(0, C(\varepsilon))$.*

For $\beta b > \beta\lambda_1 > d$, Proposition 4.4 gives an interesting information on the branch of positive solutions for (SP-3); this branch bifurcates from the semi-trivial solution curve $\{(0, \theta_b, a;\ a > \lambda_1\}$ and connects the other semi-trivial solution curve $\{\theta_a, 0, a > \lambda_1\}$.

THEOREM 4.2. *Let S_2 and S_3 be two curves defined by* (1.9) *and* (1.10)*, respectively, and assume $\beta b > \beta\lambda_1 > d$. For any $c > 0$, there exist a large number M and an open set*

$$\mathcal{O}_2 = \mathcal{O}_2(c) \subset \{(\beta, b, d);\ \beta \geq M, 0 < \lambda_1 - d/\beta, b - \lambda_1 \leq M^{-1}\}$$

such that, if $(\alpha, \beta, b, d) \in \mathcal{O}_2$, then S contains a bounded smooth curve

$$\Gamma_2 = \{(u(s), v(s), a(s)) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \times (\lambda_1, \infty);\ s \in (0, C)\},$$

which possesses the following properties:

(i) $(u(0), v(0)) = (0, \theta_b), a(0) = a^* > \lambda_1, a'(0) > 0$, *where a^* is defined by* (3.24) *and satisfies $(a^*, b) \in S_2$.*

(ii) $(u(C), v(C)) = (\theta_{a(C)}, 0), a(C) = a_* > \lambda_1$, *where a_* is defined by* (3.18) *and satisfies $(a_*, b) \in S_3$.*

(iii) *Γ_2 has at least one turning point with respect to a. Furthermore, there exists an open set $\mathcal{O}_2' \subset \mathcal{O}_2$ such that, if $(\beta, b, d) \in \mathcal{O}_2'$, then Γ_2 has at least two turning points with respect to a.*

4.5. *Stability analysis of multiple solutions of (SP-3)*

In this subsection we will study the stability of multiple positive solutions of (SP-3), which are obtained in the previous subsection. We follow the method used by Kuto [23]. The nonstationary problem associated with (SP-3) is

$$\begin{cases} u_t = \Delta u + u(a - u - cv) & \text{in } \Omega \times (0, \infty), \\ \rho v_t = \Delta[(1 + \beta u)v] + v(b + du - v) & \text{in } \Omega \times (0, \infty), \\ u = v = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(\cdot, 0) = u_0, \quad v(\cdot, 0) = v_0 & \text{in } \Omega, \end{cases} \tag{4.45}$$

where ρ is a positive constant and u_0, v_0 are nonnegative functions. In the same way as Section 4.4, we introduce a pair of functions (w, z) by

$$u = \varepsilon w \quad \text{and} \quad (1 + \beta u)v = \varepsilon z \tag{4.46}$$

and a_1, b_1, τ, γ by (4.2). Then (4.45) is rewritten as

$$\begin{cases} w_t = \Delta w + \lambda_1 w + \varepsilon B_1(w, z, a_1) & \text{in } \Omega \times (0, \infty), \\ \rho\left[-\dfrac{\gamma z w_t}{(1 + \gamma w)^2} + \dfrac{z_t}{1 + \gamma w}\right] & \\ \qquad = \Delta z + \lambda_1 z + \varepsilon B_2(w, z) & \text{in } \Omega \times (0, \infty), \\ w = z = 0 & \text{on } \partial\Omega \times (0, \infty), \\ w(\cdot, 0) = u_0/\varepsilon, \quad z(\cdot, 0) = (1 + \beta u_0)v_0/\varepsilon & \text{in } \Omega, \end{cases} \tag{4.47}$$

where

$$\begin{aligned} B_1(w, z, a_1) &= w\left(a_1 - w - \frac{cz}{1 + \gamma w}\right), \\ B_2(w, z) &= \frac{z}{1 + \gamma w}\left(b_1 + \tau\gamma w - \frac{z}{1 + \gamma w}\right), \end{aligned} \tag{4.48}$$

(see (4.5)). Recall that (PP) is the stationary problem corresponding to (4.47).

By Propositions 4.1 and 4.3, all positive solutions of (PP) with $a \in (0, A_1)$ are parameterized as

$$\Gamma^\varepsilon = \{(w(\xi,\varepsilon), z(\xi,\varepsilon), a_1(\xi,\varepsilon)); \xi \in (0, C(\varepsilon))\}$$

for sufficiently small $\varepsilon > 0$. Let X and Y be Banach spaces defined by (2.16) with $p > N$. For each $(w(\xi,\varepsilon), z(\xi,\varepsilon), a_1(\xi,\varepsilon)) \in \Gamma^\varepsilon$, define a linear operator $L(\xi,\varepsilon)$ from X to Y by

$$L(\xi,\varepsilon)\begin{pmatrix}\varphi\\ \psi\end{pmatrix} := -H\begin{pmatrix}\varphi\\ \psi\end{pmatrix} - \varepsilon B_{(w,z)}(w(\xi,\varepsilon), z(\xi,\varepsilon), a_1(\xi,\varepsilon))\begin{pmatrix}\varphi\\ \psi\end{pmatrix},$$

where H and B are mappings defined by (4.5) and $B_{(w,z)}$ denotes the Fréchet derivative of B with respect to (w, z). By taking account of the left-hand side of (4.47) it is better to define the following matrix

$$J(\xi,\varepsilon) := \begin{pmatrix} 1 & 0 \\ -\dfrac{\rho\gamma z(\xi,\varepsilon)}{(1+\gamma w(\xi,\varepsilon))^2} & \dfrac{\rho}{1+\gamma w(\xi,\varepsilon)} \end{pmatrix}.$$

The linearized eigenvalue problem associated with $(w(\xi,\varepsilon), z(\xi,\varepsilon))$ is given by

$$L(\xi,\varepsilon)\begin{pmatrix}\varphi\\ \psi\end{pmatrix} = \mu J(\xi,\varepsilon)\begin{pmatrix}\varphi\\ \psi\end{pmatrix}. \tag{4.49}$$

For this eigenvalue problem it is possible to show the following result.

LEMMA 4.6. *Assume that $\varepsilon > 0$ is sufficiently small. Then all eigenvalues $\{\mu_j(\xi,\varepsilon)\}_{j=1}^{\infty}$ of* (4.49) *satisfy*

$$\lim_{\varepsilon\to 0}\mu_1(\xi,\varepsilon) = \lim_{\varepsilon\to 0}\mu_2(\xi,\varepsilon) = 0 \tag{4.50}$$

and

$$\operatorname{Re}\mu_j(\xi,\varepsilon) > \kappa \quad \textit{for all } i \geq 3 \textit{ and } \xi \in (0, C(\varepsilon)) \tag{4.51}$$

with a positive constant κ independent of (ξ,ε).

PROOF. Let $\xi \in (0, C(\varepsilon)$ be fixed. By Propositions 4.1 and 4.3

$$\lim_{\varepsilon\to 0}(w(\xi,\varepsilon), z(\xi,\varepsilon), a_1(\xi,\varepsilon)) = (\xi\phi_1, h(\gamma\xi)\phi_1, k(\xi)) \quad \text{in } C(\Omega)\times C(\Omega)\times \boldsymbol{R},$$

where h and k are functions defined by (4.14). Hence letting $\varepsilon \to 0$ in (4.49) leads to

$$\begin{cases} -\Delta\varphi - \lambda_1\varphi = \mu\varphi & \text{in } \Omega,\\ -\Delta\psi - \lambda_1\psi = \rho\mu\left[-\dfrac{\gamma h(\gamma\xi)\phi_1}{(1+\gamma\xi\phi_1)^2}\varphi + \dfrac{1}{1+\gamma\xi\phi_1}\psi\right] & \text{in } \Omega,\\ \varphi = \psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.52}$$

It is easy to see that zero is a double eigenvalue for (4.52); this fact implies (4.50). If $\varphi \not\equiv 0$, it follows from the first equation of (4.52) that any eigenvalue μ ($\neq 0$) is real and positive. If $\varphi \equiv 0$, the second equation of (4.52) enables us to conclude that any eigenvalue μ ($\neq 0$) is also real and positive. These results imply that all eigenvalues of (4.52) except for zero eigenvalue are real and positive. Therefore, we can prove (4.51) by employing the perturbation theory of Kato [22, Chapter VIII]. □

It should be noted that, if μ is an eigenvalue of (4.49), then its complex conjugate $\overline{\mu}$ is also an eigenvalue of (4.49). Therefore, $\mu_i(\xi,\varepsilon)$ $(i=1,2)$ fulfill the following properties:

(i) both $\mu_1(\xi,\varepsilon)$ and $\mu_2(\xi,\varepsilon)$ are real numbers,

(ii) $\overline{\mu_1}(\xi,\varepsilon)=\mu_2(\xi,\varepsilon)$.

In what follows, we take $\mu_1(\xi,\varepsilon)$ and $\mu_2(\xi,\varepsilon)$ so that they satisfy $\mu_1(\xi,\varepsilon)\le\mu_2(\xi,\varepsilon)$ in case (i) and that Im $\mu_1(\xi,\varepsilon)\ge$ Im $\mu_2(\xi,\varepsilon)$ in case (ii). We can study the stability of positive solutions of (PP) by employing the abstract linearization principle of Potier-Ferry [47] (for details, see Kuto [23, Section 4]).

Let $(w(\xi,\varepsilon),z(\xi,\varepsilon),a_1(\xi,\varepsilon))\in\Gamma^\varepsilon$ for sufficiently small $\varepsilon>0$. If Re $\mu_1(\xi,\varepsilon)>0$, then it follows from Lemma 4.6 that real parts of all eigenvalues of (4.49) are positive for sufficiently small $\varepsilon>0$. In this case, one can show that $(w(\xi,\varepsilon),z(\xi,\varepsilon),a_1(\xi,\varepsilon))$ is asymptotically stable. On the other hand, if Re $\mu_1(\xi,\varepsilon)<0$, then it can be shown that $(w(\xi,\varepsilon),z(\xi,\varepsilon),a_1(\xi,\varepsilon))$ is unstable. Therefore, we will concentrate ourselves on investigating the sign of Re $\mu_1(\xi,\varepsilon)$.

It is convenient to define matrices $K(s)$ and $M(s)$ by

$$\begin{aligned} K(s)&=\begin{pmatrix} 1 & 0\\ -\rho\gamma h(\gamma s)\displaystyle\int_\Omega\frac{\phi_1^3}{(1+\gamma s\phi_1)^2}dx & \rho\displaystyle\int_\Omega\frac{\phi_1^2}{1+\gamma s\phi_1}dx\end{pmatrix},\\ M(s)&=-K(s)^{-1}F^0_{(s,t)}(s,h(\gamma s),k(s)),\end{aligned}\tag{4.53}$$

where $F^0(s,t,a_1)$ is a mapping defined by (4.13) and $F^0_{(s,t)}$ denotes the derivative of F^0 with respect to s,t. The following lemma plays an important role in determining the sign of Re $\mu_1(\xi,\varepsilon)$.

LEMMA 4.7. *Let $\nu_1(s)$ and $\nu_2(s)$ be eigenvalues of $M(s)$ satisfying* Re $\nu_1(s)\le$ Re $\nu_2(s)$ *and* Im $\nu_1(s)\ge$ Im $\nu_2(s)$. *Then for any $s\in(0,C(0))$, it holds that*

$$\lim_{(\xi,\varepsilon)\to(s,0)}\frac{\mu_i(\xi,\varepsilon)}{\varepsilon}=\nu_i(s)\quad\text{for } i=1,2.\tag{4.54}$$

PROOF. For any sequence $\{(\xi_n,\varepsilon_n)\}_{n=1}^\infty$ such that $\lim_{n\to\infty}(\xi_n,\varepsilon_n)=(s,0)$, set

$$\begin{aligned}&(w_n,z_n,a_1^n):=(w(\xi_n,\varepsilon_n),z(\xi_n,\varepsilon_n),a_1(\xi_n,\varepsilon_n))\in\Gamma^{\varepsilon_n},\\ &\mu_i^n:=\mu_i(\xi_n,\varepsilon_n)\quad(i=1,2).\end{aligned}$$

Let (φ_i^n,ψ_i^n) $(\|\varphi_i^n\|+\|\psi_i^n\|=1)$ be eigenfunctions of (4.49) associated with eigenvalues μ_i^n. Then (4.49) can be written as

$$\begin{cases}-\Delta\varphi_i^n-\lambda_1\varphi_i^n-\varepsilon_n[B_{1w}(w_n,z_n,a_1^n)\varphi_i^n+B_{1z}(w_n,z_n,a_1^n)\psi_i^n]\\ \qquad=\mu_i^n\varphi_i^n & \text{in }\Omega,\\ -\Delta\psi_i^n-\lambda_1\psi_i^n-\varepsilon_n[B_{2w}(w_n,z_n)\varphi_i^n+B_{2z}(w_n,z_n)\psi_i^n]\\ \qquad=\rho\mu_i^n\left[-\dfrac{\gamma z_n}{(1+\gamma w_n)^2}\varphi_i^n+\dfrac{1}{1+\gamma w_n}\psi_i^n\right] & \text{in }\Omega,\\ \varphi_i^n=\psi_i^n=0 & \text{on }\partial\Omega,\end{cases}$$

where $B_1(w, z, a_1)$ and $B_2(w, z)$ are defined by (4.48). We take L^2-inner product of the above differential systems with ϕ_1 to get

$$\begin{cases} \displaystyle\int_\Omega B_{1w}(w_n, z_n, a_1^n)\varphi_i^n\phi_1 dx + \int_\Omega B_{1z}(w_n, z_n, a_1^n)\psi_i^n\phi_1 dx \\ \qquad\qquad \displaystyle = -\frac{\mu_i^n}{\varepsilon_n}\int_\Omega \varphi_i^n\phi_1\, dx, \\ \displaystyle\int_\Omega B_{2w}(w_n, z_n)\varphi_i^n\phi_1 dx + \int_\Omega B_{2z}(w_n, z_n)\psi_i^n\phi_1 dx \\ \qquad\qquad \displaystyle = -\frac{\rho\mu_i^n}{\varepsilon_n}\left[-\gamma\int_\Omega \frac{z_n}{(1+\gamma w_n)^2}\varphi_i^n\phi_1 dx + \int_\Omega \frac{1}{1+\gamma w_n}\psi_i^n\phi_1 dx\right]. \end{cases} \tag{4.55}$$

From the proof of Lemma 4.6, we may assume that subject to a subsequence,

$$\lim_{n\to\infty}(\varphi_i^n, \psi_i^n) = (p_i\phi_1, q_i\phi_1) \quad \text{in } C(\overline{\Omega})\times C(\overline{\Omega})$$

with some (p_i, q_i) satisfying $|p_i| + |q_i| = 1$ $(i = 1, 2)$. Moreover, it follows from the assumption that

$$\lim_{n\to\infty}(w_n, z_n, a_1^n) = (s\phi_1, h(\gamma s)\phi_1, k(s)) \quad \text{in } C(\overline{\Omega})\times C(\overline{\Omega})\times \boldsymbol{R}.$$

Hence

$$\begin{aligned} &\lim_{n\to\infty}\begin{pmatrix} \displaystyle\int_\Omega B_{1w}(w_n, z_n, a_1^n)\varphi_i^n\phi_1 dx & \displaystyle\int_\Omega B_{1z}(w_n, z_n, a_1^n)\psi_i^n\phi_1 dx \\ \displaystyle\int_\Omega B_{2w}(w_n, z_n)\varphi_i^n\phi_1 dx & \displaystyle\int_\Omega B_{2z}(w_n, z_n)\psi_i^n\phi_1 dx \end{pmatrix} \\ &= \begin{pmatrix} \displaystyle p_i\int_\Omega B_{1w}(s\phi_1, h(\gamma s)\phi_1, k(s))\phi_1^2 dx & \displaystyle q_i\int_\Omega B_{1z}(s\phi_1, h(\gamma s)\phi_1, k(s))\phi_1^2 dx \\ \displaystyle p_i\int_\Omega B_{2w}(s\phi_1, h(\gamma s)\phi_1)\phi_1^2 dx & \displaystyle q_i\int_\Omega B_{2z}(s\phi_1, h(\gamma s)\phi_1)\phi_1^2 dx \end{pmatrix}. \end{aligned}$$

Here, by virtue of (4.12) and (4.13), it should be noted that

$$\begin{aligned} &F^0_{(s,t)}(s, h(\gamma s), k(s))* \\ &= \begin{pmatrix} \displaystyle\int_\Omega B_{1w}(s\phi_1, h(\gamma s)\phi, k(s))\phi_1^2 dx & \displaystyle\int_\Omega B_{1z}(s\phi_1, h(\gamma s)\phi, k(s))\phi_1^2 dx \\ \displaystyle\int_\Omega B_{2w}(s\phi_1, h(\gamma s)\phi_1)\phi_1^2 dx & \displaystyle\int_\Omega B_{2z}(s\phi_1, h(\gamma s)\phi_1)\phi_1^2 dx \end{pmatrix}. \end{aligned}$$

Therefore, letting $n \to \infty$ in (4.55) leads us to

$$F^0_{(s,t)}(s, h(\gamma s), k(s))\begin{pmatrix} p_i \\ q_i \end{pmatrix} = -\left(\lim_{n\to\infty}\frac{\mu_i^n}{\varepsilon_n}\right)K(s)\begin{pmatrix} p_i \\ q_i \end{pmatrix}. \tag{4.56}$$

Since $|p_i| + |q_i| = 1$ $(i = 1, 2)$, we see from (4.53) and (4.56) that $\lim_{n\to\infty}\mu_i^n/\varepsilon_n$ $(i = 1, 2)$ become eigenvalues of $M(s)$. Hence μ_i^n satisfy (4.54) for each $i = 1, 2$. Here note that $\nu_i(s)$ is independent of subsequences; so that μ_i^n/ε_n itself converges to $\nu_i(s)$ for each $i = 1, 2$. Thus the proof is complete. □

The following lemma is due to Du and Lou [15, Theorem 3.13 and Appendix] and gives an important result on the degeneracy of $(w(\xi,\varepsilon), z(\xi,\varepsilon), a_1(\xi,\varepsilon) \in \Gamma^\varepsilon$.

LEMMA 4.8. *Assume that $\varepsilon > 0$ is sufficiently small. Thus all zeros of $\mu_1(\xi,\varepsilon)$ coincide with those of $\partial_\xi a_1(\xi,\varepsilon)$ for $\xi \in (0, C_\varepsilon)$.*

This lemma asserts that any positive solution on Γ^ε is degenerate if and only if $a_1(\xi,\varepsilon)$ is critical with respect to ξ provided that $\varepsilon > 0$ is sufficiently small.

Since k is analytic, k' possesses at most a finite number of zeros in $(0, C_0)$. Furthermore, by virtue of (4.14)), any zero of k' must be a strictly critical point of k for almost every $(\tau,\gamma) \in [-\tilde{\tau},\tilde{\tau}] \times [\tilde{\gamma},\infty)$ with some positive numbers $\tilde{\tau}$ and $\tilde{\gamma}$. For such $(\tau,\gamma) \in [-\tilde{\tau},\tilde{\tau}] \times [\tilde{\gamma},\infty)$ and sufficiently small $\varepsilon > 0$, denote all zeros of $\partial_\xi a_1(\xi,\varepsilon)$ by

$$0 < \xi_1(\varepsilon) < \xi_2(\varepsilon) < \cdots < \xi_{m-1}(\varepsilon) < C(\varepsilon).$$

Then

$$(w_i, z_i, a_1^i) := (w(\xi_i(\varepsilon),\varepsilon), z(\xi_i(\varepsilon),\varepsilon), a_1(\xi_i(\varepsilon),\varepsilon)) \in \Gamma^\varepsilon \quad (i = 1, 2, \ldots, m-1)$$

are all turning points on curve Γ^ε with respect to a_1. We recall that $\lim_{\varepsilon\downarrow 0} a_1(\,\cdot\,,\varepsilon) = k$ in $C^2([0, C_0])$ by Propositions 4.1 and 4.3. We now define

$$\Gamma_i^\varepsilon := \{((w(\xi,\varepsilon), z(\xi,\varepsilon), a_1(\xi,\varepsilon)) : \xi \in (\xi_{i-1}(\varepsilon), \xi_i(\varepsilon))\} \quad \text{for each } 1 \le k \le m,$$

where $\xi_0(\varepsilon) := 0$ and $\xi_m(\varepsilon) = C(\varepsilon)$. Then

$$\bigcup_{i=1}^{m} \Gamma_i^\varepsilon = \Gamma^\varepsilon \setminus \bigcup_{i=1}^{m-1} \{(w_i, z_i, a_1^i)\}.$$

We are ready to discuss stability properties of positive solutions on Γ_i^ε.

PROPOSITION 4.5. *For almost every $(\tau,\gamma) \in [-\tilde{\tau},\tilde{\tau}] \times [\tilde{\gamma},\infty)$, there exist small positive constants δ and ε_0 such that, if $\rho \le \delta$ and $\varepsilon \le \varepsilon_0$, then any positive solution on $\Gamma_{2j-1}^\varepsilon$ $(j = 1, 2, \ldots, [(m+1)/2])$ is asymptotically stable, while any steady-state solution on Γ_{2j}^ε $(j = 1, 2, \ldots, [m/2])$ is unstable.*

PROOF. We take the trace of $M(s)$. After some calculations, it is possible to show

$$\begin{aligned} &\nu_1(s) + \nu_2(s) \\ &\quad = \frac{h(\gamma s)}{\rho}\left[\int_\Omega \frac{\phi_1^3}{(1+\gamma s\phi_1)^2}dx \left(\int_\Omega \frac{\phi_1^2}{1+\gamma s\phi_1}dx\right)^{-1} - \rho c\gamma s \int_\Omega \frac{\phi_1^4}{(1+\gamma s\phi_1)^2}dx\right] \\ &\qquad + m^* s + c\gamma s h(\gamma s) \int_\Omega \frac{\phi_1^3}{1+\gamma s\phi_1}dx \int_\Omega \frac{\phi_1^3}{(1+\gamma s\phi_1)^2}dx \left(\int_\Omega \frac{\phi_1^2}{1+\gamma s\phi_1}dx\right)^{-1}. \end{aligned} \tag{4.57}$$

We set $\ell(s) := \int_\Omega s\phi_1^4/(1+s\phi_1)^2 dx$. Since $\ell(0)=0$ and $\ell(s)=O(s^{-1})$ as $s\to\infty$, we see $\ell(\hat{s}) = \sup_{s>0}\ell(s)$ with some $\hat{s}>0$. Recall $h(\gamma s)>0$ for $s\in[0,C_0)$ with $C_0=C(0)$. Then it follows from (4.57) that

$$\begin{aligned}
&\nu_1(s)+\nu_2(s)\\
&\quad\geq \frac{h(\gamma s)}{\rho}\left[\int_\Omega \frac{\phi_1^3}{(1+\gamma s\phi_1)^2}dx\left(\int_\Omega \frac{\phi_1^2}{1+\gamma s\phi_1}dx\right)^{-1} - \rho c\ell(\gamma s)\right] + m^* s\\
&\quad> \frac{h(\gamma s)}{\rho}\left[\int_\Omega \frac{\phi_1^3}{(1+\gamma C_0\phi_1)^2}dx - \rho c\ell(\hat{s})\right] + m^* s
\end{aligned}$$

for all $s\in[0,C_0]$. Therefore, making use of $h(\gamma s)>0$ ($s\in[0,C_0)$) we see that, if

$$\rho < \frac{1}{2c\ell(\hat{s})}\int_\Omega \frac{\phi_1^3}{(1+\gamma C_0\phi_1)^2}dx,$$

then $\nu_1(s)+\nu_2(s)>0$ for all $s\in[0,C_0]$. Hence Lemma 4.7 implies that for sufficiently small $\varepsilon>0$

$$\mu_1(\xi,\varepsilon)+\mu_2(\xi,\varepsilon)>0 \quad \text{for all } \xi\in[0,C_\varepsilon]. \tag{4.58}$$

Since Re $\mu_1(\xi,\varepsilon)\leq$ Re $\mu_2(\xi,\varepsilon)$, it follows from (4.58) that Re $\mu_2(\xi,\varepsilon)>0$ for all $\xi\in[0,C_\varepsilon]$ provided that $\varepsilon>0$ is sufficiently small.

We will study the sign of Re $\nu_1(s)$ by using (4.13), (4.14) and (4.53) (note that Re $\nu_2(s)>0$). After some calculations one can derive

$$\begin{aligned}
\nu_1(s)\nu_2(s) &= \det M(s)\\
&= \frac{sh(\gamma s)k'(s)}{\rho}\int_\Omega \frac{\phi_1^3}{(1+\gamma s\phi_1)^2}dx\left(\int_\Omega \frac{\phi_1^2}{1+\gamma s\phi_1}dx\right)^{-1}.
\end{aligned} \tag{4.59}$$

Therefore, we see sign $\nu_1(s)\nu_2(s) =$ sign $k'(s)$ for every $s\in(0,C(0))$.

Let $s_0\in(0,C_0)$ be any point. If $k'(s_0)>0$, then it follows from (4.54) and (4.59) that $\mu_1(\xi,\varepsilon)\mu_2(\xi,\varepsilon)>0$ if (ξ,ε) is sufficiently close to $(s_0,0)$. This fact, together with (4.58), yields Re $\mu_1(\xi,\varepsilon)>0$. Similarly, if $k'(s_0)<0$ and (ξ,ε) is sufficiently close to $(s_0,0)$, then Re $\mu_1(\xi,\varepsilon)<0$. Furthermore, it follows from Lemma 4.8 that $\mu_1(\xi,\varepsilon)=0$ if and only if $\xi=\xi_i(\varepsilon)$ with some $1\leq i\leq m-1$ when $\varepsilon>0$ is sufficiently small. Here we recall Re $\mu_2(\xi,\varepsilon)>0$ for all $\xi\in[0,C_\varepsilon]$. Therefore, it is possible to prove that Re $\mu_1(\xi,\varepsilon)=0$ if and only if $\xi=\xi_i(\varepsilon)$ for some $1\leq i\leq m-1$. Since $k'(0)>0$ if $(\tau,\gamma)\in[-\tilde{\tau},\tilde{\tau}]\times[\tilde{\gamma},\infty)$ (see Lemma 4.2 and its proof), the above considerations enable us to conclude

$$\begin{cases}
\text{Re}\,\mu_1(\xi,\varepsilon)>0 & \text{if } (w(\xi,\varepsilon),z(\xi,\varepsilon),a_1(\xi,\varepsilon))\in\Gamma_{2j-1}^\varepsilon,\\
\text{Re}\,\mu_1(\xi,\varepsilon)<0 & \text{if } (w(\xi,\varepsilon),z(\xi,\varepsilon),a_1(\xi,\varepsilon))\in\Gamma_{2j}^\varepsilon.
\end{cases}$$

Thus the proof is complete. □

We will return to (SP-3). Let S be a set of (u, v, a) such that (u, v) is a positive solution of (SP-3) with $a > \lambda_1$. Theorem 4.1 asserts that, if $(\beta, b, d) \in \mathcal{O}_1$, then S contains an unbounded smooth curve $\Gamma_1 = \{(u(s), v(s), a(s)) \in C(\overline{\Omega}) \times C(\overline{\Omega}) \times (\lambda_1, \infty);\ s \in (0, \infty)\}$ and that $a(s)$ has multiple critical points for $0 < s < C$ with sufficiently large C. It is easy to prove the following result with use of Proposition 4.5.

THEOREM 4.3. *Assume* $\min\{\beta b, d\} > \beta\lambda_1$. *Let* $0 < s_1 < s_2 < \cdots < s_{m-1} < C$ *be all strict local maximum or minimum points of* $a(s)$ *in* $(0, C)$ *and define* $\Gamma_{1,i} := \{(u(s), v(s), a(s)) \in \Gamma_1;\ s \in (s_{i-1}, s_i)\}$ $(1 \leq i \leq m)$, *where* $s_0 := 0$ *and* $s_m := C$. *Then for almost every* $(\beta, b, d) \in \mathcal{O}_1$, *there exists a positive constant* δ *such that, if* $\rho \leq \delta$ *and* $s \in (0, C)$, *then any positive solution on* $\Gamma_{1,2j-1}$ $(j = 1, 2, \ldots, [(m+1)/2])$ *is asymptotically stable, while any positive solution on* $\Gamma_{1,2j}$ $(j = 1, 2, \ldots, [m/2])$ *is unstable.*

Note that $(u(0), v(0)) = (0, \theta_b)$. Theorem 4.3 implies that stable positive solutions bifurcate from $(0, \theta_b)$ and that the stability of solutions on Γ_1 changes at every turning point with respect to a. For a bounded smooth curve Γ_2 obtained in Theorem 4.2, similar results to Theorem 4.3 also hold true (see [23]).

REMARK 4.3. In case $(\beta, b, d) \in \mathcal{O}_1$, it is possible to show that, if ρ is sufficiently large, then the Hopf bifurcation occurs at certain $(u(s^*), v(s^*), a(s^*)) \in \Gamma_{1,1}$. For details, see Kuto [23].

Acknowledgement

Part of this work has been supported by Grant-in-Aid for Scientific Research (C-18540223), Ministry of Education, Science, Sports and Culture in Japan.

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CHAPTER 7

Fixed Point Theory and Elliptic Boundary Value Problems

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Abstract

Let $\Omega \subset \mathbf{R}^n$ be a bounded smooth domain. For $k > 1$, consider the following non-linear elliptic boundary value problem

$$\begin{aligned} &\div \mathbf{A}^l(x, \mathbf{u}, D\mathbf{u}) + f^l(x, \mathbf{u}, D\mathbf{u}) = 0 \text{ in } \Omega, \quad l = 1, \ldots, k, \\ &\mathbf{u}(\mathbf{x}) = \mathbf{u}_0(\mathbf{x}), \quad \text{on } \partial\Omega, \end{aligned} \tag{I}$$

where

$$\begin{aligned} &\mathbf{u} : \Omega \to \mathbf{R}^k, \quad f^l : \Omega \times \mathbf{R}^k \times \mathbf{R}^{nk} \to R, \\ &\quad A^l : \Omega \times \mathbf{R}^k \times \mathbf{R}^{nk} \to \mathbf{R}^n, \quad l = 1, \ldots, k, \end{aligned}$$

are vector-valued functions. Under appropriate conditions on the functions f^l and $\mathbf{A}^l$, we investigate the question of existence of positive solutions to the boundary value problem (I) via the fixed point theory. The a priori estimates play a key role.

Keywords: Reaction-diffusion system, Competition model, Prey–predator model, Cross-diffusion, Degree theory, Bifurcation theory, Maximum principle

AMS Subject Classifications: primary 35K57; secondary 35B32, 35B50, 35J65, 37C25, 92D25

Contents

HANDBOOK OF DIFFERENTIAL EQUATIONS
Stationary Partial Differential Equations, volume 6
Edited by M. Chipot

1. Fixed point theorems

The fixed point theory has been a classical and powerful tool in nonlinear analysis. The subject has been extensively studied and there is an extremely rich literature. Our interest on the subject primarily stems from its broad applications in studying existence of solutions to nonlinear differential equations. In this section, we only state several basic forms of fixed point theorems which we believe can characterize the essential features of the fixed point theory, and are sufficient for treating the nonlinear elliptic boundary value problems we are interested in.

1.1. *The Banach fixed point theorem*

We first give the Banach fixed point theorem on a complete metric space, which is perhaps the simplest fixed point theorem. Let X be a Hausdorff space and M a subset of X. Let $\Lambda : M \mapsto X$ be a map. We say that $x \in M$ is a fixed point of Λ, provided

$$x = \Lambda(x).$$

Let (X, d) be a complete metric space and M a closed subset of X. We say that a map $\Lambda : M \mapsto M$ is a δ-contraction, provided there exists a nonnegative number $\delta \in [0, 1)$ such that

$$d(\Lambda(x), \Lambda(y)) \leq \delta d(x, y) \quad \forall x,\ y \in M,$$

where $d(\cdot, \cdot)$ is the distance function on X.

THEOREM 1.1.1 (Banach fixed point theorem). *Let (X, d) be a complete metric space and M a closed subset of X. Assume that $\Lambda : M \mapsto M$ is a δ-contraction for some $\delta \in [0, 1)$. Then Λ has a unique fixed point $x \in M$.*

The Banach fixed point theorem is proved by an elementary iterative procedure (successive approximation). Taking any starting-point $x_0 \in M$, one defines a sequence

$$x_{n+1} = \Lambda(x_n) \in M, \quad n = 0, 1, \ldots.$$

Then the δ-contractiveness of Λ ensures that the sequence $\{x_n\}$ converges to a point $x \in M$ (M being closed in X being complete). Moreover, x is unique in M.

The above simple iterative procedure also provides additional useful estimates on errors and rate of convergence. The Banach fixed point theorem is particularly effective in obtaining local existence, say, for initial value problems. But it has limited applications in boundary value problems, partly because the required δ-contractiveness is usually not a trivial matter to be verified.

To illustrate the importance of the δ-contractiveness, we state the following fixed point theorem due to Browder, Göhde and Kirk.

THEOREM 1.1.2 (Browder–Göhde–Kirk fixed point theorem). *Let H be a Hilbert space and M a closed bounded convex subset of H. Assume that $\Lambda : M \mapsto M$ is nonexpansive, that is,*

$$(\Lambda(x), \Lambda(y)) \leq (x, y) \quad \forall x, y \in M,$$

where $(\cdot, \cdot)$ is the inner product on H. Then Λ has at least one fixed point $x \in M$.

We refer the reader to [3] for a proof of Theorem 1.1.2, which is based on the Banach fixed point theorem. By enriching the structure of the underlying space H, Theorem 1.1.2 slightly relaxes the δ-contractive requirement. However, both the uniqueness and the useful estimates are lost.

1.2. *The Schauder fixed point theorem*

In this section, we discuss a fundamental fixed point theorem on Banach spaces—complete normed spaces, the Schauder fixed point theorem. The following Brouwer fixed point theorem on $\mathbb{R}^n$ lays the foundation in this direction.

THEOREM 1.2.1 (Brouwer fixed point theorem). *Let M be a convex compact subset of $\mathbb{R}^n$. Assume that $\Lambda : M \mapsto M$ is a continuous map. Then Λ has a fixed point $x \in M$.*

The proof of the Brouwer fixed point theorem uses the following deep topological result.

THEOREM 1.2.2 (Borsuk). *Let B be the unit ball in $\mathbb{R}^n$ and $\partial B = S^{n-1}$ its boundary. Then there is no continuous map $f : B \mapsto S^{n-1}$ such that $f|_{S^{n-1}} = id|_{S^{n-1}}$.*

This result follows directly from the Borsuk Antipodal Theorem, which shows that the map $id|_{S^{n-1}}$ cannot be homotopic to a constant map.

The Schauder fixed point theorem extends the Brouwer fixed point theorem to infinite-dimensional Banach spaces. Rather different from the Banach fixed point theorem, the Schauder fixed point theorem relies on the compactness of the operator. Let X be a Banach space and M a subset of X. An operator $\Lambda : M \mapsto X$ is called compact if it is continuous and maps bounded subsets to relative compact sets. Below is the Schauder fixed point theorem.

THEOREM 1.2.3 (Schauder fixed point theorem). *Let M be a closed bounded convex subset of a Banach space X. Assume that $\Lambda : M \mapsto M$ is compact. Then Λ has at least one fixed point in M.*

Utilizing the compactness of Λ, the proof of 1.2.3 relies on the Brouwer fixed point theorem and a finite dimension approximation. The following alternative version of the Schauder fixed point theorem is an immediate corollary of Theorem 1.2.3.

THEOREM 1.2.4. *Let M be a compact convex subset of a Banach space X. Assume that $\Lambda : M \mapsto M$ is a continuous operator. Then Λ has at least one fixed point in M.*

PROOF. We use a proof from [57]. Denote $\overline{C(\Lambda(M))}$ the closure of the convex hull of $\Lambda(M)$. By the Mazur theorem, $\overline{C(\Lambda(M))} \subset M$ is compact and convex since $\Lambda(M)$ is relatively compact (Λ being compact). Clearly $\Lambda(\overline{C(\Lambda(M))}) \subset \overline{C(\Lambda(M))}$. Now by Theorem 1.2.3, Λ has a fixed point in $\overline{C(\Lambda(M))}$, which is a fixed point of Λ in X. □

To apply Theorem 1.2.3, the key is to construct the pair (Λ, M), with $M \subset X$ being a closed bounded convex subset of a Banach space X and $\Lambda : M \mapsto M$ being compact. The following Leray–Schauder fixed point theorem perfectly illustrates such a notion.

THEOREM 1.2.5 (Leray–Schauder fixed point theorem). *Let X be a Banach space and assume that $\Lambda : X \mapsto X$ is compact. Suppose that there exists a positive number $R > 0$ such that $\|x\| \leq R$ for all x's satisfying $x = t\Lambda(x)$ with $t \in [0, 1]$. Then Λ has at least one fixed point $x \in X$.*

PROOF. As mentioned above, the proof is to construct a new operator Γ on X and a closed bounded convex subset M of X such that $\Gamma : M \mapsto M$ is compact. Then the conclusion follows immediately from Theorem 1.2.3.

Define

$$\Gamma(x) = \begin{cases} \Lambda(x), & \text{if } \|\Lambda(x)\| \leq 2R; \\ 2R\|\Lambda(x)\|^{-1}\Lambda(x), & \text{if } \|\Lambda(x)\| > 2R, \end{cases}$$

and put

$$M = \{x \mid \|x\| \leq 2R\}.$$

Clearly $\Gamma : M \mapsto M$ is continuous and M is closed, bounded and convex. It remains to verify that Γ maps bounded subsets of M to relatively compact sets. Let $\{x_n\}$ be a sequence in M. Thus $\{x_n\}$ is bounded. Evidently, there is a subsequence of $\{x_n\}$ such that its imagine under Λ is either contained inside of M, or outside of M, both still denoted by $\{x_n\}$. If the first occurs, $\Gamma(x_n) = \Lambda(x_n)$ has a convergent subsequence in M since Λ is compact by assumption. If the latter occurs, clearly there is a subsequence $\{y_n\}$ of $\{x_n\}$ such that both $\{1/\|\Lambda(y_n)\|\}$ and $\{\Lambda(y_n)\}$ converge, so does $\Gamma(y_n)$.

Now Theorem 1.2.3 implies that Γ has a fixed point $x \in M$. If $\|\Lambda(x)\| > 2R$, then

$$x = \Gamma(x) = 2R\Lambda(x)\|\Lambda(x)\|^{-1} = t\Lambda(x)$$

with $t = 2R\|\Lambda(x)\|^{-1} \in (0, 1)$, an immediate contradiction. Hence $\|\Lambda(x)\| \leq 2R$ and $\Lambda(x) = \Gamma(x) = x$. The proof is complete. □

1.3. *The Leray–Schauder fixed point index*

In this section, we discuss a more sophisticated fixed point theory on a Banach space, namely, the Leray–Schauder fixed point index theory (the Leray–Schauder degree theory), which is based on the topological degree theory on $\mathbb{R}^n$. The central piece of the extension is preserving of the compactness structure of $\mathbb{R}^n$ in someway.

Let X be a Banach space and M an open bounded subset of X. Let $\Lambda, \Gamma : \overline{M} \mapsto X$ be compact operators which have no fixed points on the boundary ∂M. We say that Λ and Γ

are compactly homotopic (or simply homotopic) to each other, denoted by $\Lambda \sim \Gamma : \partial M$, provided there exists a compact mapping $H : M \times [0, 1] \mapsto X$ such that

(1) $H(x, t) \neq x$ for all $(x, t) \in \partial M \times [0, 1]$.
(2) $H(x, 0) = \Lambda(x)$ and $H(x, 1) = \Gamma(x)$ for all $x \in \partial M$.

We point out that the compact mapping H can be defined only on the boundary ∂M, i.e., $H : \partial M \times [0, 1] \mapsto X$ in the above definition.

Now we are ready to introduce the Leray–Schauder fixed point index. Let X be a Banach space and define

$$Y = Y(X) = \{(\Lambda, M) \mid M \text{ is an open bounded subset of } X \text{ and } \Lambda : \overline{M} \mapsto X \text{ is compact with } \Lambda(x) \neq x \text{ on } \partial M\}.$$

Then the Leray–Schauder fixed point index is a function $i : Y \to \mathbb{Z}$ ($\mathbb{Z}$ is the set of whole numbers) which satisfies

(1) $i(\Lambda, M) = 1$ if $\Lambda(x) = x_0$ (i.e., constant map) for some $x_0 \in M$.
(2) If $i(\Lambda, M) \neq 0$, then there exists an $x \in M$ such that $\Lambda(x) = x$.
(3) Let $(\Lambda, M) \in Y$ and let $\{M_j\}_{j=1}^k$ be a mutually disjoint partition of M. Suppose that Λ has no fixed point on ∂M_j for $j = 1, \dots, k$. Then there holds
$$i(\Lambda, M) = \sum_{j=1}^{k} i(\Lambda, M_j).$$
(4) If $\Lambda \sim \Gamma : \partial M$, then $i(\Lambda, M) = i(\Gamma, M)$.

The Leray–Schauder fixed point index defined above also possesses further useful properties, such as the excision property and the continuity property.

For a given Banach space, there exists a unique Leray–Schauder fixed point index and concrete formulas can be constructed to compute the index. Again, whether the dimension of the space is finite remains of central importance. We first have the Brouwer fixed point index on $\mathbb{R}^n$.

THEOREM 1.3.1 (Brouwer fixed point index). *Let $X = \mathbb{R}^n$. Then there exists a unique fixed point index i on $Y = Y(\mathbb{R}^n)$ satisfying 1–4 above.*

The second is the Leray–Schauder fixed point index on general Banach spaces.

THEOREM 1.3.2 (Leray–Schauder fixed point index). *Let X be Banach space. Then there exists a unique fixed point index i on $Y = Y(X)$ satisfying 1–4 above.*

We shall omit the proof of both Theorems 1.3.1 and 1.3.2, which are somewhat involved. We refer the reader to [3,57] and the references therein.

Various applications of the Leray–Schauder fixed point index theory have been developed and one of them is the Krasnosle'skii fixed point theorem on a wedge.

Let X be a Banach space. We say that a convex subset W of X is a wedge if

$$tW = \{tx \mid x \in W\} \subset W, \quad \forall t \geq 0.$$

A wedge is proper if it is not a linear subspace of X. A wedge W is called a cone if $W \cap (-W) = \{0\}$. Then we have the Krasnosle'skii fixed point theorem.

THEOREM 1.3.3 (Krasnosle'skii fixed point theorem). *Let W be a proper wedge of a Banach space X. Assume that $\Lambda : W \mapsto W$ is compact. Suppose that there exist $r > 0$ and $R > 0$ such that $0 < r < R$ and one of the following conditions hold*

(1) $\|\Lambda(x)\| \le \|x\|$ *for* $x \in W$ *with* $\|x\| = r$, *and* $\|\Lambda(x)\| \ge \|x\|$ *for* $x \in W$ *with* $\|x\| = R$.

(2) $\|\Lambda(x)\| \ge \|x\|$ *for* $x \in W$ *with* $\|x\| = r$, *and* $\|\Lambda(x)\| \le \|x\|$ *for* $x \in W$ *with* $\|x\| = R$.

Then Λ has at least one fixed point $x \in W$ with $r \le \|x\| \le R$.

The operator Λ is sometimes called wedge-extending in Part 1 and wedge-compression in Part 2. The proof of Theorem 1.3.3 uses the Leray–Schauder fixed point index [30].

The Krasnosle'skii fixed point Theorem 1.3.3 has been extensively used to derive existence of positive solutions on a (positive) cone. In this regard, the following refined Krasnosle'skii fixed point theorem, due to de Figueiredo, Lions and Naussbaum [18], is often more convenient.

THEOREM 1.3.4 (Refined Krasnosle'skii fixed point theorem). *Let $\mathcal{C}$ be a cone in a Banach space X and $\Lambda : \mathcal{C} \to \mathcal{C}$ a compact mapping such that $\Lambda(0) = 0$. Suppose that there exist positive numbers $t_0, r, R > 0$ and a vector $u_0 \in \mathcal{C} - \{0\}$ such that*

(1) $u \ne t\Lambda(u)$ *for* $t \in [0, 1]$ *and* $u \in \mathcal{C}$ *with* $\|u\| = r$,

(2) $u \ne \Lambda(u) + tu_0$ *for* $t \ge 0$ *and* $u \in \mathcal{C}$ *with* $\|u\| = R$.

(3) $u \ne \Lambda(u) + tu_0$ *for* $t \ge t_0$ *and* $u \in \mathcal{C}$ *with* $\|u\| \le R$.

Then if $U = \{u \in \mathcal{C} : r < \|u\| < R\}$ and $B_\rho = \{u \in \mathcal{C} : \|u\| < \rho\}$, one has

$$i_{\mathcal{C}}(\Lambda, B_R) = 0, \qquad i_{\mathcal{C}}(\Lambda, B_r) = 1, \qquad i_{\mathcal{C}}(\Lambda, U) = -1.$$

In particular, Λ has a fixed point in U.

2. Elliptic boundary value problems

The solvability of nonlinear elliptic boundary value problems (in proper spaces) is one of the central issues in the classical studies on nonlinear partial differential equations. The topic has been extensively studied with an extremely rich literature. In this section, we shall formulate an elliptic boundary value problem of second order in a general format and set up its equivalent operator form. To address the issue of its solvability, we then establish several general existence results by applying the fixed point theorems.

2.1. *Elliptic boundary value problems*

Let $n \geq 2$ and $k \geq 1$ be two integers and let $\Omega \subset \mathbb{R}^n$ be a connected smooth domain. Let

$$\mathbf{A}^l = (A_1^l, A_2^l, \ldots, A_n^l) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n; \qquad l = 1, 2, \ldots, k$$

and

$$\mathbf{f} = (f^1, f^2, \ldots, f^k) : \Omega \times \mathbb{R}^k \times \mathbb{R}^{nk} \to \mathbb{R}^k$$

be continuous (vector-valued or scalar-valued) functions.[1] We shall use bold face letters, such as $\mathbf{u}$, $\mathbf{p}$, $\mathbf{q}$ for vector values in $\mathbb{R}^k$ or $\mathbb{R}^n$, Q for $n \times k$-matrix values in $\mathbb{R}^{nk}$, and Q_l for the lth column vector (thus an n-vector) of Q.

Throughout the entire paper it is assumed that the functions $\mathbf{A}^l$ and $\mathbf{f}$ satisfy the following ellipticity and growth conditions.

(A1) $\mathbf{A}^l \in C(\Omega \times \mathbb{R} \times \mathbb{R}^n) \cap C^1(\Omega \times \mathbb{R} \times (\mathbb{R}^n \setminus \{0\}))$, $l = 1, 2, \ldots, k$.

(A2) $\mathbf{A}^l(x, u, \mathbf{0}) = \mathbf{0}$, $l = 1, 2, \ldots, k$.

(A3) For each $l = 1, 2, \ldots, k$, there exist positive numbers $m_l > 1$ and $K > 0$ such that for all $\xi \in \mathbb{R}^n$ and $(x, u, \mathbf{p}) \in \Omega \times \mathbb{R} \times (\mathbb{R}^n \setminus \{0\})$

$$\sum_{i,j=1} \frac{\partial A_j^l}{\partial p_i}(x, u, \mathbf{p})\xi_i\xi_j \geq K^{-1}(\delta_l + |\mathbf{p}|)^{m_l - 2}|\xi|^2, \tag{2.1.1}$$

$$\sum_{i,j=1} \left| \frac{\partial A_j^l}{\partial p_i}(x, u, \mathbf{p}) \right| \leq K(\delta_l + |\mathbf{p}|)^{m_l - 2}, \tag{2.1.2}$$

and

$$\sum_{i,j=1} \left| \frac{\partial A_j^l}{\partial x_i}(x, u, \mathbf{p}) \right| + \sum_{j=1} \left| \frac{\partial A_j^l}{\partial u}(x, u, \mathbf{p}) \right| \leq K(\delta_l + |\mathbf{p}|)^{m_l - 2}|\mathbf{p}|, \tag{2.1.3}$$

where either $\delta_l = 0$ or $\delta_l = 1$.

(F) For each $l = 1, 2, \ldots, k$, in addition to the numbers $m_l >$ and $K > 0$ from (A3), there exist positive number $p_l > 0$ such that for all $(x, \mathbf{u}, Q) \in \Omega \times \mathbb{R}^k \times \mathbb{R}^{nk}$

$$|f^l(x, \mathbf{u}, Q)| \leq K\left(1 + |\mathbf{u}|^{p_l} + \sum_j |Q_j|^{m_l}\right). \tag{2.1.4}$$

For convenience, we shall assume that the boundary of Ω (if nonempty) is smooth, say, C^2, and also omit the subscripts l for δ and p.

[1] It is understood that all relations involving vectors are in the component-wise sense.

Consider the following boundary value problem of systems of quasi-linear elliptic equations of second order

$$\operatorname{div} \mathbf{A}^l(x, u_l, \nabla u_l) + f^l(x, \mathbf{u}, \nabla \mathbf{u}) = 0 \quad \text{in } \Omega;\ l = 1, 2, \ldots, k, \tag{2.1.5}$$

where

$$\mathbf{u} = (u_1, u_2, \ldots, u_k) : \Omega \mapsto \mathbb{R}^k, \qquad \nabla \mathbf{u} = \left\{ \frac{\partial u_j}{\partial x_i} \right\} : \Omega \mapsto \mathbb{R}^{nk}.$$

The terms $\operatorname{div} \mathbf{A}^l(x, u_l, \nabla u_l)$, $l = 1, 2, \ldots, k$, are called principal parts of the system. Note in (2.1.5) the principal parts are *de-coupled*. When $m_l \neq 2$, $(2.1.5)_l$ is called regular if $\delta_l = 1$, and is said to be degenerate or singular if $\delta_l = 0$.

A function $\mathbf{u} \in W^{1,m_1}_{\text{loc}}(\Omega) \cap C(\Omega) \times \cdots \times W^{1,m_k}_{\text{loc}}(\Omega) \cap C(\Omega)$ is said to be a *weak solution*, or simply a *solution*, of (2.1.5) if

$$-\int \mathbf{A}^l(x, u_l, \nabla u_l) \cdot \nabla \varphi_l + \int f^l(x, \mathbf{u}, \nabla \mathbf{u}) \varphi_l = 0, \tag{2.1.6}$$

for all $\varphi_l \in C_0^\infty(\Omega)$, $l = 1, 2, \ldots, k$. All solutions considered are weak solutions in the above sense.

Whenever Ω is bounded, we shall associate (2.1.5) with the following boundary condition

$$\mathbf{u} = \mathbf{u}_0 \quad \text{on } \partial\Omega \tag{2.1.7}$$

where, for simplicity, $\mathbf{u}_0 \in C^2(\overline{\Omega})$.

We are concerned with the question of existence of a vector-valued function $\mathbf{u}(x)$ solving the boundary value problem (2.1.5) together with (2.1.7).

Throughout what follows, without further mentioning, the term "BVP" will be used to stand for boundary value problem and $C > 0$ will be used to denote generical constants which may vary from one to another and depend only on the arguments inside the parentheses (or clear from the context). Moreover, for $m \in (1, n)$, we denote

$$m_* := \frac{(m-1)n + m}{n - m} > 0, \qquad m^* := m_* + 1 = \frac{mn}{n - m},$$

being the Sobolev embedding numbers. For $m \geq n$, it is understood $m_* = m^* = \infty$.

2.2. *Maximum principles and regularity*

In dealing with the differential equation (2.1.5), we shall frequently use some forms of the classical regularity theory and the maximum principles for quasi-linear elliptic equations, which are summarized in this subsection.

The following inequalities are elementary.

LEMMA 2.2.1. *Let* $\mathbf{a}, \mathbf{b}$ *be vectors in* $\mathbb{R}^n$ *with* $|\mathbf{a}| + |\mathbf{b}| > 0$ *and let* $\mathbf{A}$ *satisfy* (A1)–(A3) *with* $m > 1$. *Then there exists a constant* $C = C(K; m; n) > 0$ *such that*

$$|\mathbf{A}(x, u, \mathbf{a}) - \mathbf{A}(x, u, \mathbf{b})| \leq C(\delta + |\mathbf{a}| + |\mathbf{b}|)^{m-2} |\mathbf{a} - \mathbf{b}|,$$

and

$$(\mathbf{A}(x,u,\mathbf{a})-\mathbf{A}(x,u,\mathbf{b}))(\mathbf{a}-\mathbf{b})\geq C(\delta+|\mathbf{a}|+|\mathbf{b}|)^{m-2}|\mathbf{a}-\mathbf{b}|^2.$$

PROOF. Write

$$\mathbf{A}(x,u,\mathbf{a})-\mathbf{A}(x,u,\mathbf{b})=\int_0^1\frac{\partial\mathbf{A}}{\partial p_i}(x,u,\mathbf{a}+t(\mathbf{b}-\mathbf{a}))(b_i-a_i)dt.$$

Then the conclusion follows immediately from the assumption (A3) and the fact

$$C^{-1}(\delta+|\mathbf{a}|+|\mathbf{b}|)^{m-2}\leq\int_0^1(\delta+|\mathbf{a}+t(\mathbf{b}-\mathbf{a})|)^{m-2}dt\leq C(\delta+|\mathbf{a}|+|\mathbf{b}|)^{m-2}$$

for some $C>0$ depending only on m. □

The following weak comparison principle is well known.

LEMMA 2.2.2 (Weak comparison principle). *Let* $\mathbf{A}$ *satisfy* (A1)–(A3) *with* $m>1$ *and* $\mathbf{f}=f(x;u)$ *be nonincreasing in* u. *Let* u *and* v *be functions in the Sobolev space* $W^{1,m}_{\text{loc}}(\Omega)$ *which satisfy the distribution inequality*

$$\operatorname{div}\mathbf{A}(x,u,\nabla u)+f(x;u)-[\operatorname{div}\mathbf{A}(x,v,\nabla v)+f(x;v)]\leq 0 \tag{2.2.1}$$

in a domain Ω *of* $\mathbb{R}^n$. *Suppose that* $u\geq v$ *on* $\partial\Omega$, *in the sense that the set* $\{u-v+\varepsilon\leq 0\}$ *has compact support in* Ω *for every* $\varepsilon>0$. *Then* $u\geq v$ *in* Ω a.e.

PROOF. The proof is a direct integration by parts, but we include it here for the reader's convenience. Suppose the contrary. For $\varepsilon>0$, put

$$w(x)=w_\varepsilon(x)=u(x)-v(x)+\varepsilon.$$

By assumption,

$$w_-(x):=\min\{0,w(x)\}\not\equiv 0\quad\text{a.e.}$$

is in the space $W^{1,m}_0(\Omega)$ for $\varepsilon>0$ sufficiently small. Multiply (2.2.1) by w_- and integrate over Ω to obtain

$$\int_\Omega[\mathbf{A}(x,u,\nabla u)-\mathbf{A}(x,v,\nabla v)]\nabla w_-\leq\int_\Omega[f(x;u)-f(x;v)]w_-\leq 0,$$

since $[f(x;u)-f(x;v)]w_-\leq 0$ a.e. on Ω by the fact $f(x;u)$ is nonincreasing in u. But by Lemma 2.2.1, there exists $C>0$ such that

$$\int_\Omega[\mathbf{A}(x,u,\nabla u)-\mathbf{A}(x,v,\nabla v)]\nabla w_-\geq C\int_\Omega(\delta+|\nabla u|+|\nabla v|)^{m-2}|\nabla w_-|^2.$$

It follows $w_-\equiv 0$, a.e., a contradiction. □

We next have the classical $W^{2,2}$ and $C^{1,\beta}$-regularity results for (2.1.5).

LEMMA 2.2.3. *Let* $\mathbf{u}$ *be a solution of (2.1.5). Then the following conclusions hold.*

(1) (Weighted $W^{2,2}$-Regularity) *Assume* $\mathbf{u} \in L^\infty_{\mathrm{loc}}(\Omega)$. *Then the function* $|u_l|^{m_l-2}|D^2 u_l|$ *is in the space* $L^2_{\mathrm{loc}}(\Omega)$ *for* $k = 1, \ldots, k$.

(2) (Interior Hölder Estimates for Gradients) *Let* $B_R = B_R(x_0)$ *be a ball with radius* $R > 0$ *such that* $B_{2R}(x_0) \subset \Omega$. *Then there exist two positive constants* $\beta = \beta(K, M, p, m, n, R) \in (0, 1)$ *and* $C = C(K, M, p, m, n, R)$ *such that*

$$\sup_{x \in B_{R/2}} |\nabla \mathbf{u}(x)| + \sup_{x_1, x_2 \in B_{R/2}; x_1 \neq x_2} \frac{|\nabla \mathbf{u}(x_1) - \nabla \mathbf{u}(x_2)|}{|x_1 - x_2|^\beta} \leq C \tag{2.2.2}$$

where K *is given in* (A3) *and*

$$M = \sup_{x \in B_R} |\mathbf{u}(x)|.$$

(3) (Global Hölder Estimates for Gradients) *If in addition* $\partial\Omega$ *is nonempty and* $\mathbf{u}$ *satisfies the boundary condition* (2.1.7), *then there exist two positive constants* $\beta = \beta(K, M, p, m, n, \|\partial\Omega\|_{1,1}) \in (0, 1)$ *and* $C = C(K, M, p, m, n, \|\partial\Omega\|_{1,1}) > 0$ *such that*

$$\sup_{x \in \Omega \cap B_{R/2}} |\nabla \mathbf{u}(x)| + \sup_{x_1, x_2 \in \Omega \cap B_{R/2}; x_1 \neq x_2} \frac{|\nabla \mathbf{u}(x_1) - \nabla \mathbf{u}(x_2)|}{|x_1 - x_2|^\beta} \leq C, \tag{2.2.3}$$

where $B_R = B_R(x_0) \subset \mathbb{R}^n$ *and*

$$M = \sup_{x \in \Omega \cap B_R} |\mathbf{u}(x)|.$$

Lemma 2.2.3 follows directly from the classical regularity theory for scalar equations, since the principal parts of (2.1.5) are de-coupled and the lower-order term f obeys the growth condition (F). For scalar equations, the interior estimate (2.2.2) is due to [21, 53], see also [23,31,32,55] and the references therein for earlier results in this direction. The global estimate (2.2.3) then follows by combining (2.2.2) with the boundary estimate later obtained in [33]. The weighted $W^{2,2}$-regularity was proved in [50], Proposition 8.1, p. 132 (the fact $m \in (1, 2)$ is neither needed, nor used in the proof). For $m \in (1, 2)$, a standard $W^{2,2}$-regularity was obtained in [1]. When $m > 2$, one can show that the function $|u|^{(m-2)/2}|D^2 u|$ is in the space $L^2_{\mathrm{loc}}(\Omega)$, see for example [21] and the references therein. But Lemma 2.2.3(1) suffices for our purposes.

The next result is a strong maximum principle for (2.1.5). For $1 \leq l \leq k$, we say that the pair $(\mathbf{A}^l, f^l)$ has a positivity property, provided there exists a matrix function $\{a^l_{i,j}(x)\} \in C^1(\Omega)$ and nonnegative continuous functions $A^l, h^l, \omega^l \in C(\mathbb{R}_+)$ such that

(a1) There holds

$$\mathbf{A}^l(x, u, \mathbf{p}) = A^l(|\mathbf{p}|) \left(\sum_j a^l_{1,j}(x) p_j, \ldots, \sum_j a^l_{n,j}(x) p_j \right).$$

(a2) There exists $C > 0$ such that for $\xi \in \mathbb{R}^n$

$$a^l_{i,j}(x) \xi_i \xi_j \geq C|\xi|^2.$$

(h1) There exists $C > 0$ such that

$$\omega^l(t) \le C \int_0^t sA^l(s)ds, \quad t \in [0,1],$$

and

$$h^l(u_l) + \omega^l(|Q_l|) + f^l(x, \mathbf{u}, Q) \ge 0,$$
$$(x, \mathbf{u}, Q) \in \Omega \times (\mathbb{R}_+ \times \cdots \times \mathbb{R}_+) \times \mathbb{R}^{nk}.$$

(h2) There exists $\delta > 0$ such that h^l is nondecreasing on $[0, \delta)$ and is nonnegative.

(h3) There holds

$$\int_0^\delta \frac{ds}{H_l^{-1}(\bar{h}^l(s))} = \infty,$$

where

$$H_l(t) = t^2 A^l(t) - \int_0^t sA^l(s)ds, \qquad \bar{h}^l(t) = \int_0^t h^l(s)ds; \quad t \ge 0.$$

(It is understood that (h3) holds automatically if $h^l(t) \equiv 0$ near $t = 0^+$ by agreement.)

By (A3), one readily sees that for $t \ge 0$

$$C^{-1}t^2(\delta + t)^{m_l-2} \le H_l(t) \le Ct^2(\delta + t)^{m_l-2},$$

whence $h^l(0) = 0$ (in general, one could use $h_l(t) \sim t^{m_l-1}$ when $\delta = 0$). One may also take appropriately larger h^l and ω^l to enforce the positivity if necessary, facts being used later.

We have the following strong maximum principle.

LEMMA 2.2.4 (Strong maximum principle). *Assume for some $1 \le l \le k$ the pair $(\mathbf{A}^l, f^l)$ satisfies a positivity property. Then the l-th component u_l of all nonnegative solutions $\mathbf{u}$ of (2.1.5) is either identically zero or strictly positive on Ω.*

PROOF. This is simply Theorem 1′ of [42] for scalar equations (condition (2.5) in [42] was later removed in [41]). Indeed, (2.1.5) can be de-coupled into scalar equations for each component, in view of the fact that its principal parts are de-coupled and the nature of the positivity property. □

2.3. *An auxiliary existence lemma*

Let Ω be bounded and let

$$\mathbf{A} = (A_1, A_2, \dots, A_n) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$$

be a function satisfying the conditions (A1)–(A3) with $m > 1$. Suppose that $h \in C(\mathbb{R})$ is a nondecreasing function satisfying the growth condition

$$|h(u)| \le C(1 + |u|^{m_*}), \qquad |h(u) - h(u)| \le L(|u| + |v|)^{m-2}|u - v|, \qquad (2.3.1)$$

where $L > 0$ is constant. For $\varphi \in C(\overline{\Omega})$, consider the BVP

$$\begin{aligned} -\operatorname{div}\mathbf{A}(x,u,\nabla u) + h(u) = \varphi(x) \quad & \text{in } \Omega, \\ u = u_0 \qquad & \text{on } \partial\Omega, \end{aligned} \tag{2.3.2}$$

where $u_0 \in C^2(\overline{\Omega})$. Without loss of generality, we shall assume $u_0 \equiv 0$ (for otherwise one simply considers $v = u - u_0$).

The Banach space $X = W_0^{1,m}(\Omega)$ is separable and sequentially compact since $m > 1$, whence X is reflexive (Bberlein–Shmulyan). Define an operator

$$\mathcal{A}\colon X \mapsto X', \qquad \mathcal{A}(u) := \mathbf{A}(x,u,\nabla u) + h(u),$$

where X' is the dual space of X, in the sense

$$(\mathcal{A}(u), v) = \int \mathbf{A}(x,u,\nabla u)\cdot\nabla v + \int h(u)v; \quad v \in X.$$

We claim that the operator $\mathcal{A}$ is bounded, continuous and monotone.

(1) Boundedness. By the Sobolev embedding [2], (2.3.1) and (A3),

$$\begin{aligned} |(\mathcal{A}(u), v)| &\le \int |\mathbf{A}(x,u,\nabla u)\cdot\nabla v| + C\int (1+|u|^{m_*})|v| \\ &\le \left(\int |\mathbf{A}(x,u,\nabla u)|^{m'}\right)^{1/m'}\left(\int |\nabla v|^m\right)^{1/m} \\ &\quad + C\left(\int |u|^{m^*}\right)^{m_*/m^*}\left(\int |v|^{m^*}\right)^{1/m^*} + C\|v\|_X \\ &= \|\nabla u\|_{L^m}^{m-1}\|\nabla v\|_{L^m} + \|u\|_{L^{m^*}}^{m_*}\|v\|_{L^{m^*}} \le C(\|u\|_X^{m_*}+1)\|v\|_X, \end{aligned}$$

where $m' = m/(m-1)$ is the conjugate.

(2) Continuity. With the aid of (A3) (and of course Lemma 2.2.1) and (2.3.1), one also readily verifies that

$$\begin{aligned} \|\mathcal{A}(u)-\mathcal{A}(v)\|_{X'}^{m'} &\le C\int |\mathbf{A}(x,u,\nabla u) - \mathbf{A}(x,v,\nabla v) + h(u) - h(v)|^{m'} \\ &\le C\int [(\delta + |\nabla u| + |\nabla v|)^{m-2}|\nabla u - \nabla v|]^{m'} \\ &\quad + C\int [(|u|+|v|)^{m-2}|u-v|]^{m'} \\ &\le C(\|u\|_X, \|v\|_X)\|u-v\|_X^{m'}, \end{aligned}$$

That is, $\mathcal{A}$ is continuous.

(3) Monotonicity. By (A3) and the monotonicity of h, we have for $u, v \in X$

$$(\mathcal{A}(u) - \mathcal{A}(v), u - v) = \int (\mathbf{A}(x, u, \nabla u) - \mathbf{A}(x, v, \nabla v)) \cdot (\nabla u - \nabla v) + \int (h(u) - h(v))(u - v) \geq C \int (\delta + |\nabla u| + |\nabla v|)^{m-2} |\nabla u - \nabla v|^2 \geq 0.$$

It follows that $\mathcal{A}$ is monotone (strongly monotone if $m \geq 2$).

We need the following existence result due to [51] (Corollary 2.2, see also [20]).

LEMMA 2.3.1. *Let X be a separable and reflexive Banach space. Assume that $\mathcal{A} : X \mapsto X'$ is bounded, continuous, monotone and coercive, i.e.,*

$$\lim_{\|u\|_X \to \infty} \frac{(\mathcal{A}(u), u)}{\|u\|_X} = \infty.$$

Then $\mathcal{A}$ is surjective.

We now can prove the existence theorem of this subsection.

LEMMA 2.3.2. *For every $\varphi \in C(\overline{\Omega})$ and $u_0 \in C^2(\overline{\Omega})$, the BVP (2.3.2) has a unique solution $u \in X \cap C(\overline{\Omega})$, where $X = W_0^{1,m}(\Omega)$.*

PROOF. We first show the existence in X. For any $\varphi \in C(\overline{\Omega})$, define

$$L(v) = \int_\Omega \varphi v, \quad v \in X.$$

Clearly $L \in X'$. Therefore, it suffices to show that $\mathcal{A} : X \mapsto X'$ is surjective. In view of Lemma 2.3.1, it remains to show that $\mathcal{A}$ is coercive since $\mathcal{A}$ is bounded, continuous and monotone. By the Poincare inequality, there exists $C = C(\Omega, n, m) > 0$ such that

$$\|u\|_X^m \leq C \int |\nabla u|^m.$$

By the monotonicity of h,

$$[h(u) - h(0)]u \geq 0.$$

Finally, by the Holder inequality,

$$|h(0)| \int |u| \leq C \left(\int |u|^m \right)^{1/m} \leq C \|u\|_X.$$

It follows that, by Lemma 2.2.1, for $\|u\|_X$ sufficiently large

$$\begin{aligned}(\mathcal{A}(u), u) &= \int \mathbf{A}(x, u, \nabla u) \cdot \nabla u + \int [h(u) - h(0)]u + h(0) \int u \\ &\geq \int (\mathbf{A}(x, u, \nabla u) - \mathbf{A}(x, u, 0)) \cdot \nabla u - C\|u\|_X \\ &\geq C \int (\delta + |\nabla u|)^{m-2} |\nabla u|^2 - C\|u\|_X \geq C\|u\|_X^m - C\|u\|_X,\end{aligned}$$

where we have used (A2). In turn,

$$\lim_{\|u\|_X \to \infty} \frac{(\mathcal{A}(u), u)}{\|u\|_X} \geq \liminf_{\|u\|_X \to \infty} [C\|u\|_X^{m-1} - C] = \infty,$$

since $m > 1$. Hence the BVP (2.3.2) has a solution $u \in X$.

Next, with the aid of Lemma 2.2.1, using $|u|^\gamma u$ as a test function in (2.3.2) to get

$$\begin{aligned}C\gamma \int (\delta + |\nabla u|)^{m-2} |\nabla u| |u|^\gamma &\leq \int \mathbf{A}(x, u, \nabla u) \cdot \nabla u \\ &= \int |u|^\gamma u\varphi - \int h(u)|u|^\gamma u \\ &\leq \int |u|^\gamma u[\varphi - h(0)] \leq C \int |u|^{\gamma+1},\end{aligned}$$

since $[h(u) - h(0)]|u|^\gamma u \geq 0$ by the monotonicity of h. One then readily derives an a priori $W_0^{1,m}(\Omega)$-bound for u and a standard boot-strap argument (cf. [47], see also Section 3.2 for details and it may require slightly more work when $m < 2$ and $\delta = 1$) implies that there exists $C = C(n, m, K, h, \|\varphi\|) > 0$ such that

$$\|u\|_{L^\infty(\Omega)} \leq C.$$

Now the classical De Giorgi estimate implies that u is actually Holder-continuous on $\overline{\Omega}$, i.e., $u \in X \cap C(\overline{\Omega})$.

The uniqueness follows directly from the weak comparison Lemma 2.2.2, in view of the monotonicity of h. □

2.4. *Equivalent operator form*

In this subsection, we establish an equivalent operator form for the BVP (2.1.5) and (2.1.7).

Let Ω be bounded. For $l = 1, \ldots, k$, suppose $h^l \in C(\mathbb{R})$ is a nondecreasing function satisfying the growth condition (2.3.1). For each $\mathbf{A} = \mathbf{A}^l$, paired with h^l and $(u_0)_l$ (the boundary data), Lemma 2.3.2 shows that for each $\varphi \in C(\overline{\Omega})$ the BVP (2.3.2) has a unique solution $u = u_{\varphi,l} \in W^{1,m_l}(\Omega) \cap C(\overline{\Omega})$ satisfying $u_{\varphi,l} = (u_0)_l$ on $\partial\Omega$. Define an operator

$$\Phi_l : C(\overline{\Omega}) \mapsto W^{1,m_l}(\Omega) \cap C(\overline{\Omega}), \qquad \Phi_l(\varphi) := u_{\varphi,l}.$$

We have the following result.

LEMMA 2.4.1. *For $l = 1, \ldots, k$, the operators*

$$\Phi_l : C(\overline{\Omega}) \mapsto \{u \in C^1(\overline{\Omega}) \mid u(x) = (u_0)_l(x) \text{ on } \partial\Omega\}, \qquad \Phi_l(\varphi) := u_{\varphi,l}$$

are well defined and compact.

If in addition the pair $(\mathbf{A}^l, f^l)$ satisfies a positivity condition, with $\omega^l \equiv 0$ and h^l given in the definition of Φ_l, then the operator Φ_l is also positive, that is, $\Phi_l(\varphi) > 0$ provided $\varphi \gneqq 0$.

PROOF. Fix l. The well-posedness of Φ_l in $W^{1,m_l}(\Omega) \cap C(\overline{\Omega})$ is simply Lemma 2.3.2. As mentioned earlier, the classical De Giorgi estimate yields $C^{0,\beta}$ estimates for some $\beta > 0$. It follows that, by Lemma 2.2.3, there exists $\beta_l = \beta_l(n, m_l, K, p, h, \Omega) \in (0, 1)$, such that all solutions of the BVP (2.3.2) are actually in the space $C^{1,\beta_l}(\overline{\Omega})$, whence the well-posedness of Φ_l in the space $\{u \in C^1(\overline{\Omega}) \mid u(x) = (u_0)_l(x) \text{ on } \partial\Omega\}$ follows.

Next, we show that Φ_l is compact. Plainly, it suffices to show Φ_l is continuous since the embedding $C^{1,\beta_l}(\overline{\Omega}) \hookrightarrow C^1(\overline{\Omega})$ is compact.

To this end, by Lemma 2.2.3, there exists $C = C(\beta_1, h, n, m_l, K, \delta, h, \varphi, \Omega) > 0$ such that

$$\|u\|_{1,\beta_1,\overline{\Omega}} \leq C \tag{2.4.1}$$

for all $\varphi \in C(\overline{\Omega})$. Suppose for contradiction that

$$\Phi_l : C(\overline{\Omega}) \mapsto \{u \in C^1(\overline{\Omega}) \mid u(x) = (u_0)_l(x) \text{ on } \partial\Omega\}$$

is not continuous. Then there exist $\varphi_0(x) \in C(\overline{\Omega})$, a sequence of functions $\{\varphi_j(x)\} \subset C(\overline{\Omega})$, and $u_{\varphi_j,l}(x) = \Phi_l(\varphi_j(x))$ and $\bar{u}(x) = \Phi_l(\varphi_0(x))$ such that

$$\lim_{j\to\infty} \|\varphi_j - \varphi_0\|_{C(\overline{\Omega})} = 0, \qquad \liminf_{j\to\infty} \|u_{\varphi_j,l} - \bar{u}\|_{1,\overline{\Omega}} > 0. \tag{2.4.2}$$

Without loss of generality, we assume

$$\|\varphi_j\|_{C(\overline{\Omega})} \leq 2\|\varphi_0\|_{C(\overline{\Omega})} + 1.$$

Then, by (2.4.1), the set $\{u_{\varphi_j,l}(x)\}$ is bounded in $C^{1,\beta_1}(\overline{\Omega})$. By the Ascoli-Arzela theorem, $\{u_{\varphi_j,l}\}$ converges to some $u_l \in C^1(\overline{\Omega})$ in $C^1(\overline{\Omega})$ (up to a subsequence). Fix any function $\phi \in C_0^\infty(\Omega)$. Using ϕ as a test function in the corresponding equations $-\operatorname{div} \mathbf{A}^l(x, u_{\varphi_j,l}, \nabla u_{\varphi_j,l}) + h(u_{\varphi_j,l}) = \varphi_j$ and letting $j \to \infty$, one readily deduces that $-\operatorname{div} \mathbf{A}^l(x, u_l, \nabla u_l) + h(u_l) = \varphi_0$ (cf. the continuity of $\mathcal{A}$ in Section 2.3), namely, $u_l = \Phi(\varphi_0(x)) = \bar{u}$. This is an immediate contradiction to (2.4.2).

Finally, the positivity of Φ_l is a direct consequence of the comparison principle and the strong maximum principle, in view of the positivity condition. This completes the proof. □

In the sequel, we shall say that the operator Φ_l is associated with the pair $(\mathbf{A}^l, h^l)$.

We now are able to introduce an equivalent operator form for the BVP (2.1.5) and (2.1.7). Put

$$X := \{\mathbf{u} \in C^1(\overline{\Omega})\}, \qquad Y = \{\mathbf{u} \in C(\overline{\Omega})\}$$

being Banach spaces equipped with the standard norms. For $l = 1, \ldots, k$, let $h^l \in C(\mathbb{R})$ be monotone functions satisfying (2.3.1) with $m = m_l$. Set

$$T_l(\mathbf{u}) := f^l(x, \mathbf{u}, \nabla\mathbf{u}) + h^l(u_l), \qquad \mathbf{T}(\mathbf{u}) = (T_1(\mathbf{u}), \ldots, T_k(\mathbf{u})) : X \mapsto Y.$$

One readily verifies that $\mathbf{T}$ is continuous and bounded.

Let Φ_l be the operator given in Lemma 2.4.1, $l = 1, \ldots, k$, associated with $(\mathbf{A}^l, h^l)$. Denote

$$Z := \{\mathbf{u} \in C^1(\overline{\Omega}) \mid \mathbf{u}(x) = \mathbf{u}_0(x) \text{ on } \partial\Omega\}.$$

Define

$$\Lambda_l = \Phi_l \circ T_l \qquad \Lambda = \Phi \circ \mathbf{T} := (\Lambda_1(\mathbf{u}), \ldots, \Lambda_k(\mathbf{u})) : X \to Y \to Z.$$

We have the following result.

LEMMA 2.4.2. *The operator $\Lambda : X \to Y \to Z \hookrightarrow X$ is compact. Moreover, the BVP* (2.1.5) *and* (2.1.7) *is equivalent to the operator equation*

$$\mathbf{u} = \Lambda(\mathbf{u}). \tag{2.4.3}$$

The operator Λ is said to be associated with the pair $(\mathbf{A}, \mathbf{h}) = [(\mathbf{A}^1, \ldots, \mathbf{A}^k), (\mathbf{h}^1, \ldots, \mathbf{h}^k)]$.

PROOF. The compactness of Λ is trivial since each component of Λ is compact by Lemma 2.4.1. It also follows directly from the definition of Λ that the BVP (2.1.5) and (2.1.7) is equivalent to the operator equation (2.4.3). □

Plainly, a fixed point $\mathbf{u} \in X$ of Λ is a solution of the BVP (2.1.5) and (2.1.7) in Z. By Lemma 2.2.3, then $\mathbf{u}$ is actually in $C^{1,\beta}(\overline{\Omega})$ for some $\beta > 0$. Thus all solutions under consideration will be assumed at least in the space $C^{1,\beta}(\overline{\Omega})$ for some $\beta > 0$.

2.5. *Existence theorems*

In this subsection, we apply the fixed point theorems to prove two general existence results for the BVP (2.1.5) and (2.1.7). Specifically, we show the existence of a fixed-point of the equivalent operator equation (2.4.3).

We say that the BVP (2.1.5) and (2.1.7) has an a priori estimate property (AP1), provided

(AP1). There exists $C > 0$ (independent of t and $\mathbf{u}$) such that

$$\|\mathbf{u}\|_{L^\infty(\Omega)} \leq C$$

for all solutions $\mathbf{u}$ of

$$\operatorname{div} \mathbf{A}^l(x, u_l/t, \nabla u_l/t) + f^l(x, \mathbf{u}, \nabla \mathbf{u}) = 0 \quad \text{in } \Omega; \qquad l = 1, 2, \ldots, k,$$
$$\mathbf{u} = \mathbf{u}_0 \quad \text{on } \partial\Omega,$$

where $t \in (0, 1]$ is a parameter.

Our first existence result is a direct consequence of the Leray–Schauder fixed point Theorem 1.2.5.

THEOREM 2.5.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then there exists $\beta \in (0, 1)$ such that the BVP* (2.1.5) *and* (2.1.7) *has a solution $\mathbf{u} \in C^{1,\beta}(\overline{\Omega})$, provided it satisfies an a priori estimate property* (AP1).

PROOF. In Lemma 2.4.2, take $h^l \equiv 0$ for $l = 1, 2, \ldots, k$. By Lemma 2.4.2, it suffices to show the operator equation (2.4.3) admits a solution in

$$X := \{\mathbf{u} \in C^1(\overline{\Omega}) | \mathbf{u} = \mathbf{u}_0 \text{ on } \partial\Omega\}.$$

To this end, we apply the Leray–Schauder fixed point Theorem 1.2.5 to the compact operator $\Lambda : X \to Y \to Z \hookrightarrow X$ and it remains to verify the a priori estimate condition: there exists a positive number $R > 0$ such that $\|\mathbf{u}\| \le R$ for all $\mathbf{u} \in X$ satisfying $\mathbf{u} = t\Lambda(\mathbf{u})$ with $t \in [0, 1]$. But one readily sees that this is equivalent to (AP1) (noting $t = 0$ implies $\mathbf{u} \equiv 0$), in view of the regularity Lemma 2.2.3. □

Next, we consider the question of existence of a positive function $\mathbf{u}$ (i.e., every component being positive!) satisfying the BVP (2.1.5) and (2.1.7) with homogeneous boundary data, namely, $\mathbf{u}_0 \equiv 0$. We shall assume that all pairs $(\mathbf{A}^l, f^l)$, $l = 1, \ldots, k$ satisfy a positivity condition with

$$h^l(t) = L|t|^{m_l - 2}t, \tag{2.5.1}$$

where $L > 0$ is constant (usually large). Clearly the growth condition (2.3.1) is observed.

The BVP (2.1.5) and (2.1.7) is said to satisfy an a priori estimate property (AP2), provided

(AP2). There exists $C > 0$ (independent of t and $\mathbf{u}$) such that

$$t + \|\mathbf{u}\|_{L^\infty(\Omega)} \le C. \tag{2.5.2}$$

for all solutions $\mathbf{u}$ of

$$\begin{aligned} \operatorname{div} \mathbf{A}^l(x, u_l, \nabla u_l) + f^l(x, \mathbf{u}, \nabla \mathbf{u}) + \Phi_l^{-1}(t\Phi_l(1)) = 0 \quad &\text{in } \Omega; \\ l = 1, 2, \ldots, k, \\ \mathbf{u} = 0 \quad &\text{on } \partial\Omega \end{aligned} \tag{2.5.3}$$

where $t \ge 0$ is a parameter and Φ_l is the operator given in Lemma 2.4.1, $l = 1, \ldots, k$, associated with $(\mathbf{A}^l, h^l)$.

We also need a "superlinearity" at the origin for the function $\mathbf{f}$. That is, for each $l = 1, \ldots, k$, there exist nonnegative continuous functions b^l and c^l such that

(1) For $l = 1, \dots, k$, there holds for $(x, \mathbf{u}, Q) \in \Omega \times (\mathbb{R}_+ \times \cdots \times \mathbb{R}_+) \times \mathbb{R}^{nk}$

$$f^l(x, \mathbf{u}, Q) + b^l(x, \mathbf{u}, Q) + c^l(x, \mathbf{u}, Q) = o((|u_l| + |Q_l|)^{\sigma_l}),$$

as $(\mathbf{u}, Q) \to 0$ uniformly on Ω, where

$$\sigma_l = \begin{cases} m_l - 1, & \text{if } \delta = 0, \\ \max(1, m_l - 1) & \text{if } \delta = 1. \end{cases}$$

(2) For $l = 1, \dots, k$, $b^l(x, 0, 0) = c^l(x, 0, 0) = 0$.

Remarks. 1. The “superlinearity” condition often can be replaced by a slightly weaker assumption related to the first eigenvalue of the principal part, see details in the sections below.

2. The superlinearity implies that $f^l(x, 0, 0) = 0$, $l = 1, \dots, k$.

Now we are ready to give our second existence result.

THEOREM 2.5.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Assume that all pairs* $(\mathbf{A}^l, f^l)$, $l = 1, \dots, k$, *satisfy a positivity condition with h^l being given in* (2.5.1) *and $\omega^l \equiv 0$. Suppose that each f^l is superlinear near the origin and that the BVP* (2.1.5) *and* (2.1.7) *has an a priori estimate property* (AP2). *Then the BVP* (2.1.5) *and* (2.1.7) *has a nonnegative solution $\mathbf{u}$ with at least one nonvanishing component. Moreover, every component of $\mathbf{u}$ is either identically zero or strictly positive on Ω.*

PROOF. Let (A^l, h^l), $l = 1, \dots, k$, be the pair given in (2.5.1) from the positivity property for $(\mathbf{A}^l, f^l)$. Let Λ_l be the operator given in Lemma 2.4.1 associated with the pair (A^l, h^l), $l = 1, \dots, k$, and Λ the compact operator given in Lemma 2.4.2. We shall apply the refined Krasnosle’skii fixed point Theorem 1.3.4 to the compact operator Λ on the cone

$$\mathcal{C} := \{\mathbf{u} \in X \mid \mathbf{u} \geq 0\}, \quad X := \{\mathbf{u} \in C^1(\bar{\Omega}) \mid \mathbf{u} = 0 \text{ on } \partial\Omega\}.$$

We need to verify all conditions of Theorem 1.3.4 and divide the proof into five steps.

Step 1. To show $\Lambda(0) = 0$ and Λ is $\mathcal{C}$-preserving. First, for $l = 1, \dots, k$, $f^l(x, 0, 0) = 0$ by the superlinearity near of $\mathbf{f}$, and $h^l(0) = 0$ by definition. Hence $\mathbf{T}(0) = 0$. We claim $\Phi_l(0) = 0$ for $l = 1, \dots, k$. To this end, we use $u_l = \Phi_l(0)$ as a test function in $(2.1.6)_l$ to obtain

$$\begin{aligned} 0 &= \int_\Omega A^l(|\nabla u_l|) a^l_{i,j} \partial_i u_l \partial_j u_l + L \int_\Omega |u_l|^{m_l} \\ &\geq C \int_\Omega (\delta + |\nabla u_l|)^{m_l - 2} |\nabla u_l|^2 + L \int_\Omega |u_l|^{m_l}, \end{aligned}$$

since $u_l = \Phi_l(0)$ satisfies (2.3.2) with $\varphi = 0$. Thus $u_l \equiv 0$, i.e., $\Phi_l(0) = 0$, $l = 1, \dots, k$, whence $\Lambda(0) = 0$. By the positivity condition on $\mathbf{f}$ and the definition of $\mathbf{T}$, $\mathbf{T}(\mathbf{u}) \geq 0$ for all $\mathbf{u} \in \mathcal{C}$. It follows that $\Lambda(\mathbf{u}) \in \mathcal{C}$ for all $\mathbf{u} \in \mathcal{C}$ since Φ_l, $l = 1, \dots, k$, is positive by Lemma 2.4.1.

Step 2. For $t \in [0, 1]$, there exists a positive number r such that $\mathbf{u} \neq t\Lambda(\mathbf{u})$ for $\mathbf{u} \in \mathcal{C}$ with $\|\mathbf{u}\| = r$. Consider $\mathbf{u} = t\Lambda(\mathbf{u})$ for $t \in (0, 1]$ ($t = 0$ implies $\mathbf{u} = 0$). That is, for $l = 1, \dots, k$ and $t \in (0, 1]$,

$$-\operatorname{div} \mathbf{A}^l(x, u_l/t, \nabla u_l/t) + [h^l(u_l/t) - h^l(u_l)] = f^l(x, \mathbf{u}, \nabla \mathbf{u}).$$

By the monotonicity of h^l, we have

$$h^l(u_l/t) - h^l(u_l) \geq 0,$$

since $t \in (0, 1]$ and $u_l \geq 0$. In turn

$$\begin{aligned}&-\operatorname{div} \mathbf{A}^l(x, u_l/t, \nabla u_l/t) + b^l(x, \mathbf{u}, \nabla \mathbf{u}) + c^l(x, \mathbf{u}, \nabla \mathbf{u}) \\ &\quad \leq f^l(x, \mathbf{u}, \nabla \mathbf{u}) + b^l(x, \mathbf{u}, \nabla \mathbf{u}) + c^l(x, \mathbf{u}, \nabla \mathbf{u}),\end{aligned} \tag{2.5.4}$$

where b^l and c^l are nonnegative functions to satisfy the superlinear condition on f^l near the origin. Multiply (2.5.4) by $t^{m_l-1}u_l \geq 0$ and integrate over Ω to obtain

$$\begin{aligned}C\int_\Omega (t\delta + |\nabla u_l|)^{m_l-2}|\nabla u_l|^2 &\leq t^{m_l-2}\int_\Omega A^l(|\nabla u_l|/t)a^l_{i,j}\partial_i u_l \partial_j u_l \\ &\leq t^{m_l-1}\int_\Omega [f^l(x, \mathbf{u}, \nabla \mathbf{u}) + b^l(x, \mathbf{u}, \nabla \mathbf{u}) \\ &\quad + c^l(x, \mathbf{u}, \nabla \mathbf{u})]u_l - t^{m_l-1}\int_\Omega [b^l(x, \mathbf{u}, \nabla \mathbf{u}) \\ &\quad + c^l(x, \mathbf{u}, \nabla \mathbf{u})]u_l \\ &\leq t^{m_l-1}\int_\Omega o(|u_l|^{\sigma_l+1} + |\nabla u_l|^{\sigma_l+1}) \\ &= t^{m_l-1}\int_\Omega o(|\nabla u_l|^{\sigma_l+1}),\end{aligned}$$

as $\|\mathbf{u}\| \to 0$, where we have used (A3), the positivity of b^l and c^l and the superlinearity of f^l. It follows that there exists $r_0 > 0$ such that the equation $\mathbf{u} = t\Lambda(\mathbf{u})$ has no positive solutions in $B_{r_0}(0) - \{0\}$ for all $t \in [0, 1]$.

Step 3. There exist positive numbers t_0 and R and a $\mathbf{u}_0 \in \mathcal{C} - \{0\}$ such that

$$\mathbf{u} \neq \Lambda(\mathbf{u}) + t\mathbf{u}_0 \tag{2.5.5}$$

for $t \geq t_0$ and $\mathbf{u} \in \mathcal{C}$ with $\|\mathbf{u}\| \leq R$.

Take $\mathbf{u}_0 = \Phi(1)$ and consider the equation

$$\mathbf{u} = \Lambda(\mathbf{u}) + t\Phi(1)$$

in $\mathcal{C}$, that is, $\mathbf{u}$ is a nonnegative solution of the equation (2.5.3). By the assumption (AP2), (2.5.2) holds. Take $t_0 = C + 1$, where $C > 0$ is given in (2.5.2). Then (2.5.5) holds as long as $t \geq t_0$. Note particularly that the choice of $R > 0$ can be arbitrary.

Step 4. There exists a positive number R such that (2.5.5) holds for all $t \geq 0$ and $\mathbf{u} \in \mathcal{C}$ with $\|\mathbf{u}\| = R$. By Lemma 2.2.3, there exists $R_0 > 0$ (independent of $\mathbf{u}$ and t by (2.5.2)) such that

$$\|\mathbf{u}\|_X \leq R_0.$$

Just take $R = R_0 + 1$ and the result follows.

Step 5. Now we can finish the proof by applying Theorem 1.3.4. Plainly, taking $X = X$ and $\Lambda = \Lambda$, one readily verifies that all conditions of Theorem 1.3.4 are satisfied by Steps

1–4 above. Therefore the mapping Λ has a fixed point $\mathbf{w} \in \mathcal{C}$ with $\|\mathbf{w}\| \in [r, R]$, which is a nonnegative solution of the BVP (2.1.5) and (2.1.7) with $\|\mathbf{w}\| \geq r > 0$. Clearly $\mathbf{w} \geq 0$ must have at least one nontrivial component since $\|\mathbf{w}\| \geq r > 0$. By the strong maximum principle Lemma 2.2.4, each component of $\mathbf{w}$ must be either strictly positive or identically zero. The proof is complete. □

The existence Theorems 2.5.1 and 2.5.2 given above provide numerous applications in the existence theory of elliptic boundary value problems. Here we mainly focus on applications of existence of a positive solution, given in Theorem 2.5.2. We shall consider semi-linear scalar equations, semi-linear systems of equations, and quasi-linear equations and systems separately and the results are given in the remaining sections.

3. Single semi-linear equations

In this section, we apply Theorem 2.5.2 to a prototype of single semi-linear equations, in which

$$k = 1, \qquad \mathbf{A}^1(x, u, \mathbf{p}) = \mathbf{p}.$$

Clearly the conditions (A1)–(A3) are valid. Throughout this section, we assume that $\mathbf{f} = f(x, u, \mathbf{p})$ is a nonnegative and continuous function. We shall also assume that Ω is bounded and the boundary condition (2.1.7) is homogeneous, i.e., $u_0 = 0$. We are looking for a positive solution to the BVP (2.1.5) and (2.1.7).

3.1. *Existence*

Consider the BVP

$$\begin{aligned} \Delta u + f(x, u, \nabla u) &= 0, \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{3.1.1}$$

We wish to apply Theorem 2.5.2 to show that the BVP (3.1.1) admits a positive solution. We have the following existence result.

THEOREM 3.1.1. *Assume that f satisfies the limit condition*

$$\liminf_{u\to\infty} \frac{f(x, u, \mathbf{p})}{u} > \lambda_1, \qquad \limsup_{u\to 0^+} \frac{f(x, u, \mathbf{p})}{u} < \lambda_1 \tag{3.1.2}$$

uniformly for $(x, \mathbf{p}) \in \Omega \times \mathbb{R}^n$, where λ_1 is the first eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary conditions. Then the BVP (3.1.1) has a positive solution, provided that one of the following alternatives holds:

(i) *$n > 2$ and there holds*

$$\lim_{u\to\infty} \frac{f(x, u, \mathbf{p})}{u^\gamma} = 0,$$

where $\gamma = (n + 1)/(n - 1)$.

(ii) *Assume* $f(x,u,\mathbf{p}) = f(u)$ *is locally Lipschitz on* $[0,\infty)$ *and the function* $f(u)u^{-2_*}$ *is nonincreasing for* $u > 0$. *Also suppose that*

$$\lim_{u\to\infty} \frac{f(u)}{u^{2_*}} = 0, \qquad \limsup_{u\to\infty} \frac{uf(u) - \theta F(u)}{u^2 f^{2/n}(u)} \le 0,$$

where F *is the primitive of* f *and* $\theta \in [0, 2^*)$.

(iii) *Assume* $f(x,u,\mathbf{p}) = f(u)$ *is continuously differentiable and there exist* $\delta, \gamma \in (0, 2_*)$ *such that*

$$\lim_{u\to\infty} \frac{f(u)}{u^{\gamma}} = u_0 > 0, \qquad \delta f(u) \ge uf'(u) \quad \text{for } u > 0.$$

Moreover, the solutions are in the space $W^{2,q}$ *for all* $q > 0$ *in Case* (i), *and in the space* $C^{2,\beta}_{\mathrm{loc}}(\Omega) \cap C_0(\Omega)$ *for all* $\beta \in (0,1)$ *in Cases* (ii) *and* (iii).

Part (i) is due to [9], (ii) to [18] and (iii) to [26,27]. In Part (i), the Laplace operator can be replaced by a linear (uniform) elliptic operator of second order. In Parts (ii) and (iii), the nonlinearity f can also depend on the independent variable x, or even on the gradient ∇u, see [9,18,26,27] (and also Section 5) for details.

PROOF OF THEOREM 3.1.1. We apply Theorem 2.5.2. First, one readily verifies that all conditions of Theorem 2.5.2, except the super-linearity at the origin and the a priori estimate (AP2), are valid. Next, we show that (3.1.2) implies Step 2 in the proof of Theorem 2.5.2, replacing the superlinearity. To this end, consider $u = t\Lambda(u)$ with $u \in X$, namely (taking $h = 0$),

$$\Delta u + tf(x,u,\nabla u) = 0.$$

In turn, by (3.1.2), there exists $\varepsilon > 0$ such that for $\|u\|_X$ small,

$$\lambda_1 \int_\Omega |u|^2 \le \int_\Omega |\nabla u|^2 = t\int_\Omega uf(x,u,\nabla u) \le (\lambda_1 - \varepsilon)\int_\Omega u^2.$$

Hence the only solution of $u = t\Lambda(u)$ with small $\|u\|_X$ is $u \equiv 0$, completing Step 2.

To prove (AP2), we need to show that (2.5.2) holds for all nonnegative solutions of the BVP

$$\begin{aligned} \Delta u + f(x,u,\nabla u) + t &= 0, && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.1.3}$$

since $\Phi^{-1}(t\Phi(1)) = t$, by taking $h(u) = 0$. The proofs will be carried out in the next three subsections, under the assumptions (i), (ii) and (iii) separately. □

For simplicity, we shall only give the proofs of (AP2) for the prototype $f(x,u,\mathbf{p}) \equiv u^p$ with $p > 1$. Needless to say, all three methods apply to more general settings, and the proofs for general f's are essentially similar to the ones demonstrated here.

3.2. *A priori estimate I: The Brezis–Turner approach*

In this subsection, we derive the a priori supremum bound (2.5.2) under the assumption (i), using an approach introduced in [9]. The key is to establish an a priori H^1-bound. Then the desired a priori supremum bounds follow from a standard boot-strap argument.

THEOREM 3.2.1 (Brezis–Turner). *Suppose that condition* (i) *holds. Then the a priori supremum bound* (2.5.2) *is valid for all nonnegative solutions of* (3.1.3).

The proof of Theorem 3.2.1 is divided into several lemmas. The first is an a priori bound for the parameter t and a weighted L^1 norm of $f(u) = u^p$.

LEMMA 3.2.1. *There exists a positive constant* $C = C(n, p, |\Omega|) > 0$ *such that*

$$t \le C\lambda_1^{p/(p-1)}, \qquad \int_\Omega \delta(x)u^p \le C\int_\Omega u^p\phi_1 \le C, \tag{3.2.1}$$

where $\delta(x) = \operatorname{dist}(x, \partial\Omega)$ *is the distance from* x *to the boundary.*

PROOF. Let $\phi_1 > 0$ be an associated first eigenfunction to λ_1, i.e.,

$$-\int_\Omega \phi\Delta\phi_1 = \lambda_1\int_\Omega \phi\phi_1,$$

for all $\phi \in H_0^1$. Multiply (3.1.3) by the test function ϕ_1 and integrate over Ω to obtain

$$\int_\Omega (t + u^p)\phi_1 = -\int_\Omega u\Delta\phi_1 = \lambda_1\int_\Omega u\phi_1. \tag{3.2.2}$$

Applying the Young inequality, one infers that there exists $C = C(p) > 0$ such that for $t \ge 0$

$$\inf_{u>0}(t + u^p) \ge Ct^{(p-1)/p}u. \tag{3.2.3}$$

Combining (3.2.2) and (3.2.3) immediately yields the conclusions of the lemma, noting

$$\lambda_1\int_\Omega u\phi_1 \le \frac{1}{2}\int_\Omega u^p\phi_1 + C|\Omega|,$$

by the Hölder inequality, since $p > 1$. □

The next result is the key H^1-bound.

LEMMA 3.2.2. *There exists a positive constant* $C = C(n, p, |\Omega|) > 0$ *such that*

$$\|u\|_{H^1} \le C.$$

PROOF. Multiply (3.1.3) by the test function u and integrate over Ω to obtain, with the aid of the Hölder inequality and Lemma 3.2.1 (both will be frequently used without further mentioning),

$$\int_\Omega |\nabla u|^2 = \int_\Omega (tu + u^{p+1}) \le C\int_\Omega (1 + u^{p+1}).$$

For $\gamma = 2/(N+1) \in (0,1)$ and $\varepsilon > 0$, we have

$$\begin{aligned}\int_\Omega u^{p+1} &\le \left(\int_\Omega \delta u^p\right)^\gamma \left(\int_\Omega u^{p+1/(1-\gamma)}/\delta^{\gamma/(1-\gamma)}\right)^{1-\gamma}\\ &\le \left(\int_\Omega u^{p+1/(1-\gamma)}/\delta^{\gamma/(1-\gamma)}\right)^{1-\gamma}\\ &\le \varepsilon\left(\int_\Omega u^{2/(1-\gamma)}/\delta^{\gamma/(1-\gamma)}\right)^{1-\gamma} + C\left(\int_\Omega u^{1/(1-\gamma)}/\delta^{\gamma/(1-\gamma)}\right)^{1-\gamma}\\ &= \varepsilon\|u\delta^{-\gamma/2}\|^2_{L^q} + C\|u\delta^{-\gamma}\|_{L^{q/2}},\end{aligned}$$

where $q = 2/(1-\gamma) = 2(n+1)/(n-1) > 0$ and we have used the Young inequality

$$u^{p+1/(1-\gamma)} \le \varepsilon u^{\tau+1/(1-\gamma)} + Cu^{1/(1-\gamma)} = \varepsilon u^{2/(1-\gamma)} + Cu^{1/(1-\gamma)},$$

since $p < \tau = 1/(1-\gamma) = (n+1)/(n-1)$ by (i). It follows that by choosing $\varepsilon > 0$ sufficiently small

$$\int_\Omega |\nabla u|^2 \le \varepsilon\|u\delta^{-\gamma/2}\|^2_{L^q} + C\|u\delta^{-\gamma}\|_{L^{q/2}} + C \le \frac{1}{2}\int_\Omega |\nabla u|^2 + C, \tag{3.2.4}$$

where we have used a Hardy–Littlewood–Sobolev-type inequality (cf. Lemma 2.2 of [9])

$$\|u\delta^{-r}\|_{L^q} \le C\|\nabla u\|_{L^2}, \qquad \frac{1}{q} = \frac{1}{2} - \frac{1-r}{n}.$$

Now the Lemma follows immediately from (3.2.4). □

PROOF OF THEOREM 3.2.1. We use Lemma 3.2.2 and a standard boot-strap argument. For any $\gamma > 0$, use u^γ as a test function in (3.1.3) to obtain

$$\|u^{2^*(\gamma+1)/2}\|_{L^1} \le C\|\nabla u^{(\gamma+1)/2}\|^{2^*}_{L^2} \le C\gamma\|u^{p+\gamma}\|^{2^*/2}_{L^1} + C\gamma.$$

Take

$$\gamma_1 = 2^* - p > 0, \qquad \gamma_{k+1} = 2^*(\gamma_k+1)/2 - p, \quad k = 1, \ldots,$$

Then direct computations yield

$$\gamma_{k+1} = (2^*/2)^k\gamma_1 + ((2^*/2)^k - 1)q = (2^*/2)^k(\gamma_1 + q) - q > 0,$$

where $q = (n-2)(2^* - 2p)/4 > -\gamma_1$ for all $p < 2_*$, and

$$\begin{aligned}\|u^{p+\gamma_{k+1}}\|_{L^1} &\le C\gamma_k\left(\|u^{p+\gamma_k}\|^{2^*/2}_{L^1} + 1\right) \le \cdots\\ &\le C^{\sum_{i=0}^k (2^*/2)^i}(\gamma_1\cdots\gamma_k)^{2^*/2}\left(\|u^{p+\gamma_1}\|^{(2^*/2)^k}_{L^1} + 1\right)\\ &\le C^{(2^*/2)^{k+1}}\left(\|u^{p+\gamma_1}\|^{(2^*/2)^k}_{L^1} + 1\right) \le C^{(2^*/2)^{k+1}}\left(C^{(2^*/2)^k} + 1\right).\end{aligned}$$

In turn

$$\|u^{p+\gamma_{k+1}}\|_{L^{p+\gamma_{k+1}}} \le C$$

and the lemma follows by letting $k \to \infty$ (noting $\gamma_k \to \infty$). □

3.3. *A priori estimate II: The de Figueiredo–Lions–Naussbaum approach*

In this subsection, we derive the a priori supremum bound (2.5.2) under the assumption (ii), using an approach introduced in [18]. The assumption (ii) (and (iii)) is essential optimum (at least) for the prototype $f(u) = u^p$ on star-shaped domains.

THEOREM 3.3.1 (de Figueiredo–Lions–Naussbaum). *Suppose that condition* (ii) *holds. Then the a priori supremum bound (2.5.2) holds for all nonnegative solutions of (3.1.3).*

The proof of Theorem 3.3.1 is spiritually similar to that of Theorem 3.2.1, namely, to establish an a priori H^1-bound. However, the proof is more involved and also requires a variational structure, among others. Note that (ii) implies $p < 2_*$.

PROOF OF THEOREM 3.3.1. The proof is divided into several steps.

Step I. There holds (3.2.1). This is simply Lemma 3.2.1.

Step II. There exist positive constants $\delta_0 = \delta_0(n, p, \Omega) > 0$ and $C = C(n, p, \Omega) > 0$ such that

$$\sup_{\delta(x)<\delta_0} |u(x)| \le C, \qquad \sup_{\delta(x)<\delta_0} |\nabla u(x)| \le C. \tag{3.3.1}$$

We first show that every solution u of (3.1.3) possesses the following monotone property. (M). There exist positive numbers ε, γ and C such that for all $x \in \Omega_\varepsilon = \{x \in \overline{\Omega}, \delta(x) < \varepsilon\}$, there exists a measurable set I_x such that

(1) $\operatorname{meas}(I_x) \ge \gamma$.
(2) $I_x \subset \Omega - \Omega_\varepsilon$.
(3) $u(x) \le Cu(\xi)$ for all $\xi \in I_x$.

We shall omit the proof of property (M), which is based on the well-known moving-plane method. We refer the reader to [18], or see Section 4.4, for details.

Assuming (M), let us prove $(3.3.1)_1$. By (M2), there exists a positive constant $C = C(\varepsilon, \phi_1) > 0$ such that

$$\phi_1(\xi) \ge C, \qquad \xi \in I_x$$

since $I_x \subset \Omega - \Omega_\varepsilon$. Using (M) and Step I, one readily infers that for $x \in \Omega_\varepsilon$

$$\begin{aligned} u^p(x)\operatorname{meas}(I_x) &= \int_{I_x} u^p(x) \le C \int_{I_x} u^p(\xi)d\xi \\ &\le C \int_{I_x} \phi_1 u^p(\xi)d\xi \le C \int_\Omega \phi_1 u^p(\xi)d\xi = C. \end{aligned}$$

In turn,

$$u(x) \le C, \quad x \in \Omega_\varepsilon,$$

which is $(3.3.1)_1$ with $\delta_0 = \varepsilon$. $(3.3.1)_2$ follows from $(3.3.1)_1$ and standard regularity theory.

Step III. There exist positive constant $C = C(n, p, \Omega) > 0$ such that

$$\int_\Omega |\nabla u|^2 \le C. \tag{3.3.2}$$

In proving (3.2.2), the Pohozaev identity [38] below plays a key role, which however requires a variational structure. Denote ν the unit outer-normal to $\partial\Omega$ and let (t, u) be a solution of (3.1.3). Then for any $y \in \mathbb{R}^n$, there holds

$$\int_{\partial\Omega} |\nabla u|^2 (x - y) \cdot \nu = \left(\frac{2n}{p+1} - (n-2)\right) \int_{\Omega} u^{p+1} + (n+2)t \int_{\Omega} u. \quad (3.3.3)$$

Now combining (3.3.1) and (3.3.3), we infer that

$$\int_{\Omega} u^{p+1} \le \int_{\partial\Omega} |\nabla u|^2 (x-y) \cdot \nu \le C,$$

since

$$\frac{2n}{p+1} - (n-2) > \frac{2n}{2^*} - (n-2) = 0.$$

It follows that, by using the test function u in (3.1.3)

$$\int_{\Omega} |\nabla u|^2 = \int_{\Omega} u^{p+1} + t \int_{\Omega} u \le C \int_{\Omega} u^{p+1} + C \le C,$$

since $t \le C\lambda_1^{p/(p-1)}$ by Lemma 3.2.1.

Step IV. There exist positive constant $C = C(n, p, \Omega) > 0$ such that

$$\sup_{x \in \Omega} |u(x)| \le C.$$

This is precisely the proof of Theorem 3.2.1, which remains valid for all $p < 2_*$. Now the proof is complete by combining Steps I and IV. □

3.4. *A priori estimate III: The Gidas–Spruck approach*

In this subsection, we derive the a priori supremum bound (2.5.2) under the assumption (iii), using an approach introduced in [26,27]. This approach is distinctly different from the Brezis–Turner approach or the de Figueiredo–Lions–Naussbaum approach. It blows up a sequence of positive solutions at points where the solutions assume their (global) maximums. Then one can show that the a priori supremum bound (2.5.2) holds if the limiting equation does not admit any positive solutions on either the entire space or the half-space. Such a nonexistence is usually referred to as Liouville theorems and plays a central role in the arguments.

THEOREM 3.4.1 (Gidas–Spruck). *Suppose that condition* (iii) *holds. Then the a priori supremum bound* (2.5.2) *is valid for all nonnegative solutions of* (3.1.3).

PROOF OF THEOREM 3.4.1. The proof is by contradiction. Suppose that (2.5.2) is false. Then there exists a sequence of solutions $\{t_j, u_j(x)\}$ of (3.1.3) such that

$$\lim_{j \to \infty} \|u_j\|_{L^\infty(\Omega)} = \infty,$$

since $t_j \le C$ for some $C > 0$ by (3.2.1). Put

$$M_j = \max_{x\in\Omega} u_j(x) = u_j(\xi^j) \to \infty, \quad \xi^j \in \Omega.$$

Denote

$$v(y) = M_j^{-1} u_j(x), \quad y = (x - \xi^j) M_j^{\alpha}, \quad \alpha = (p-1)/2$$

and

$$\Omega_j = \{y \in \mathbb{R}^n \mid x = \xi^j + M_j^{-\alpha} y \in \Omega\}.$$

By direct calculations, v_j satisfies

$$\begin{aligned} \Delta v_j + v_j^p &= 0, && \text{in } \Omega_j \\ v_j &= 0 && \text{on } \partial\Omega_j, \end{aligned} \tag{3.4.1}$$

and

$$0 < v_j(y) \le 1, \quad y \in \Omega_j; \qquad v_j(0) = 1. \tag{3.4.2}$$

Invoking Lemma 2.2.3, one then infers that there exist positive constants $\beta = \beta(n, \partial\Omega) \in (0,1)$ and $C = C(\partial\Omega, n, p) > 0$ such that

$$\|v_j\|_{C^{1,\beta}(\overline{\Omega}_j)} \le C. \tag{3.4.3}$$

In particular, by combining $(3.4.1)_2$, (3.4.2) and (3.4.3), one deduces that there exists $C = C(\partial\Omega, n, p) > 0$ such that

$$dist(0, \partial\Omega_j) = d_j M_j^{\alpha} \ge C, \qquad d_j = dist(\xi^j, \partial\Omega). \tag{3.4.4}$$

Next we consider two cases.

Case I: The sequence $\{d_j M_j^{\alpha}\}$ is unbounded, say (without loss of generality), $d_j M_j^{\alpha} \to \infty$ as $j \to \infty$. Plainly $\Omega_j \to \mathbb{R}^n$ as $j \to \infty$. With the aid of (3.4.3), one can apply the Ascoli-Arzela theorem to derive that there exists a nonnegative function $v \in C^{1,\beta/2}(\mathbb{R}^n)$ such that[2]

$$\lim_{j\to\infty} v_j(y) = v(y) \ge 0, \qquad v(0) = 1, \tag{3.4.5}$$

uniformly on any compact subset of $\mathbb{R}^n$ in $C^{1,\beta/2}$-topology.

Now fix any function $\phi \in C_0^{\infty}(\mathbb{R}^n)$. Taking ϕ as a test function in (3.4.1) (for j large so Ω_j contains the support of ϕ) and letting $j \to \infty$, one readily verifies that the limiting function $v \ge 0$ satisfies the following limiting equation

$$\Delta v + v^p = 0 \quad \text{in } \mathbb{R}^n$$

with $p \in (1, 2_*)$ by (iii). Thus $v \equiv 0$ by Lemma 4.2.1. This contradicts the fact $v(0) = 1$.

[2]Here and in the sequel, the convergence is understood up to a subsequence.

Case II: The sequence $\{d_j M_j^{\alpha}\}$ is bounded as $j \to \infty$. In this case, with a proper rotation, the sequence of domains Ω_j converges (up to a subsequence) to the half-space $\mathbb{R}^n_\varepsilon = \{y \in \mathbb{R}^n \mid y_n > -\varepsilon\}$ for some $\varepsilon > 0$, in view of (3.4.4). Similarly as in Case I), one deduces that there exists a nonnegative function $v \in C^{1,\beta/2}(\mathbb{R}^n_\varepsilon)$ such that

$$\lim_{j\to\infty} v_j(y) = v(y) \geq 0, \qquad v(0) = 1,$$

uniformly on any compact subset of $\mathbb{R}^n_\varepsilon$ in $C^{1,\beta/2}$-topology. Moreover, v vanishes on $\partial\mathbb{R}^n_\varepsilon$ since v_j vanishes on $\partial\Omega_j$ for every j. It follows that $v \geq 0$ satisfies the

$$\begin{aligned} \Delta v + v^p &= 0 \qquad \text{in } \mathbb{R}^n_\varepsilon \\ v &= 0 \qquad \text{on } \partial\mathbb{R}^n_\varepsilon \end{aligned}$$

with $p \in (1, 2_*)$. Thus $v \equiv 0$ by Lemma 4.2.1, again a contradiction to the fact $v(0) = 1$. □

4. Systems of semi-linear equations

In this section, we turn our attention to systems of semi-linear elliptic equations, i.e., $k \geq 2$. As in Section 3, we consider that the principal parts of (2.1.5) are the Laplacians, namely,

$$\mathbf{A}^l(x, u, \mathbf{p}) = \mathbf{p}, \quad l = 1, \ldots, k.$$

We shall assume that the function $\mathbf{f}$ satisfies the following conditions.

(1) For simplicity, $\mathbf{f}(x, \mathbf{u}, Q) = \mathbf{f}(x, \mathbf{u})$ is independent of Q.

(2) $\mathbf{f}$ satisfies a positivity condition of the form

$$\mathbf{f}(x, \mathbf{u}) + L\mathbf{u} \geq 0,$$

where $L > 0$ is a (large) constant.

(3) $\mathbf{f}$ is superlinear near the origin, namely, there holds the limit

$$\limsup_{|\mathbf{u}|\to 0^+} \frac{\mathbf{u} \cdot \mathbf{f}(x, \mathbf{u})}{|\mathbf{u}|^2} < \lambda_1$$

uniformly on Ω.

Finally, we assume that the boundary condition (2.1.7) is homogeneous, i.e., $\mathbf{u}_0 = 0$ whenever Ω is bounded. We are concerned with the question of existence of a nonnegative and nontrivial vector-valued function $\mathbf{u}$ satisfying the BVP (2.1.5) and (2.1.7) on bounded Ω's. This issue was raised as an open question in the survey article [33] and has since been studied by many authors, see for example [14,16,19,37,43,44,52,58,59] and the references therein. Needless to say, due to the presence of multiple components, the structure of systems is more complicated than that of scalar equations. For instance, (2.1.5) generically does not admit a variational structure when $k > 1$, even with $\mathbf{f}$ being independent of $\mathbf{p}$. Consequently variational methods typically do not apply. Here we shall exclusively focus ourselves on systems *without* variational structure.

4.1. *Existence*

We are seeking positive solutions of the BVP

$$\begin{aligned} \Delta u_l + f^l(x, \mathbf{u}) &= 0, \quad \text{in } \Omega \text{ for } l = 1, \dots, k, \\ \mathbf{u} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{4.1.1}$$

The system (4.1.1) is said to be *fully coupled* if for each $j \in J$, there exists $i \in I$ such that $f^j(x, \mathbf{u}) = 0$ for $\mathbf{u} \geq 0$ with $u_i = 0$, where $\{I, J\}$ is a partition of the set $\{1, \dots, k\}$.

We say that (4.1.1) is *strongly coupled* if it is fully coupled and it admits a nonnegative solution with (at least) a nontrivial component. A fully coupled (4.1.1) need not to be strongly coupled. Let $k = 2$ and $\Omega = \mathbb{R}^n$ and consider

$$f^1(u_1, u_2) = u_2^a, \qquad f^2(u_1, u_2) = u_1^b, \quad 0 < a,\ b < 2^*.$$

Then one readily verifies that (4.1.1) is fully coupled, but has no nonnegative solutions with nontrivial components and thus is not strongly coupled. On the other hand, a strongly coupled (4.1.1) need not to have *positive* solutions. For example, consider $\Omega = \mathbb{R}^n$, $k = 2$, $n = 3$ and

$$f^1(u_1, u_2) = u_1 u_2^2, \qquad f^2(u_1, u_2) = u_1^2 u_2.$$

Then (4.1.1) is fully coupled and has a constant solution $\mathbf{u} = (1, 0)$, so is strongly coupled. But it does not admit a positive solution (Lemma 4.2.4).

In this section, we shall study two canonical prototype 2-systems (i.e., $k = 2$). We first consider a fully, but not strongly, coupled (i.e., weakly coupled), being a variation of the so-called Lane–Emden system. We say that $\mathbf{f} = (f^1, f^2)$ satisfies a growth-limit condition (GL), provided.

(GL). There exist positive numbers $p, q, r, s \geq 1$ and continuous functions $a(x), b(x), c(x)$ and $d(x)$ on $\overline{\Omega}$ such that

(1) $r, s < 2_*$ and

$$s - 1 \neq \frac{pq - 1}{p + 1}, \qquad r - 1 \neq \frac{pq - 1}{q + 1}.$$

(2) The functions $a(x)$ and $d(x)$ are either identically zero or strictly positive on $\overline{\Omega}$, while $b(x)$ and $c(x)$ are strictly positive on $\overline{\Omega}$.

(3) There hold the limits

$$\lim_{u_1 + u_2 \to \infty} \frac{f^1(x, \mathbf{u})}{a(x) u_1^r + b(x) u_2^q} = 1, \qquad \lim_{u_1 + u_2 \to \infty} \frac{f^2(x, \mathbf{u})}{c(x) u_1^p + d(x) u_2^s} = 1,$$

and

$$\lim_{u_1 + u_2 \to \infty} \frac{f^1(x, \mathbf{u}) + f^2(x, \mathbf{u})}{u_1 + u_2} = \infty$$

uniformly on Ω and for $\mathbf{u} \geq 0$.

In other words, the nonlinearity $\mathbf{f}$ behaves like sums of powers at infinity if $\mathbf{f}$ satisfies a (GL) condition. Under this (GL) assumption, then the BVP (4.1.1) admits a positive solution $\mathbf{u}$. We present our existence results for $n = 3$ and $n > 3$ separately. The first one is for $n = 3$.

THEOREM 4.1.1. *Let $k = 2$ and $n = 3$. Suppose that $\mathbf{f}$ satisfies a growth-limit condition* (GL) *with either $pq = 1$ or $pq > 1$ and*

$$\frac{2(p+1)}{pq-1} + \frac{2(q+1)}{pq-1} > 1.$$

Then (4.1.1) *has a nonnegative solution $\mathbf{u}$ with a nonvanishing component.*

If, in addition, $\mathbf{f}$ is fully coupled, then (4.1.1) *has a positive solution $\mathbf{u}$.*

Moreover, all solutions $\mathbf{u}$ are in the space $W^{2,l}(\Omega) \cap C_0(\Omega)$ for all $l > 0$.

The conclusion of Theorem 4.1.1 is optimum for the Lane–Emden system in which $f^1 = u_2^q$ and $f^2 = u_1^p$. When $n > 3$, a weaker conclusion holds.

THEOREM 4.1.2. *Let $k = 2$ and $n > 3$. Suppose that $\mathbf{f}$ satisfies a growth-limit condition* (GL) *with either $pq = 1$ or*

(1) *$pq > 1$ and*

$$\max\left\{\frac{2(p+1)}{pq-1}, \frac{2(q+1)}{pq-1}\right\} \geq n-2;$$

or

2. $\max\{p, q\} \leq 2_*, \qquad \min\{p, q\} < 2_*.$

Then the BVP (4.1.1) *has a nonnegative solution $\mathbf{u}$ with a nonvanishing component.*

If, in addition, $\mathbf{f}$ is fully coupled, then (4.1.1) *has a positive solution $\mathbf{u}$.*

Moreover, all solutions $\mathbf{u}$ are in the space $W^{2,l}(\Omega) \cap C_0(\Omega)$ for all $l > 0$.

PROOF OF THEOREMS 4.1.1 AND 4.1.2. The proofs for Theorems 4.1.1 and 4.1.2 are similar to that of Theorem 3.1.1. Indeed, one readily verifies that all conditions of Theorem 2.5.2 hold (including the superlinearity). Hence the existence of a nonnegative solution $\mathbf{u}$ with at least one strictly positive component is a direct consequence of Theorem 2.5.2, once the a priori estimate (AP2) is obtained. The positivity of $\mathbf{u}$ follows from Lemma 4.1.1 below since each component of $\mathbf{u}$ is either identically zero or strictly positive on Ω by Theorem 2.5.2. The proof of (AP2) will be given in Subsection 4.5, by using the blow-up method. □

LEMMA 4.1.1. *Suppose that $\mathbf{f}$ is fully coupled. Then all nonnegative solutions of* (4.1.1) *must be either strictly positive or identically zero on Ω.*

PROOF. Let $\mathbf{u}$ be a nonnegative solution of (4.1.1). By Lemma 2.2.4, each component of $\mathbf{u}$ is either identically zero or strictly positive. We want to show that $\mathbf{u}$ is identically zero, if $\mathbf{u}$ has a vanishing component. Suppose the contrary. Then $\mathbf{u}$ has both a strictly positive component and a vanishing component, say $u_2 > 0$ and $u_1 \equiv 0$, respectively. Choose $I = \{1\}$ and $J = \{2, \ldots, k\}$. By the definition of $\mathbf{f}$ being fully coupled, for $j = 2 \in J$,

$f^2(x, \mathbf{u}) = 0$ for $\mathbf{u} \geq 0$ with $u_1 = 0$ (i.e., $1 \in I$). In particular, $f^2(x, \mathbf{u}) = 0$ on Ω. It implies that $u_2 \equiv 0$, a contradiction. □

It is worth remarking that this approach also applies to systems of general nonlinearities $\mathbf{f}(x, \mathbf{u}, \nabla\mathbf{u})$ under suitable restrictions, see for example [19]. One may also see Section 5 for a flavor of handling the dependence on the gradient $\nabla\mathbf{u}$.

For an integer $k \geq 1$, denote Π the first (open) octant of $\mathbb{R}^k$

$$\Pi = \{\mathbf{u} = (u_1, u_2, \ldots, u_k) \in \mathbb{R}^k \mid u_i > 0\}$$

and $\Xi = \overline{\Pi}$ the closure of Π. For $\mathbf{p},\ \mathbf{u} \in \Xi$, we write

$$|\mathbf{p}| = p_1 + p_2 + \cdots + p_k, \qquad \mathbf{u}^{\mathbf{p}} = u_1^{p_1} \cdot u_2^{p_2} \cdots u_k^{p_k},$$

where we have used the notation $0^0 = 1$. Denote $\mathcal{P}$ the set of all real $k \times k$ matrices $P = \{p_{ij}\}_{k\times k}$ with nonnegative entries. For $P \in \mathcal{P}$, put its row vectors

$$\mathbf{p}_l := (p_{l1}, \ldots, p_{lk}) \in \Xi, \quad l = 1, \ldots, k.$$

For $k > 1$, let $\mathbf{p}_l \in \Xi$ and

$$f^l(\mathbf{u}) = f^l(x)\mathbf{u}^{\mathbf{p}_l}, \quad l = 1, \ldots, k, \tag{4.1.2}$$

where $f^l(x)$ are positive continuous functions on $\overline{\Omega}$.

Now consider $k = 2$ and

$$\min\{p_{21}, p_{12}\} > 0, \qquad p_{11} + p_{22} > 0.$$

This ensures that (4.1.1)–(4.1.2) is strongly coupled.

Our canonical prototype of strongly coupled 2-systems is then (4.1.1) with $k = 2$ and $\mathbf{f}$ given by (4.1.2). We have the following existence result.

THEOREM 4.1.3. *Suppose $k = 2$ and that the exponents $\mathbf{p}_1$ and $\mathbf{p}_2$ satisfy*

$$1 < \min\{|\mathbf{p}_1|, |\mathbf{p}_2|\} < 2_*, \qquad \max\{|\mathbf{p}_1|, |\mathbf{p}_2|\} \leq 2_*,$$

(*recalling $2_* = \infty$ for $n = 2$*) *and*

$$p_{21}p_{12} > (1 - p_{11})(1 - p_{22}).$$

Then the BVP (4.1.1)–(4.1.2) has a positive solution $\mathbf{u}$ in the space $W^{2,l}(\Omega) \cap C_0(\Omega)$ for all $l > 0$.

PROOF. One readily verifies all conditions of Theorem 2.5.2 are satisfied, whence the existence of a nonnegative solution $\mathbf{u}$ with a strictly positive component is ensured once the a priori estimate (AP2) is obtained, whose derivation is deferred to in Section 4.6.

The positivity of $\mathbf{u}$ is from Lemma 4.1.1. □

In [44], the authors also considered the BVP (4.1.1)–(4.1.2) with $k = 2$. For $a, b \in \mathbb{R}$, denote

$$\hat{b} = \frac{(n+1)b}{n+1-(n-1)a}, \qquad \hat{c} = \frac{(n+1)c}{n+1-(n-1)d}.$$

Below is an existence result obtained in [44].

THEOREM 4.1.4. *Suppose $k = 2$ and $\mathbf{f}$ is given by (4.1.2) with $a = p_{11}$, $b = p_{12}$, $c = p_{21}$ and $d = p_{22}$, and $f^1(x) = f^2(x) \equiv 1$. Assume*

$$\max(a, d) < \frac{n+1}{n-1},$$

and either $\hat{b}\hat{c} \le 1$, or

$$\max\left\{\frac{2(\hat{b}+1)}{\hat{b}\hat{c}-1}, \frac{2(\hat{c}+1)}{\hat{b}\hat{c}-1}\right\} > n-1.$$

Also assume one of the following alternatives holds

(1) $\max(a, d) < 1$ *and* $bc > (1-a)(1-d)$.
(2) $a \le 1/2$, $d \le c+1$, $c+d > a+b \ge 1$ *and* $\hat{c} \ge 1$.
(3) $a \le \min(b+1, c+1/2)$, $d \le c+1$, $c+d = a+b > 1$ *and* $\min(\hat{b}, \hat{c}) \ge 1$.

Then the BVP (4.1.1) has a positive solution $\mathbf{u}$ in the space $C^2_{\mathrm{loc}}(\Omega) \cap C_0(\Omega)$.

Moreover, the existence continues to hold if

$$d \in (0, 1), \qquad 1 \le a, c < \frac{n+1}{n-1},$$

and one of the following alternatives holds

(1) $b \ge d$, $c \le a$ *and* $\hat{b} \ge 1$.
(2) $b < d$, $c \le a(1-d)/(1-b)$ *and* $\hat{b} \ge 1$.
(3) $c < (n+1)(1-d)/(n-1)$ *and* $\hat{b} \ge 1$.

To conclude this section, we present a general existence theorem for (4.1.1) due to [14]. It applies to arbitrary $k > 1$ and requires no special forms of $\mathbf{f}$.

THEOREM 4.1.5. *Let $\mathbf{f} = \mathbf{f}(\mathbf{u})$ be a smooth function. Suppose that $\mathbf{f} \ge 0$ for $\mathbf{u} \ge 0$ and $\mathbf{f}$ is quasi-monotone, i.e., $\partial f^l/\partial u_j \ge 0$ for $l \ne j$ and $\mathbf{u} \ge 0$. Also assume that $\mathbf{f}$ is superlinear at infinity and sublinear at the origin, that is,*

$$\liminf_{u_l \to \infty} \frac{f^l(\mathbf{u})}{u_l} > \lambda_1, \qquad \liminf_{|\mathbf{u}| \to 0^+} \frac{\mathbf{u} \cdot \mathbf{f}(\mathbf{u})}{|\mathbf{u}|^2} < \lambda_1$$

(uniformly for $u_j \ge 0$ with $j \ne l$). Then the BVP (4.1.1) has a nonnegative solution $\mathbf{u}$ with a nonvanishing component, provided one of the following alternatives holds

(i) *There holds*

$$\lim_{|\mathbf{u}| \to \infty} \frac{|\mathbf{f}(\mathbf{u})|}{|\mathbf{u}|^{\gamma}} = 0,$$

where $\gamma = (n+1)/(n-1)$.

(ii) *Assume Ω is strictly convex (with positive sectional curvature) and there holds*

$$\lim_{|\mathbf{u}| \to \infty} \frac{|\mathbf{f}(\mathbf{u})|}{|\mathbf{u}|^{\gamma'}} = 0,$$

where $\gamma' < n/(n-2)$ for $n \ge 3$ and $\gamma' < \infty$ for $n = 2$.

If, in addition, $\mathbf{f}$ *is fully coupled, then* (4.1.1) *has a positive solution* $\mathbf{u}$*.*
Moreover, all solutions $\mathbf{u}$ *are in the space* $C^2_{\mathrm{loc}}(\Omega) \cap C_0(\Omega)$*.*

By using Theorem 2.5.2, the proofs of Theorems 4.1.4 and 4.1.5 are again essentially reduced to deriving the a priori estimate (AP2). The derivation, though, is an extension of the Brezis–Turner method to systems (see also [11]). That is, one obtains an a priori H^1-bound first and then uses it to derive the desired estimate (AP2) via boot-strap. We shall omit the proofs.

The remaining subsections will be devoted to establish (AP2) for Theorem 4.1.1–Theorem 4.1.3 and the blow-up method will be employed. The proofs are relatively straight for the weakly coupled Lane–Emden system (Theorems 4.1.1 and 4.1.2), but become somewhat technical for the strongly coupled case (Theorem 4.1.3).

4.2. *Liouville theorems*

As mentioned earlier, Liouville theorems play a central role in the blow-up procedure. In this subsection, we gather several Liouville theorems which will be used later. We shall omit the proofs, since the references are readily available.

By a Liouville theorem, we mean that all nonnegative nontrivial solutions must vanish identically (on unbounded domains Ω). We first consider scalar equations. Let $\Omega \subset \mathbb{R}^n$ and suppose $f(x,u) \in C(\Omega \times \mathbb{R})$ and $c(x), g(x) \in C(\overline{\Omega})$. Consider

$$\Delta u + c(x)u + f(x,u) + g(x) = 0 \quad \text{in } \Omega. \tag{4.2.1}$$

As is custom, $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_n > 0\}$ denotes the half-space.

LEMMA 4.2.1. *Suppose*

$$c(x) \equiv 0, \quad g(x) \equiv \kappa \geq 0, \quad f(x,u) \geq 0 \text{ for } u \geq 0,$$

where κ *is constant. Then there hold*

(A) *All nonnegative solutions* $u \geq 0$ *of* (4.2.1) *on* $\Omega = \mathbb{R}^n$ *must be constant if* $\kappa = 0$ *and* $f \equiv 0$.
(B) *If* $\kappa > 0$*, then* (4.2.1) *admits no nonnegative solution on* $\Omega = \mathbb{R}^n$.
(C) *Suppose* $\kappa = 0$ *and* $f(x,u) = u^p$*. Then all nonnegative solutions* $u \geq 0$ *of* (4.2.1) *on* $\Omega = \mathbb{R}^n$ *must be identically zero if* $p \in (0, 2_*)$.
(D) *Suppose* $\kappa = 0$ *and* $f(x,u) = u^p$*. Then all nonnegative solutions* $u \geq 0$ *of* (4.2.1) *on* $\Omega = \mathbb{R}^n_+$*, vanishing on* $\partial\mathbb{R}^n_+$*, must be identically zero if* $p \in [1, 2_*)$.

Part (A) is the classical Liouville theorem. Part (B) follows from Lemmas 2.7 and 2.8, [50].[3] Part (C) is due to [26] for $p \in [1, 2_*)$ and later generalized to the full range in [10]. Part (D) was obtained in [27] (a wider range of p was obtained in [15] for bounded solutions).

[3] Lemma 2.8 was proved for $\Delta u + u^p \leq 0$ with $p \in (0,1)$ in [50]. A slight modification shows the arguments are valid for the inequality $\Delta u + \kappa \leq 0$ with $\kappa > 0$.

We next consider the so-called Lane–Emden system in which

$$k = 2, \qquad f^1(\mathbf{u}) = v^p; \quad f^2(\mathbf{u}) = u^q,$$

where $p,\ q > 0$ are positive numbers and $\mathbf{u}(x) = (u(x), v(x))$. Then (2.1.5) becomes

$$\begin{aligned} \Delta u + v^p &= 0, \\ \Delta v + u^q &= 0, \end{aligned} \qquad x \in \Omega. \tag{4.2.2}$$

We shall state Liouville theorems for (4.2.2) on $\Omega = \mathbb{R}^n$ and $\Omega = \mathbb{R}^n_+$ separately.

LEMMA 4.2.2. *Let $\Omega = \mathbb{R}^n$. Then the following conclusions hold.*

(A) *The only nonnegative solution* $\mathbf{u}$ *of (4.2.2) is the trivial solution, provided either $pq \le 1$ or*

$$pq > 1 \quad \textit{and} \quad \max\left\{\frac{2(p+1)}{pq-1}, \ \frac{2(p+1)}{pq-1}\right\} \ge n-2.$$

(B) *The only nonnegative solution* $\mathbf{u}$ *of (4.2.2) is the trivial solution, provided $n = 3$ and (p, q) satisfies*

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{1}{3}.$$

(C) *Suppose that the pair p and q satisfy*

$$\min(p, q) < 2_*, \qquad \max(p, q) \le 2_*.$$

Then the only nonnegative solution $\mathbf{u}$ *of (4.2.2) is the trivial solution.*

Part A) of Lemma 4.2.2 was first proved in [36] for $p, q > 1$, and later generalized in [49] to the full range. Part B) was obtained in [49] for solutions having algebraic growth at infinity and generalized in [39] for all nonnegative solutions. Finally, Part C) is due to [17].

We also have the following Liouville theorem for solutions of (4.2.2) on $\Omega = \mathbb{R}^n_+$.

LEMMA 4.2.3. *Let $p, q \ge 1$ and $\Omega = \mathbb{R}^n_+$. Then the only nonnegative solution* $\mathbf{u}$ *of (4.2.2) is the trivial solution, provided one of the following alternatives holds*

(A) $n = 3$.

(B) $n = 4$ *and*

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{1}{3}.$$

(C) $n \ge 5$ *and either*

$$\max\left\{\frac{2(p+1)}{pq-1}, \ \frac{2(p+1)}{pq-1}\right\} \ge n-3,$$

or

$$\min(p, q) < 2_*, \qquad \max(p, q) \le 2_*.$$

Lemma 4.2.3 was first proved in [8] for bounded solutions and generalized in [39] for all nonnegative solutions.

Now we turn our attention to a general (2.1.5) when $\mathbf{f}$ is given by (4.1.2) with arbitrary $k > 1$ and $f^l(x) \equiv 1$. We have the following Liouville theorem for (2.1.5) on the entire space $\Omega = \mathbb{R}^n$.

LEMMA 4.2.4. *Suppose $k > 1$ and $\Omega = \mathbb{R}^n$. Let $\mathbf{f}$ be given by (4.1.2). Assume that $\mathbf{f}$ is strictly pseudo-subcritical, namely,*

$$\max_l |\mathbf{p}_l| < 2_*.$$

Then all nonnegative solutions of (2.1.5) are necessarily constant. In particular, (2.1.5) does not admit any positive solution.

If, in addition, (2.1.5) is fully coupled, then the conclusion continues to hold with the weaker assumption that $\mathbf{f}$ is pseudo-subcritical, namely,

$$\min_l |\mathbf{p}_l| < 2_*, \qquad \max_l |\mathbf{p}_l| \le 2_*.$$

These are special cases of early results of [24,45].

When $\Omega = \mathbb{R}^n_+$, we need to consider

$$\begin{aligned} \Delta u_l + \kappa_l x_n^{\sigma_l} \mathbf{u}^{\mathbf{p}_l} + \delta_l &= 0 \quad \text{in } \mathbb{R}^n_+, \\ \mathbf{u} &= 0 \quad \text{on } \partial\mathbb{R}^n_+, \end{aligned} \tag{4.2.3}$$

where, for $l = 1, \ldots, k$ $(k \ge 1)$, $\mathbf{p}_l = (p_{l1}, \ldots, p_{lk}) \ge 0$ are constant vectors and $\kappa_l > 0$, $\sigma_l, \delta_l \ge 0$ are nonnegative numbers satisfying

$$\min_l \{\sigma_l + |\mathbf{p}_l|\} \ge 1 \text{ and } \max_l |\mathbf{p}_l| \le 2_* \quad (<\infty \text{ if } n = 2). \tag{4.2.4}$$

LEMMA 4.2.5. *Suppose* (4.2.4) *holds. Then the only nonnegative solutions* $\mathbf{u}$ *of* (4.2.3) *are $u = \mathbf{h} x_n$, where $\mathbf{h}$ is a nonnegative constant vector. Moreover, necessarily $\delta_l = \kappa_l = 0$, $l = 1, \ldots, k$.*

Remarks: 1. If $|\mathbf{p}_l| = 0$, then the condition $\sigma_l + |\mathbf{p}_l| = \sigma_l \ge 1$ is superfluous.

Lemma 4.2.5 is due to [62].

The following Harnack inequality for (4.2.1) will be used later.

LEMMA 4.2.6 (Harnack inequality). *Let $u \ge 0$ be a weak solution of (4.2.1) and suppose $f \equiv 0$. Then for any subdomain $\Omega' \subset\subset \Omega$, there exists*

$$C = C(n, \Omega', \Omega, \min\{1, dist(\Omega', \partial\Omega)\}, \|c\|_\Omega) > 0$$

such that

$$\sup_{\Omega'} u \le C \left(\inf_{\Omega'} u + \|g\|_\Omega \right).$$

This is a slight variation of the combination of Theorems 9.20 and 9.22, [28]. □

4.3. *A priori estimates I: The Lane–Emden system*

In this subsection, we derive the a priori estimate property (AP2) for (2.5.3) by employing the blow-up method. We have the following estimate.

THEOREM 4.3.1. *Suppose that all conditions of either Theorem 4.1.1 or Theorem 4.1.2 are satisfied. Then the a priori supremum bound (2.5.2) is valid for all nonnegative solutions $(t, \mathbf{u})$ of (2.5.3).*

We shall use the blow-up method and only prove Theorem 4.3.1 for

$$f^1 = u_1^r + u_2^q, \qquad f^2 = u_1^p + u_2^s.$$

The case for general **f**'s can be proved essentially the same, and is left to the reader (or see [58] for details). Then (2.5.3) becomes

$$\begin{aligned} \Delta u_1 + u_1^r + u_2^q + t &= 0 \quad \text{in } \Omega, \\ \Delta u_2 + u_1^p + u_2^s + t &= 0 \quad \text{in } \Omega, \\ \mathbf{u} &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{4.3.1}$$

where $t \geq 0$ is a parameter. The standard blow-up procedure for scalar equations can be directly generalized to treat systems of equations with a suitable structure (i.e., asymptotically weakly coupled, like the Lane–Emden system). The blow-up points are thus plainly at a point of $\max_l \sup_x u_l(x)$, where u_l are components of the solution.

The following formula is by direct calculations.

LEMMA 4.3.1. *Let $\mathbf{u}$ be a positive solution of (4.3.1). For $z \in \Omega$ and $S > 0$, $\tau_1, \tau_2 \geq 0$, put*

$$v_1(y) = S^{-1} u_1(x), \quad \bar{v}_2(y) = S^{-\tau_1} u_2(x), \qquad y = (x - z) S^{\tau_2}. \tag{4.3.2}$$

Then $\mathbf{v}$ satisfies

$$\begin{aligned} &\Delta v_1 + S^{r-1-2\tau_2} v_1^r + S^{q\tau_1 - 1 - 2\tau_2} v_2^q + t S^{-1-2\tau_2} = 0, \\ &\Delta v_2 + S^{p-\tau_1-2\tau_2} v_1^p + S^{s\tau_1 - \tau_1 - 2\tau_2} v_2^s + t S^{-\tau_1 - 2\tau_2} = 0. \end{aligned}$$

Using (GL), one readily infers that the parameter t is bounded from above.

LEMMA 4.3.2. *There exists $t_0 > 0$ (independent of $\mathbf{u}$ and t) such that*

$$t \leq t_0$$

for all $(t, \mathbf{u}) \geq 0$ satisfying (4.3.1).

PROOF. By (GL), there exists $M > 0$ (independent of $\mathbf{u}$ and t) such that for $\mathbf{u} \geq 0$ and $x \in \Omega$

$$u_1^r + u_2^q + u_1^p + u_2^s \geq \lambda_1 (u + v) - M. \tag{4.3.3}$$

Let ϕ_1 be a (normalized) first eigenfunction corresponding to λ_1. Using (ϕ_1, ϕ_1) as a test function in (4.3.1), we have

$$2t \int_\Omega \phi_1 + \int_\Omega \phi_1[u_1^r + u_2^q + u_1^p + u_2^s] = \lambda_1 \int_\Omega \phi_1(u + v). \tag{4.3.4}$$

Combining (4.3.3) and (4.3.4), we immediately get $t \le M/2$, completing the proof. □

PROOF OF THEOREM 4.3.1. Suppose that Theorem 4.3.1 is false. Then there exists a sequence of solutions $\{t_j, \mathbf{u}_j(x)\}_{j=1}^\infty$ of (4.3.1) such that

$$\lim_{j\to\infty} \|\mathbf{u}_j\|_{L^\infty(\Omega)} = \infty, \tag{4.3.5}$$

since $t_j \le t_0$ by Lemma 4.3.2. For $j = 1, 2, \ldots$ and $l = 1, 2$, put

$$U_{l,j} = \sup_{x\in\Omega} u_{l,j}(x) = u_{l,j}(\xi_{l,j}),$$

where $\xi_{l,j} \in \Omega$.

In Lemma 4.3.1, we take

$$\tau_1 = \frac{p+1}{q+1} > 0, \qquad \tau_2 = \frac{pq-1}{2(q+1)} \ge 0,$$

and for $j = 1, 2, \ldots$ and $l = 1, 2$,

$$v_l = v_{l,j}, \qquad S = S_j = U_{1,j} + U_{2,j}^{1/\tau_1} \to \infty.$$

The choice of z will be determined later. A direct calculation yields

$$q\tau_1 - 1 - 2\tau_2 = p - \tau_1 - 2\tau_2 = 0.$$

In turn,

$$\begin{aligned} \Delta v_{1,j} + S_j^{r-1-2\tau_2} v_{1,j}^r + v_{2,j}^q + t_j S_j^{-1-2\tau_2} &= 0,\\ \Delta v_{2,j} + v_{1,j}^p + S_j^{s\tau_1-\tau_1-2\tau_2} v_{2,j}^s + t_j S_j^{-\tau_1-2\tau_2} &= 0, \end{aligned} \tag{4.3.6}$$

where

$$r - 1 - 2\tau_2 = r - 1 - \frac{pq-1}{q+1} \ne 0,$$

$$s\tau_1 - \tau_1 - 2\tau_2 = \tau_1\left(s - 1 - \frac{pq-1}{p+1}\right) \ne 0$$

by our assumption. Moreover, the function $\mathbf{v}_j$ given by (4.3.2) satisfies

$$0 \le v_{1,j} \le 1, \qquad 0 \le v_{2,j} \le 1. \tag{4.3.7}$$

We next consider several cases.

Case 1. $r - 1 - 2\tau_2 > 0$. We first show

$$\lim_{j\to\infty} \frac{U_{1,j}}{U_{2,j}^{1/\tau_1}} = 0. \tag{4.3.8}$$

Taking $z = \xi_{1,j}$, then obviously it is equivalent to show

$$\lim_{j\to\infty} v_{1,j}(0) = 0.$$

Suppose for contradiction this is not true. Then there exist $\varepsilon_0 > 0$ and a subsequence (still using same subscripts) such that

$$v_{1,j}(0) \geq \varepsilon_0, \quad j = 1, 2, \ldots. \tag{4.3.9}$$

Put (abusing notation!)

$$\mathbf{w}_j(x) = \mathbf{v}_j(y), \quad x = yS_j^{(r-1-2\tau_2)/2}.$$

Therefore, by (4.3.6)$_1$, the functions $\{w_{1,j}(x)\}$ are bounded and satisfy

$$w_{1,j}(0) \in [\varepsilon_0, 1), \qquad \Delta w_{1,j} + w_{1,j}^r + S_j^{-(r-1-2\tau_2)/2} w_{2,j}^q + t_j S_j^{-r} = 0.$$

For each j, denote

$$d_j = dist(\xi_{1,j}, \partial\Omega).$$

Then

$$dist(0, \partial\Omega_j) = d_j S_j^{(r-1)/2}.$$

There are two possibilities. First, assume that the sequence $\{d_j S_j^{(r-1)/2}\}$ is unbounded. Then $\Omega_j \to \mathbb{R}^n$ and, by standard elliptic theory, the sequence $\{w_{1,j}\}$ (extracting a subsequence if necessary) converges uniformly to a nonnegative function $w \in C^2(\mathbb{R}^n)$ on any compact subset $\Sigma \subset \mathbb{R}^n$. By (4.3.3), (4.3.7), Lemma 4.3.2 and the fact $(r - 1 - 2\tau_2)/2 > 0$ one has

$$\lim_{j\to\infty} S_j^{-(r-1-2\tau_2)/2} w_{2,j}^q(x) = 0, \qquad \lim_{j\to\infty} t_j S_j^{-r} = 0,$$

uniformly on compact subset $\Sigma \subset \mathbb{R}^n$. Therefore w satisfies

$$\Delta w + w^r = 0 \quad \text{in } \mathbb{R}^n.$$

Thus $w \equiv 0$ by Lemma 4.2.1 since $r \in [1, 5)$, an immediate contradiction in view of (4.3.9).

Next suppose that $\{d_j S_j^{(r-1)/2}\}$ is bounded. Thanks to the smooth (C^1) boundary condition, the sequence $\{d_j S_j^{(r-1)/2}\}$ is bounded away from zero (standard by elliptic estimates, see for example [28]). In this case, there exist $s > 0$ and a nonnegative function $w \in C^2(\mathbb{R}^n_s)$, satisfying

$$\Delta w + w^r = 0 \text{ in } \mathbb{R}^n_s = \mathbb{R}^n \cap \{x^n > -s\}, \qquad w = 0 \quad \text{on } \partial\mathbb{R}^n_s.$$

Thus $w \equiv 0$ by Lemma 4.2.1, which yields a contradiction again. And (4.3.8) is proved.

Next, we further divide the proof into two subcases.

(i) $s\tau_1 - \tau_1 - 2\tau_2 > 0$. Similarly as above, we can utilize the fact $s\tau_1 - \tau_1 - 2\tau_2 > 0$ in (4.3.6)$_2$, by taking $z = \xi_{2,j}$, to derive

$$\lim_{j\to\infty} v_{2,j}(0) = 0.$$

That is,

$$\lim_{j\to\infty} \frac{U_{2,j}^{1/\tau_1}}{U_{1,j}} = 0.$$

This is impossible, in view of (4.3.8).

(ii) $s\tau_1 - \tau_1 - 2\tau_2 < 0$. Since $r - 1 - 2\tau_2 > 0$, (4.3.8) holds. Now taking $z = \xi_{2,j}$, then clearly one has

$$v_{2,j}(0) \to 1, \qquad v_{1,j}(y) \le \sup v_{1,j} \to 0 \tag{4.3.10}$$

as $j \to \infty$. Proceeding as in (i), with the aid of the fact $s\tau_1 - \tau_1 - 2\tau_2 < 0$ and (4.3.10), we pass to a limit in $(4.3.6)_2$ to infer that there exists $v \in C^2(\mathbb{R}^n)$ such that

$$\Delta v = 0, \qquad \text{in } \mathbb{R}^n, \qquad v(0) = 1,$$

provided the sequence $\{d_j S_j^{\tau_2}\}$ is unbounded, and for some $s > 0$

$$\Delta v = 0, \qquad \text{in } \mathbb{R}^n_s, \qquad v(0) = 1, \qquad v\Big|_{\partial \mathbb{R}^n_s} = 0,$$

provided $\{d_j S_j^{\tau_2}\}$ is bounded, where

$$d_j = dist(\xi_{2,j}, \partial\Omega), \qquad dist(0, \partial\Omega_j) = d_j S_j^{\tau_2}.$$

The second case cannot happen, since the Phragmèn-Lindelöf principle [40] implies v vanishes identically, contradicting the fact $v(0) = 1$. If the first possibility occurs, then

$$v \equiv v(0) = 1,$$

since all bounded harmonic functions on $\mathbb{R}^n$ must be constant. In turn,

$$\lim_{j\to\infty} v_{2,j}(y) = 1$$

uniformly for $y \in B = B_1(0)$. Moreover, by (4.3.8)

$$\lim_{j\to\infty} v_{1,j}(y) = 0$$

uniformly on B. On the other hand, applying Green's formula to $(4.3.6)_1$ on B yields

$$\begin{aligned} 0 \leftarrow v_{1,j}(0) &= \int_{\partial B} v_{1,j}(y)\frac{\partial G}{\partial \nu}(y,0)d\sigma + \int_B [S_j^{r-1-2\tau_2} v_{1,j}^r + v_{2,j}^q] G(y,0)dy \\ &\ge \int_{\partial B} v_{1,j}(y)\frac{\partial G}{\partial \nu}(y,0)d\sigma + \int_B v_{2,j}^q G(y,0)dy \\ &\to \int_B G(y,0)dy = c_n \end{aligned}$$

as $j \to \infty$, where $G(x, y)$ is the Green function on B, an absurdity.

Case 2. $r - 1 - 2\tau_2 < 0$. Again we consider two subcases.

(i) $s\tau_1 - \tau_1 - 2\tau_2 > 0$. The proof is essentially the same as that of (ii) of Case 1 (being a mirror image) and the detail is left to the reader.

(ii) $s\tau_1 - \tau_1 - 2\tau_2 < 0$. Plainly,

$$U_{1,j}S_j^{-1} + U_{2,j}S_j^{-\tau_1} \geq c > 0.$$

In turn, without loss of generality (by taking $z = \xi_{1,j}$ or $z = \xi_{2,j}$ accordingly), we may assume

$$v_{1,j}(0) + v_{2,j}(0) \geq c > 0.$$

Letting $j \to \infty$ in (4.3.6), similarly as in Case 1, one deduces that there exists $\mathbf{v} \geq 0$ satisfying (4.2.2) either on $\Omega = \mathbb{R}^n$ or on $\Omega = \mathbb{R}^n_s$ with $\mathbf{v}\Big|_{\partial\mathbb{R}^n_s} = 0$. Moreover

$$v_1(0) + v_2(0) \geq c > 0, \qquad v_1 + v_2 \leq 1. \tag{4.3.11}$$

By our assumption, all the conditions of either Lemma 4.2.2 or Lemma 4.2.3 are satisfied. Hence either Lemma 4.2.2 or Lemma 4.2.3 applies, which implies $\mathbf{v} \equiv 0$. But this is impossible, in view of (4.3.11).

It follows that (4.3.5) cannot hold and the proof is complete. □

4.4. *A priori estimates II: The strongly coupled case with $k = 2$*

In this subsection, we want to establish the (AP2) property for (4.1.1)–(4.1.2) with $k = 2$ (so-called strongly coupled systems). Then (2.5.3) becomes

$$\begin{aligned} \Delta u_1 + \mathbf{u}^{\mathbf{p}_1} + t &= 0 \quad \text{in } \Omega, \\ \Delta u_2 + \mathbf{u}^{\mathbf{p}_2} + t &= 0 \quad \text{in } \Omega, \\ \mathbf{u} &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{4.4.1}$$

where $t \geq 0$ is a parameter ($f^1(x) = f^2(x) \equiv 1$ for simplicity).

As demonstrated in the last section, the standard blow-up scheme, which blows up a solution at a point of $\sup_x u(x)$ (or $\max_l \sup_x u_l(x)$), can be directly used to treat weakly coupled systems of equations (e.g., the Lane–Emden system). For strongly coupled systems, however, the situation is more complicated. Indeed, the limit of the blown-up solutions may be with both trivial and positive components (thus *not positive*!) if the blow-up point is at a point of $\max_l \sup_x u_l(x)$ (i.e., components need not have (global) maximum at the same location). Consequently, the standard blow-up procedure above need not apply directly to strongly coupled systems whose limiting systems admit nonnegative solutions with positive components, even though they do not have positive solutions. In this case, there is no direct contradiction to the nonexistence of positive solutions should the limiting solution be not positive. To treat such a system via blow-up, one then needs to carefully coordinate the process to ensure that the limiting solution be positive. Indeed, special care will be used in treating (4.1.1)–(4.1.2) where the blow-up point is instead chosen at a point of $\max_l \sup_x[f^l(\mathbf{u})/u_l](x)$. Under the assumption $P - I$ being nonsingular, the new blow-up procedure developed here effectively converts obtaining a priori (supremum) estimates to proving a Liouville theorem (of *positive* solutions), recovering the very original feature of the blow-up procedure. In other words, Liouville theorems imply a priori estimates.

Yet adding to the complexity, the choice of the blow up points at points of $\max_l \sup_x [f^l(\mathbf{u})/u_l](x)$ no longer implies the uniform boundedness of the blown-up solutions (or components), which is an important component of the standard blow up procedure. To overcome this, the monotonicity Lemma 4.4.3 plays a crucial role in our new blow-up process.

We begin with notations and technical lemmas whose proofs are deferred. In the sequel, the conditions of Theorem 4.4.1 are enforced.

For $\gamma \in \mathbb{R}$ and $\nu \in S^{n-1}$ (a direction in $\mathbb{R}^n$), denote by $\Gamma_{\gamma,\nu}$ the hyperplane

$$\Gamma_{\gamma,\nu} = \{x \in \mathbb{R}^n \mid x \cdot \nu = \gamma\}.$$

We define

$$\Omega_{\gamma,\nu} = \{x \in \Omega \mid x \cdot \nu > \gamma\},$$

the positive cap of Ω with respect to $\Gamma_{\gamma,\nu}$ (in the direction of ν). Let $x^{\gamma,\nu}$ be the *reflection* in $\Gamma_{\gamma,\nu}$ of a point x in $\mathbb{R}^n$, that is,

$$x^{\gamma,\nu} = x + 2(\gamma - x \cdot \nu)\nu,$$

and similarly let $\Omega^{\gamma,\nu}$ be the reflection in $\Gamma_{\gamma,\nu}$ of a set Ω in $\mathbb{R}^n$,

$$\Omega^{\gamma,\nu} = \{x^{\gamma,\nu} \mid x \in \Omega\}.$$

For $z \in \partial\Omega$, denote the (unit) outer-normal $\nu_z = \nu(z)$ at z. Put

$$\gamma_z := \sup_{x\in\Omega}\{x \cdot \nu_z\} \geq z_\nu := z \cdot \nu_z.$$

Let $\mathbf{u}$ be a positive solution of (4.4.1). Write

$$n_l(x) := \mathbf{u}^{\mathbf{p}_l}(x) u_l^{-1}(x), \qquad l = 1, 2.$$

Denote for $l = 1, 2$

$$U_l := \max_{x\in\Omega} u_l(x) > 0, \qquad N_l := \sup_{x\in\Omega} n_l(x) > 0.$$

By the homogeneous Dirichlet boundary data, plainly there exists $\xi_l \in \Omega$ such that

$$U_l = u_l(\xi_l) \in (0, \infty), \quad l = 1, 2.$$

For $|\mathbf{p}_l| \geq 1$, using Lemma 4.6.2, without loss of generality one may assume that there exists $\zeta_l \in \Omega$ such that

$$n_l(\zeta_l) = N_l, \quad l = 1, 2.$$

Our first lemma is a lower bound of the blow-up distance.

LEMMA 4.4.1. *Assume* $\max(\bar{N}_1, \bar{N}_2) = \infty$. *For* $j = 1, 2, \ldots,$ *put*

$$N_j := \max\{N_{1,j}, N_{2,j}\} \to \infty, \qquad \zeta_j := \zeta_{l,j} \text{ if } N_l = N_{l,j},$$

and

$$Q_j^2 := \max\{N_j, t_j u_{1,j}^{-1}(\zeta_j), t_j u_{2,j}^{-1}(\zeta_j)\}.$$

Then there holds

$$\lim_{j\to\infty} \tau_j = \lim_{j\to\infty} Q_j \operatorname{dist}(\zeta_j, \partial\Omega) = \infty.$$

The second one is an upper bound for the parameter t.

LEMMA 4.4.2. *There exists a constant $t_0 > 0$ independent of $\mathbf{u}$ or t such that*

$$t \le t_0.$$

The last lemma is the following monotonicity property.

LEMMA 4.4.3. *There exists $\delta_0 > 0$ such that for all $\gamma \in (\gamma_z - \delta_0, \gamma_z)$ there holds*

$$\mathbf{u}(x) < \mathbf{u}(x^{\gamma,\nu_z}), \quad x \in \Omega_{\gamma,\nu_z},$$

where ν_z is the (unit) outer-normal to $\partial\Omega$ at z.

We now are ready to prove the supremum a priori estimate (AP2) for (4.4.1). Note under the assumptions below, the above three Lemmas 4.4.1–4.4.3 are all applicable.

THEOREM 4.4.1. *Suppose that the conditions of Theorem 4.1.3 are satisfied. Then the supremum a priori estimates (2.5.2) holds for all nonnegative solutions $(t, \mathbf{u})$ of (4.4.1).*

PROOF. Suppose for contradiction that (2.5.2) is false. Then there exist a sequence of nonnegative solutions $\{(t_j, \mathbf{u}_j)\}$ of (4.4.1) such that

$$0 < U_{2,j} + U_{2,j} \to \infty,$$

as $j \to \infty$ since $t_j \le t_0$ by Lemma 4.4.2. It is understood the notations $n_{l,j}(x)$, $U_{l,j}$, $N_{l,j}$, $\xi_{l,j}$ and $\zeta_{l,j}$ are used to denote the quantities above associated with $\mathbf{u}_j$, $j = 1, 2, \ldots.$ By Lemma 2.2.4, each component of $\mathbf{u}_j$ is either strictly positive or identically zero. If $\mathbf{u}_j$ is trivial, there is nothing left to prove. If, say, $u_{1,j} \equiv 0$ and $u_{2,j} > 0$, then one readily verifies that (4.4.1) reduces to a single equation with $t = 0$ and the estimate (2.5.2) is then well known (noting $|\mathbf{p}_l| > 1$. As a result, we only need to consider $\mathbf{u}_j > 0$. Without loss of generality, we assume

$$\lim_{j\to\infty} U_{1,j} = \lim_{j\to\infty} \max\{U_{1,j}, U_{2,j}\} = \infty. \tag{4.4.2}$$

For $z^j \in \Omega$ and $Q_j \ge 1$ to be determined later, we make the following change of variables

$$v_{l,j}(y) = \frac{u_{l,j}(x)}{u_{l,j}(z^j)}, \qquad y = (x - z^j)Q_j; \quad l = 1, 2; \quad j = 1, 2, \ldots. \tag{4.4.3}$$

Put

$$\begin{aligned} &\Omega_j := \{y \in \mathbb{R}^n \mid y = (x - z^j)Q_j, x \in \Omega\};\\ &\tau_j := dist(z^j, \partial\Omega)Q_j = dist(0, \partial\Omega_j). \end{aligned} \tag{4.4.4}$$

Clearly

$$\mathbf{v}_j(0) \equiv (1, 1), \quad j = 1, 2, \ldots. \tag{4.4.5}$$

By direct calculations, $\mathbf{v}_j$ satisfies

$$\begin{aligned} \Delta v_{l,j} + Q_j^{-2}\bar{n}_{l,j}(y)v_{l,j} + Q_j^{-2}t_j u_{l,j}^{-1}(z^j) &= 0 \quad \text{in } \Omega_l, \\ \mathbf{v}_j &= 0 \quad \text{on } \partial\Omega_j, \end{aligned} \tag{4.4.6}$$

for $l = 1, 2$ and $j = 1, 2, \ldots,$ where

$$\bar{n}_{l,j}(y) = n_{l,j}(x) = \mathbf{u}_j^{\mathbf{p}_l}(x)u_{l,j}^{-1}(x) = n_{l,j}(z^j)\mathbf{v}_j^{\mathbf{p}_l}(y)v_{l,j}^{-1}(y).$$

Denote

$$\bar{N}_l = \lim_{j\to\infty} N_{l,j} \in [0, \infty], \quad l = 1, 2.$$

By Lemma 4.4.3, there exists $\delta_0 > 0$ such that

$$\max_{x\in\Omega} u_{l,j}(x) = \max_{x\in\Omega_0} u_{l,j}(x), \quad l = 1, 2; \quad j = 1, 2, \ldots, \tag{4.4.7}$$

where

$$\Omega_0 = (\Omega\backslash\Omega_{\delta_0}) := \{x \in \Omega \,|\, dist(x, \partial\Omega) \geq \delta_0\}.$$

Note that one may actually choose $\xi_{l,j} \in \Omega_0$ by (4.4.7). We shall utilize these facts without further mentioning about it. For simplicity and convenience, in the sequel, we shall also assume that all sequences (of numerals) converge to a nonnegative quantity including infinity.

We divide the proof into three cases.

Case I). $\bar{N}_1 = 0$. In (4.4.6), take

$$z^j = \xi_{1,j}, \qquad Q_j \equiv 1.$$

Since the sequence $\{t_j\}$ is bounded by Lemma 4.4.2, thanks to (4.4.2), simple computations yield

$$0 \leq v_{1,j}(y) \leq 1, \qquad Q_j^{-2}\bar{n}_{1,j}(y)v_{1,j} = o(1), \qquad Q_j^{-2}t_jU_{1,j}^{-1} = o(1)$$

uniformly for all $y \in \Omega_j$.

Without loss of generality, assume $\Omega_j = \Omega$ for all $j \geq 1$ (they possibly differ by a translation). Applying Lemma 2.2.3 to the first equations $(4.4.6)_{1,j}$, we see that there exists $\beta \in (0, 1)$ such that the sequence $v_{1,j}$ are bounded in the Banach space $C^{1,\beta}(\overline{\Omega}) \cap C_0(\overline{\Omega})$. It follows, by the Ascoli-Arzela theorem, that there exists $v \in C^{1,\beta/2}(\overline{\Omega}) \cap C_0(\Omega)$ such that

$$\lim_{j\to\infty} v_{1,j}(y) = v(y)$$

in $C^{1,\beta/2}(\overline{\Omega}) \cap C_0(\Omega)$.

Fix any function $\phi \in C_0^\infty(\Omega)$. Taking ϕ as a test function in $(4.4.6)_{1,j}$ and letting $j \to \infty$, one immediately deduces that v satisfies

$$\begin{aligned} \Delta v &= 0, \quad \text{in } \Omega, \\ v &= 0, \quad \text{on } \partial\Omega. \end{aligned}$$

Clearly $v \equiv 0$ on Ω. This contradicts the fact $v(0) = \lim_{j\to\infty} v_{1,j}(0) = 1$, in view of (4.4.5), and completes the proof of Case I).

Case II). $\bar{N}_1 = \infty$. In (4.4.6) we take

$$z^j = \zeta_j, \qquad Q_j^2 = \max\{N_j, t_j u_{1,j}^{-1}(\zeta_j), t_j u_{2,j}^{-1}(\zeta_j)\} \to \infty,$$

where

$$N_j := \max\{N_{1,j}, N_{2,j}\} \to \infty, \qquad \zeta_j := \zeta_{l,j} \text{ if } N_j = N_{l,j}.$$

By Lemma 4.4.1,

$$\tau_j := Q_j dist(\zeta_j, \partial\Omega) \to \infty, \qquad B_{\tau_j}(0) \subset \Omega_j \to \mathbb{R}^n.$$

Fixing any compact subset $\Gamma \subset \mathbb{R}^n$, we have

$$|Q_j^{-2}\bar{n}_{l,j}(y)| \le 1, \qquad |Q_j^{-2}t_j u_{l,j}^{-1}(\zeta_j)| \le 1$$

on Γ for $l = 1, 2$, for all j sufficiently large (so that Ω_j contains Γ). Applying the Harnack inequality Lemma 4.2.6 to $(4.4.6)_{l,j}$, we obtain that $\mathbf{v}_j$ are uniformly bounded on Γ for j sufficiently large since $\mathbf{v}_j(0) = (1, 1)$.

Clearly there exist $\zeta_0 \in \Omega$ such that

$$\lim_{j\to\infty} \zeta_j = \zeta_0 \in \overline{\Omega}.$$

By the choices of Q_j, there exist nonnegative numbers $\kappa_1, \kappa_2, \delta_1, \delta_2 \in [0, 1]$ such that

$$\lim_{j\to\infty} Q_j^{-2}t_j u_{l,j}^{-1}(\zeta_j) = \delta_l, \qquad \lim_{j\to\infty} Q_j^{-2}\bar{n}_{l,j}(\zeta_j) = \kappa_l; \quad l = 1, 2,$$

with $\delta_1 + \delta_2 + \kappa_1 + \kappa_2 \ge 1 > 0$.

Similarly as in I), applying Lemma 2.2.3 to the equations $(4.4.6)_{l,j}$, we deduce that there exists $\mathbf{v} \in C^{1,\beta/2}_{\text{loc}}(\mathbb{R}^n)$ such that

$$\lim_{j\to\infty} \mathbf{v}_j(y) = \mathbf{v}(y) \ge 0, \qquad \mathbf{v}(0) = (1, 1)$$

and

$$\lim_{j\to\infty} Q_j^{-2}\bar{n}_{l,j}(y)v_{l,j}(y) = \lim_{j\to\infty} Q_j^{-2}n_{l,j}(\zeta_j)\mathbf{v}_j^{\mathbf{p}_l}(y) = \kappa_l \mathbf{v}^{\mathbf{p}_l}(y)$$

uniformly on any compact subset of $\mathbb{R}^n$ (in $C^{1,\beta/2}$-topology). Moreover, one readily verifies that the limiting function $\mathbf{v}$ satisfies the following limiting equations

$$\Delta v_l + \kappa_l \mathbf{v}^{\mathbf{p}_l} + \delta_l = 0 \quad \text{in } \mathbb{R}^n.$$

Applying Lemma 4.2.1 to each equation above respectively, one sees that $\delta_1 = \delta_2 = 0$ since $\kappa_1, \kappa_2, \delta_1, \delta_2 \ge 0$ and $\mathbf{v}(y) \ge 0$. It follows that

$$\Delta v_l + \kappa_l \mathbf{v}^{\mathbf{p}_l} = 0 \quad \text{in } \mathbb{R}^n$$

with $\kappa_1 + \kappa_2 \ge 1 > 0$. If both κ_1 and $\kappa_2 > 0$, then Lemma 4.2.4 applies to the above system by our assumptions and consequently $\mathbf{v}(y) \equiv 0$, a contradiction to $\mathbf{v}(0) = (1, 1)$.

Therefore either κ_1 or κ_2 must be zero. If, say, $\kappa_1 = 0$, then $v_1(y) \equiv 1$ by Lemma 4.2.1 since $\Delta v_1 = 0$ and $v_1(0) = 1$. Thus $\kappa_2 = 1$ and the second equation ($v_1 \equiv 1!$) reduces to

$$\Delta v_2 + v_2^{p_{22}}(y) = 0 \text{ in } \mathbb{R}^n.$$

Clearly $p_{22} \in [0, 2_*)$ since $p_{22} < p_{21} + p_{22} \leq 2_*$ with $p_{21} > 0$. Hence $v_2(y) \equiv 0$ by Lemma 4.2.1. This is again impossible in view of $v_2(0) = 1$. One reaches a similar contradiction if assuming $\kappa_2 = 0$ and $\kappa_1 > 0$.

Case III). $\bar{N}_1 \in (0, \infty)$. We first show that $\bar{N}_2 = \infty$. Suppose for contradiction $\bar{N}_2 < \infty$. For constants $R_{l,j} > 0$ to be determined, make the following change of variables

$$w_{l,j}(x) = u_{l,j}(x) R_{l,j}^{-1}, \quad l = 1, 2,\ j = 1, 2, \ldots.$$

Then (4.4.1) becomes

$$\begin{aligned} \Delta w_{l,j} + n_{l,j}(x) w_{l,j} + t_j R_{l,j}^{-1} &= 0 \quad \text{in } \Omega, \\ w_{l,j} &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{4.4.8}$$

for $l = 1, 2$ and $j = 1, 2, \ldots$. By our assumptions and Lemma 4.4.2, there exists a constant $C > 0$ such that

$$|n_{l,j}(x)| \leq C, \qquad |t_j| \leq C, \qquad t_j U_{1,j}^{-1} \to 0 \tag{4.4.9}$$

uniformly on Ω for $l = 1, 2$ and $j = 1, 2, \ldots$.

Taking $R_{l,j} \equiv 1$ and applying the Harnack inequality (Lemma 4.2.6) to (4.4.8) with (4.4.9), one infers that $\forall \delta > 0$ there exists $C = C(\delta, n, \Omega) > 0$ (also depending on the constant C in (4.4.9)) such that

$$\sup_{x \in \Omega^\delta} u_{l,j}(x) \leq C \inf_{x \in \Omega^\delta} u_{l,j}(x), \quad j = 1, \ldots, k, \tag{4.4.10}$$

where $\Omega^\delta = \{x \in \Omega \,|\, dist(x, \partial\Omega) > \delta\}$. We now claim that there exists $c_0 > 0$

$$\sup_{x \in \Omega_0} n_{1,j}(x) \geq c_0 N_{1,j} \geq c_0 \bar{N}_1/2 \tag{4.4.11}$$

for j sufficiently large, see the definition after (4.4.7) for Ω_0. For otherwise, there holds

$$\sup_{x \in \Omega_0} n_{1,j}(x) \to 0 \Longrightarrow \sup_{x \in \Omega^\delta} n_{1,j}(x) \to 0 \quad \text{as } j \to \infty \tag{4.4.12}$$

for every fixed $\delta > 0$, by (4.4.10).

Taking $R_{1,j} = U_{1,j}$ and letting $j \to \infty$ in $(4.4.8)_1$, with the aid of (4.4.9) and (4.4.12), one readily infers that there exist $\beta \in (0, 1)$ and $w_1 \in C_0^{1,\beta}(\Omega)$ such that

$$\lim_{j \to \infty} w_{1,j}(x) = w_1(x) \geq 0, \qquad \max_{x \in \Omega} w_1(x) = 1$$

in $C^{1,\beta}$-topology, and

$$\Delta w_1 = 0 \quad \text{in } \Omega, \qquad w_1 = 0 \quad \text{on } \partial\Omega.$$

Therefore $w_1 \equiv 0$ on Ω. This is an immediate contradiction since $\max_{x \in \Omega} w_1(x) = 1$ and proves our claim (4.4.11).

Note $p_{12} > 0$ (otherwise $p_{11} > 1 - p_{12} = 1$ and consequently $\bar{N}_1 = \infty$, impossible). Using (4.4.11), $\bar{N}_1 \in (0, \infty)$ and the definition of $N_{l,j}$, one infers that there exists $c_j \in [c_0, 1]$ such that

$$u_{2,j}(\xi_{1,j}) = [c_l U_{1,j}^{1-p_{11}} N_{1,j}]^{1/p_{12}} = U_{1,j}^{(1-p_{11})/p_{12}}[(c_l \bar{N}_1)^{1/p_{12}} + o(1)]$$

(noting $\xi_{l,j} \in \Omega_0$ actually) and

$$\begin{aligned} N_{2,j} \geq n_{2,j}(\xi_{1,j}) &= U_{1,j}^{p_{21}} u_{2,j}^{p_{22}-1}(\xi_{1,j}) \\ &= U_{1,j}^{p_{21}-(1-p_{11})(1-p_{22})/p_{12}}[(c_l \bar{N}_1)^{(p_{22}-1)/p_{12}} + o(1)] \to \infty \end{aligned}$$

since $p_{21}p_{12} > (1 - p_{11})(1 - p_{22})$, $U_{1,j} \to \infty$ and $c_l \in [c_0, 1] \subset (0, 1]$. This is a contradiction which implies that there must hold $\bar{N}_2 = \infty$. Now take

$$z^j = \zeta_{2,j}, \qquad Q_j^2 = \max\{N_{2,j}, t_j u_{1,j}^{-1}(\zeta_{2,j}), t_j u_{2,l}^{-1}(\zeta_{2,j})\} \to \infty,$$

and this becomes an analogue of Case II).

In conclusion, the contradiction we just derived implies that the hypothesis (4.4.6) cannot be valid. Therefore the a priori estimate (2.5.2) must hold and the proof is complete. □

4.5. *A priori estimates III. Strongly coupled case with $k > 2$*

In this subsection, we derive a priori supremum estimates for nonnegative solutions of (4.1.1)–(4.1.2) for arbitrary $k > 2$. We would like to point out that the upper bound of t (Lemma 4.4.2) remains valid. Specifically, we shall consider nonnegative solutions of

$$\begin{aligned} \Delta u_l + \mathbf{u}^{\mathbf{p}_l} &= 0 \quad \text{in } \Omega, \quad l = 1, \dots, k, \\ \mathbf{u} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{4.5.1}$$

We have the following theorem.

THEOREM 4.5.1. *Suppose that the exponents $\mathbf{p}_l$ satisfy*

$$1 \leq \min_l |\mathbf{p}_l| \leq \max_l |\mathbf{p}_l| < 2_*.$$

Assume that the matrix $P - I$ is nonsingular, where P is the exponent matrix of (4.5.1) *and I is the $k \times k$ identity matrix. Then there exists a positive constant $C > 0$* (*independent of* $\mathbf{u}$) *such that*

$$\|\mathbf{u}\|_{L^\infty(\Omega)} \leq C$$

for all nonnegative solutions $\mathbf{u}$ *of* (4.5.1).

If, in addition, (4.5.1) *is fully coupled, then the conclusion continues to hold with the weaker assumption that*

$$1 \leq \min_l |\mathbf{p}_l| \leq \max_l |\mathbf{p}_l| \leq 2_*.$$

PROOF. The proof is essentially the same as that of Theorem 4.4.1, but simpler with $t = 0$. Again we argue by contradiction. Assume (4.5.1) has a sequence of solutions $\mathbf{u}_j = (u_{1,j}(x), \dots, u_{k,j}(x)) \geq 0$ such that

$$\lim_{j\to\infty} \max_{1\leq l\leq k} U_{l,j} = \infty.$$

We first claim that one may assume $\mathbf{u}_j > 0$. Indeed, without loss of generality (i.e., up to a subsequence), there exists $1 \leq k' \leq k$ such that (note $k' > 0$ for otherwise nothing to prove)

$$u_{l,j} > 0 \quad \text{for } l = 1, \dots, k'; \qquad u_{l,j} \equiv 0 \quad \text{for } l = k'+1, \dots, k,$$

since every $u_{l,j}(x)$ is either identically zero or strictly positive. It follows that

$$\Delta u_{l,j} + \bar{\mathbf{u}}_j^{\bar{\mathbf{p}}_l} = 0, \quad \text{in } \Omega,$$

where

$$\bar{\mathbf{p}}_l = (p_{l1}, \dots, p_{lk'}), \qquad \bar{\mathbf{u}}_j = (u_{1,j}, \dots, u_{k',j}), \quad l = 1, \dots, k'.$$

Then necessarily

$$\mathbf{p}_l = (\bar{\mathbf{p}}_l, 0, \dots, 0) \quad \text{for } l = 1, \dots, k'.$$

In turn, $|\bar{\mathbf{p}}_l| = |\mathbf{p}_l| \geq 1$, $l = 1, \dots, k'$, continue to satisfy all conditions of Theorem 4.5.1, and,

$$\det(P - I) = \det\begin{pmatrix} \bar{P} - I & 0 \\ P_{21} & P_{22} - I \end{pmatrix} \neq 0 \Longrightarrow \det(\bar{P} - I) \neq 0,$$

where $\bar{P}$ is the $k' \times k'$ matrix with $\bar{\mathbf{p}}_l$ as row vectors. Thus we may simply restrict ourselves to the k'-subsystem of $\bar{\mathbf{u}}_j$ since $\bar{P} - \mathbf{I}$ is nonsingular, $|\bar{\mathbf{p}}_l| > 1$ and $\bar{\mathbf{u}}_j > 0$ for $l = 1, \dots, k'$.

Next we follow the same line of the proof of Theorem 4.4.1 and use the same notations. Then the new functions $\mathbf{v}_j$ after the transformation (4.4.2) satisfy (note $t = 0$)

$$\begin{aligned} \Delta v_{l,j} + Q_j^{-2}\bar{n}_{l,j}(y) v_{l,j} &= 0 \quad \text{in } \Omega_j, \\ \mathbf{v}_j &= 0 \quad \text{on } \partial\Omega_j. \end{aligned} \tag{4.5.2}$$

One immediately concludes that $\bar{N}_l \neq 0$ for all $l = 1, \dots, k$, in view of Case I) of Theorem 4.4.1 since $t = 0$. It remains to consider only the following two possibilities.

(a) $\bar{N}_{l_0} = \infty$ for some index l_0 $(1 \leq l_0 \leq k)$. We first have the following lemma parallel to Lemma 4.4.1 for arbitrary $k \geq 2$.

LEMMA 4.5.1. *Suppose that all conditions of Theorem* 4.5.1 *are valid. Assume* $\max_l \bar{N}_l = \infty$. *For* $j = 1, 2, \dots,$ *put*

$$N_j := \max_i N_{l,j} \to \infty, \qquad \zeta_j := \zeta_{l,j} \textit{ if } N_j = N_{l,j},$$

and

$$Q_j^2 := \max\{N_j, t_j u_{1,j}^{-1}(\zeta_j), \ldots, t_j u_{k,j}^{-1}(\zeta_j)\}.$$

Then there holds

$$\lim_{j\to\infty} Q_j \operatorname{dist}(\zeta_j, \partial\Omega) = \infty.$$

The proof is essentially the same as that of Lemma 4.4.1, with the help of Lemma 4.2.4, and is left to the reader.

To continue our proof, without loss of generality, we assume

$$N_{1,j} = \max_i N_{l,j} \to \infty, \quad j = 1, 2, \ldots.$$

Take

$$z^j = \zeta_{1,j}, \qquad Q_j^2 = N_{1,j} \to \infty.$$

Plainly, we may assume that there exists $1 \le k' \le k$ such that (note $k' > 0$)

$$\kappa_l > 0 \quad \text{for } l = 1, \ldots, k'; \qquad \kappa_l = 0 \quad \text{for } l = k'+1, \ldots, k,$$

where

$$\kappa_l := \lim_{j\to\infty} \frac{N_{l,j}}{Q_j^2} \in [0, 1].$$

Proceeding exactly as in Theorem 4.4.1, noting that $\Omega_j \to \mathbb{R}^n$ by Lemma 4.5.1, one readily sees that, letting $j \to \infty$ in (4.5.2), for $l = k'+1, \ldots, k$,

$$\Delta v_l = 0, \quad \text{in } \mathbb{R}^n.$$

Thus $v_l \equiv 1$ by Lemma 4.2.1 for $l = k'+1, \ldots, k$ since $v_l(0) = 1$. It follows that

$$\Delta v_l + \kappa_l \bar{\mathbf{v}}^{\bar{\mathbf{p}}_l} = 0, \quad \text{in } \mathbb{R}^n$$

for $l = 1, \ldots, k'$, where $\kappa_l > 0$ and

$$\bar{\mathbf{p}}_l = (p_{l1}, \ldots, p_{lk'}), \qquad \bar{\mathbf{v}} = (v_1, \ldots, v_{k'}).$$

Therefore, by Lemma 4.2.3, $\mathbf{v} \equiv 0$. This contradicts the fact $\bar{\mathbf{v}}(0) = (1, \ldots, 1)$.

(b) $\bar{N}_l \in (0, \infty)$, $l = 1, \ldots, k$. It follows that

$$\sup_j \sup_{x\in\Omega} n_{l,j}(x) < \infty, \quad l = 1, 2, \ldots, k.$$

Slightly modifying the arguments used in the proof of Theorem 4.4.1, Case III), we derive the following estimates

$$\max_{x\in\Omega^\delta} u_{l,j}(x) \le C \min_{x\in\Omega^\delta} u_{l,j}(x), \quad l = 1, \ldots, k,\ j = 1, 2, \ldots \tag{4.5.3}$$

for all $\delta > 0$ and

$$\sup_{x\in\Omega_0} n_{l,j}(x) \geq c_0 N_{l,j}, \quad l = 1, \ldots, k,\ j = 1, 2, \ldots, \tag{4.5.4}$$

where $c_0 > 0$ is a constant independent of j. Note (4.5.4) is also valid for all $l = 1, \ldots, k$ this time, thanks to $t = 0$. Indeed, if (4.5.4) fails for some l_0, then (4.4.12) holds for l_0. With the aid of $(4.4.12)_{l_0}$, simply take $R_{l_0,j} = U_{l_0,j}$ and let $j \to \infty$ in $(4.5.2)_{l_0}$, which remains valid for any index l_0, but without the last term $t_j U_{l_0,j}^{-1}$. Similarly as before, one infers that there exists $w_{l_0} \in C_0^{1,\beta}(\Omega)$ such that

$$\lim_{j\to\infty} w_{l_0,j}(x) = w_{l_0}(x) \geq 0, \qquad \max_{x\in\Omega} w_{l_0}(x) = 1,$$

and

$$\Delta w_{l_0} = 0 \quad \text{in } \Omega, \qquad w_{l_0} = 0 \quad \text{on } \partial\Omega,$$

a contradiction.

Now, by (4.4.7), (4.5.3) and (4.5.4), we have

$$U_{l,j} = \max_{x\in\Omega_0} u_{l,j}(x) \leq C \min_{x\in\Omega_0} u_{l,j}(x),$$

and

$$c_0 N_{l,j} \leq \max_{x\in\Omega_0} n_{l,j}(x) \leq C \min_{x\in\Omega_0} n_{l,j}(x)$$

for $l = 1, \ldots, k$ and $j = 1, 2, \ldots$. It follows that there exists $C > 0$ such that for $l = 1, \ldots, k$ and $j = 1, 2, \ldots$

$$C^{-1} \prod_{i=1}^{k} U_{i,j}^{p_{li}-\delta_l^i} \leq N_{l,j} \leq C \prod_{j=1}^{k} U_{i,j}^{p_{li}-\delta_l^i}.$$

Therefore, we have

$$\lim_{j\to\infty} \prod_{i=1}^{k} U_{i,j}^{p_{li}-\delta_l^i} = \bar{U}_l \in (0, \infty) \quad l = 1, \ldots, k.$$

For $\bar{\mathbf{U}} = (\bar{U}_1, \ldots, \bar{U}_k) \in \Pi$, the k-system of $x = (x_1, \ldots, x_k)$

$$\prod_{i=1}^{k} x_i^{p_{li}-\delta_l^i} = \bar{U}_l, \quad l = 1, \ldots, k \tag{4.5.5}$$

has exactly one solution $x^0 = (x_1^0, \ldots, x_k^0) \in \Pi$. Indeed, the k-system of $y = \ln x = (\ln x_1, \ldots, \ln x_k)$

$$(P - I)y = \ln \bar{\mathbf{U}} = (\ln \bar{U}_1, \ldots, \ln \bar{U}_k) \in \mathbb{R}^k$$

has exactly one solution $y^0 \in \mathbb{R}^k$ since $P - I$ is nonsingular. Hence

$$x^0 = e^{y^0} = (e^{y_1^0}, \ldots, e^{y_k^0}) \in \Pi$$

is the unique solution of (4.5.5). One therefore readily infers that all $U_{l,j}$ must be bounded[4] for all $i = 1, \dots, k$ and for all $j = 1, 2, \dots,$ in view of the definition of $\bar{U}_l$. This contradicts the assumption $\max_l U_{l,j} \to \infty$ as $j \to \infty$ and the proof is complete. □

4.6. *Proof of Lemma* 4.4.3

In this subsection, we prove the monotonicity Lemma 4.4.3. For $\Omega = \mathbb{R}^n$ (i.e., without boundary), a (local) Lipschitz-continuity of **f** on Π (plus some growth condition) can be sufficient to derive monotonicity for positive solutions, see for example [10,17,24,45] for more details. When the boundary $\partial\Omega$ is nonempty, it is not known if such a (local) Lipschitz-continuity of **f** on Π is generally sufficient if the solution vanishes on the boundary $\partial\Omega$. In this case, one naturally turns to (local) Lipschitz-continuity on the closed Ξ, see for instance [45] (a 'proper monotonicity' of **f** may also be enough though). Lemma 4.4.3 treats functions **f** lacking in such properties.

We begin with a classical maximum principle for the so-called cooperative systems. For $k \geq 1$, consider

$$\begin{aligned} \Delta w_i + \textstyle\sum_{j=1}^k a_{ij}(x) w_j \leq 0 \quad & \text{in } \Omega, \\ \mathbf{w} \geq 0 \quad & \text{on } \partial\Omega, \end{aligned} \tag{4.6.1}$$

where $\mathbf{w} = (w_1, \dots, w_k)$. We say that (4.6.1) is cooperative if there holds for $i, j = 1, \dots, k$

$$a_{ij}(x) \geq 0, \quad j \neq i \text{ and } x \in \Omega.$$

LEMMA 4.6.1 (Maximum principle). *Let Ω be bounded and let $\mathbf{w}$ be a solution of (4.6.1). Suppose that (4.6.1) is cooperative. Then the following conclusions hold.*

(A) *Assume $\mathbf{w} \geq 0$. Then for each $i = 1, \dots, k$, we have either $w_i \equiv 0$ or $w_i > 0$, provided that*

$$a_{ii} \geq -C > -\infty, \quad x \in \Omega.$$

Moreover, for any $x_0 \in \partial\Omega$, there holds $\partial_\nu w_i(x_0) < 0$ provided that $w_i > 0$, $w_i(x_0) = 0$ and Ω is smooth at x_0 (say C^2). If, in addition, $\partial\Omega$ is C^1 and $w_i \in C^1(\overline{\Omega})$ is strictly positive and vanishes on $\partial\Omega$, then there exists $C = C(w_i, \partial\Omega) > 0$ such that

$$\sup_{x \in \partial\Omega} \frac{\partial w_i}{\partial \nu}(x) < 0,$$

where ν is the outer-normal of $\partial\Omega$ at x.

(B) *Assume*

$$s_i = \sup_{\Omega} \sum_{j=1}^k a_{ij}(x) < \lambda_1(\Omega), \quad i = 1, \dots, k. \tag{4.6.2}$$

Then $\mathbf{w} \geq 0$ in Ω.

---

[4] In fact, $U_{l,j} \to x_l^0$ as $j \to \infty$.

PROOF. The lemma is well known, but we include a proof here for the reader's convenience.

(A) is a straight-forward consequence of the strong maximum principle for single equations.

Next we prove (B). By (4.6.2), we can take a bounded domain Ω' such that

$$\Omega' \supset \overline{\Omega} \text{ and } \lambda_1(\Omega) > \lambda_1(\Omega') > \max_i s_i. \tag{4.6.3}$$

Let ψ be the positive first eigenfunction of $-\Delta$ on Ω', namely,

$$\Delta\psi + \lambda_1(\Omega')\psi = 0, \qquad \psi\Big|_{\partial\Omega'} = 0.$$

We set $\Psi = (\psi, \ldots, \psi)$. For sufficiently large t we have $\mathbf{h}^t(x) = \mathbf{w}(x) + t\Psi(x) \geq 0$ in Ω. Hence there is a smallest $t^* > -\infty$ such that $\mathbf{h}^{t^*}(x) = \mathbf{w}(x) + t^*\Psi(x) \geq 0$ in Ω and $h_{i_0}^{t^*}(x_0) = 0$ for some index i_0 and some point $x_0 \in \overline{\Omega}$.

We claim $t^* \leq 0$. Assume for contradiction $t^* > 0$, then necessarily $x_0 \in \Omega$, since $\mathbf{w} \geq 0$ and $t^*\Psi > 0$ on $\partial\Omega$. It follows that $\Delta h_{i_0}^{t^*}(x_0) \geq 0$ since $h_{i_0}^{t^*}(x_0)$ achieves a minimum at $x_0 \in \Omega$, and

$$\sum_{j=1}^k a_{i_0 j} h_j^{t^*}(x_0) = \sum_{j\neq i_0} a_{i_0 j} h_j^{t^*}(x_0) \geq 0$$

since (4.6.1) is cooperative and $h_j^{t^*}(x_0) \geq 0$ for $j = 1, \ldots, k$ (note $h_{i_0}^{t^*}(x_0) = 0$). That is,

$$\Delta h_{i_0}^{t^*}(x_0) + \sum_{j=1}^k a_{i_0 j} h_j^{t^*}(x_0) \geq 0. \tag{4.6.4}$$

On the other hand, using the equations for $\mathbf{w}$ and for ψ we find

$$\begin{aligned}
&\Delta h_{i_0}^{t^*}(x_0) + \sum_{j=1}^k a_{i_0 j}(x_0) h_j^{t^*}(x_0) \\
&\quad = \Delta w_{i_0}(x_0) + \sum_{j=1}^k a_{i_0 j} w_j(x_0) + t^*\left(\Delta\psi(x_0) + \psi(x_0)\sum_{j=1}^k a_{i_0 j}(x_0)\right) \\
&\quad \leq t^*\left(\Delta\psi(x_0) + \psi(x_0)\sum_{j=1}^k a_{i_0 j}(x_0)\right) \\
&\quad = t^*(\Delta\psi(x_0) + s_{i_0}\psi(x_0)) < t^*(\Delta\psi(x_0) + \lambda_1(\Omega')\psi(x_0)) = 0,
\end{aligned} \tag{4.6.5}$$

where we have used (4.6.2). Clearly (4.6.4) and (4.6.5) contradict each other and thus $t^* \leq 0$, i.e., $\mathbf{w} \geq -t^*\Psi \geq 0$ as required. □

Let $\mathbf{u}$ be a positive solution of

$$\begin{aligned}
\Delta u_l + \mathbf{u}^{\mathbf{p}_l} + t = 0, &\quad \text{in } \Omega \text{ for } l = 1, \ldots, k, \\
\mathbf{u} = 0 &\quad \text{on } \partial\Omega,
\end{aligned} \tag{4.6.6}$$

where $t \geq 0$ is a parameter. For $1 \leq i \leq k$ and $1 \leq j \leq k$, denote

$$m_{ij}(x) := \frac{u_i(x)}{u_j(x)} > 0, \quad x \in \Omega.$$

We can bound the quantities $m_{ij}(x)$ as follows.

LEMMA 4.6.2. *There exists* $C_{ij} = C(u_i, u_j, \partial\Omega, \Omega) > 0$ *such that*

$$\sup_{x \in \Omega} m_{ij}(x) \leq C_{ij} < \infty \tag{4.6.7}$$

for every $1 \leq i \leq k$ *and* $1 \leq j \leq k$.

PROOF. We argue by contradiction. Suppose for contradiction that (4.6.7) is false. Then there exist $1 \leq i \leq k$, $1 \leq j \leq k$ and a sequence $\{x^l \in \Omega\}$ such that

$$\lim_{l \to \infty} m_{ij}(x^l) = \infty. \tag{4.6.8}$$

Denote

$$\lim_{l \to \infty} x^l = x^0 \in \overline{\Omega}.$$

Clearly the point x^0 must be at $\partial\Omega$.

Clearly (4.6.6) is cooperative since $t \geq 0$ and $\mathbf{u}(x) \geq 0$. Hence by Lemma 4.6.1, we infer that there exist $\delta = \delta(u_j, \partial\Omega) > 0$ and $C = C(u_j, \partial\Omega) > 0$ such that

$$\sup_{x \in \Omega_\delta} \frac{\partial u_j}{\partial \nu}(x) \leq -C < 0$$

since $u_j \in C^1(\overline{\Omega})$, where

$$\Omega_\delta = \{x \in \Omega \mid dist(x, \partial\Omega) < \delta\}.$$

It follows that, for every $x \in \Omega_\delta$, there exists $C_j = C(u_j, \partial\Omega) > 0$ such that

$$u_j(x) = u_j(x) - u_j(x_{\partial\Omega}) \geq C_j dist(x, x_{\partial\Omega}), \quad x \in \Omega_\delta \tag{4.6.9}$$

since $u_j = 0$ on $\partial\Omega$, where $x_{\partial\Omega} \in \partial\Omega$ is such that the opposite of the outer-normal of $\partial\Omega$ at $x_{\partial\Omega}$ points to x and so $dist(x, \partial\Omega) = dist(x, x_{\partial\Omega})$.

On the other hand, there exists $C_i = C(u_i, \partial\Omega) > 0$ such that

$$u_i(x) = u_i(x) - u_i(x_{\partial\Omega}) \leq C_i dist(x, x_{\partial\Omega}), \quad x \in \Omega_\delta \tag{4.6.10}$$

since u_i is in $C^1(\overline{\Omega})$.

For l sufficiently large, we have $x^l \in \Omega_\delta$. Therefore, by (4.6.9) and (4.6.10), one has

$$m_{ij}(x^l) = \frac{u_i(x^l)}{u_j(x^l)} \leq \frac{C_i dist(x^l, (x^l)_{\partial\Omega})}{C_j dist(x^l, (x^l)_{\partial\Omega})} = \frac{C_i}{C_j} < \infty$$

for l sufficiently large. This contradicts (4.6.8) and completes the proof. □

Next we introduce the notion of normality of a domain Ω at boundary points. Using notations from Section 4.4, we say that Ω is normal at z if there exists $\gamma' < z_\nu$ such that for all $\gamma \in (\gamma', \gamma_z)$

$$(\Omega_{\gamma,\nu_z})^{\gamma,\nu_z} \subset \Omega.$$

Define

$$\begin{aligned}\beta_z &= \beta(z)\\ &:= \inf\{\gamma \in \mathbb{R} \mid (\Omega_{t,\nu_z})^{t,\nu_z} \subset \Omega \text{ for all } t \in (\gamma, \gamma_z)\} \in (-\infty, z_\nu),\end{aligned} \tag{4.6.11}$$

provided that Ω is normal at z. By the compactness of $\partial\Omega$, if Ω is normal for each point on $\partial\Omega$, then there exists a $\delta_0 > 0$ (independent of z) such that

$$z_\nu - \beta_z \geq \delta_0, \quad z \in \partial\Omega. \tag{4.6.12}$$

Since $\partial\Omega$ is C^2, it satisfies a uniform exterior ball condition. That is, for each $z \in \partial\Omega$, there exists a ball B outside of Ω, tangential to $\partial\Omega$ at z (exterior ball). Moreover, the radii of all such balls are bounded away from zero. Without loss of generality, we write $z = (1, 0)$ and $B = B_1(0)$. Define

$$\bar{\mathbf{u}}(y) = |x|^{n-2}\mathbf{u}(x), \qquad y = \frac{x}{|x|^2}, \quad x \in \Omega,$$

being the Kelvin-transform. Then the image of Ω, still denoted by Ω, is normal at $z = (1, 0)$. The Kelvin-transformed function $\bar{\mathbf{u}}$ then satisfies for $l = 1, \ldots, k$,

$$\Delta\bar{u}_l + |y|^{(n-2)|\mathbf{p}_l|-(n+2)}\bar{\mathbf{u}}^{\mathbf{p}_l} + t|y|^{-n-2} = 0, \quad y \in \Omega.$$

In light of the above discussions, we shall assume that Ω is normal at every point $z \in \partial\Omega$ and consider nonnegative solutions $\mathbf{u}$ of

$$\begin{aligned}\Delta u_l + |x|^{(n-2)|\mathbf{p}_l|-(n+2)}\mathbf{u}^{\mathbf{p}_l} + t|x|^{-n-2} &= 0, \quad \text{in } \Omega \text{ for } l = 1, \ldots, k,\\ \mathbf{u} &= 0 \quad \text{on } \partial\Omega.\end{aligned}$$

Moreover, we shall assume $1/2 \leq |x| \leq 1$ (locally near $z = (1, 0)$!) For simplicity, we write $\nu_z = \mathbf{e}_1$, being the positive x_1-direction and omit the script $\nu_z = \mathbf{e}_1$ in the sequel (i.e., $x^\gamma = x^{\gamma,\mathbf{e}_1}$, etc.). For $\gamma \in (\beta_z, \gamma_z)$ and $l = 1, \ldots, k$, denote (well-defined)

$$v_l(x) = u_l(x^\gamma), \quad \mathbf{v}(x) = (v_1(x), \ldots, v_k(x)), \quad \mathbf{w} = \mathbf{v} - \mathbf{u}; \quad x \in \Omega_\gamma.$$

Then $\mathbf{v}$ satisfies[5]

$$\Delta v_l + |x^\gamma|^{(n-2)|\mathbf{p}_l|-(n+2)}\mathbf{v}^{\mathbf{p}_l} + t|x^\gamma|^{-n-2} = 0 \quad \text{in } \Omega_\gamma.$$

---

[5] Note that both $\mathbf{v}$ and $\mathbf{w}$ also depend on (the variable) γ. But this shall not be reflected in our notation, for the sake of brevity.

Since $\mathbf{v}^{\mathbf{p}_l}, t \geq 0$ and $\max_l |\mathbf{p}_l| \leq 2_*$, the functions $t|x|^{-n-2}$ and $|x|^{(n-2)|\mathbf{p}_l|-(n+2)}\mathbf{v}^{\mathbf{p}_l}$ are nonincreasing in $|x|$ and one readily sees that $\mathbf{w}$ satisfies

$$\begin{aligned}&\Delta w_l + |x|^{(n-2)|\mathbf{p}_l|-(n+2)}(\mathbf{v}^{\mathbf{p}_l} - \mathbf{u}^{\mathbf{p}_l}) = t(|x|^{-n-2} - |x^\gamma|^{-n-2})\\ &\quad + \mathbf{v}^{\mathbf{p}_l}[|x|^{(n-2)|\mathbf{p}_l|-(n+2)} - |x^\gamma|^{(n-2)|\mathbf{p}_l|-(n+2)}] \leq 0, \quad x \in \Omega_\gamma\end{aligned}$$

and is nonnegative on $\partial\Omega_\gamma$, since $|x| \geq |x^\gamma|$ in Ω_γ. It follows that the function $\mathbf{w}$ satisfies

$$\begin{aligned}\Delta w_l + |x|^{(n-2)|\mathbf{p}_l|-(n+2)} \textstyle\sum_{j=1}^k c_{lj}(x) w_j \leq 0 \quad &\text{in } \Omega_\gamma,\\ \mathbf{w} \geq 0 \quad &\text{on } \partial\Omega_\gamma,\end{aligned} \tag{4.6.13}$$

where

$$c_{lj}(x) w_j = v_1^{p_{l1}} \cdots v_{j-1}^{p_{l,j-1}} u_{j+1}^{p_{l,j+1}} \cdots u_k^{p_{lk}} [v_j^{p_{lj}} - u_j^{p_{lj}}], \quad x \in \Omega$$

for $l, j = 1, \ldots, k$. Using the integration presentation, there holds

$$v_j^{p_{lj}}(x) - u_j^{p_{lj}}(x) = p_{lj} w_j(x) \int_0^1 [t v_j + (1-t) u_j]^{p_{lj}-1} dt.$$

In turn, we can write

$$\begin{aligned}c_{lj}(x) &= v_1^{p_{l1}} \cdots v_{j-1}^{p_{l,j-1}} u_{j+1}^{p_{l,j+1}} \cdots u_k^{p_{lk}} p_{lj} \int_0^1 [t v_j + (1-t) u_j]^{p_{lj}-1} dt\\ &\geq 0, \quad x \in \Omega_\gamma\end{aligned} \tag{4.6.14}$$

for $l, j = 1, \ldots, k$.

Using (4.6.14), we have the following estimate on the functions $c_{lj}(x)$.

LEMMA 4.6.3. *Suppose*

$$\min_i \{|\mathbf{p}_l|\} \geq 1.$$

Then for all $\gamma \in (\beta_z, \gamma_z)$, *there exists* $C = C(\mathbf{u}, \Omega, \partial\Omega) > 0$ *such that*

$$\max_{l,j} \{ \sup_{x \in \Omega_\gamma} |c_{lj}(x)| \} \leq C < \infty.$$

PROOF. By Lemma 4.6.2, there exists $C_{lj} = C(u_l, u_j, \partial\Omega, \Omega) > 0$ such that

$$\sup_{x \in \Omega} m_{lj}(x) \leq C_{lj} < \infty$$

for every $1 \leq l \leq k$ and $1 \leq j \leq k$. It follows that there exists $C_0 = C_0(\mathbf{u}, \Omega, \partial\Omega) > 0$ such that

$$\max_{lj} \{ \sup_{x \in \Omega_\gamma} m_{lj}(x^\gamma), \sup_{x \in \Omega_\gamma} m_{lj}(x), \sup_{x \in \Omega_\gamma} u_j(x^\gamma), \sup_{x \in \Omega_\gamma} u_j(x) \} \leq C_0 < \infty. \tag{4.6.15}$$

Clearly $c_{lj}(x) \equiv 0$ in Ω if $p_{lj} = 0$. When $p_{lj} \geq 1$, with the aid of (4.6.14), (4.6.15), one readily sees that there exists $C = C(\mathbf{p}_l, C_0) > 0$ such that

$$c_{lj}(x) \leq v_1^{p_{l1}} \cdots v_{j-1}^{p_{l,j-1}} u_{j+1}^{p_{l,j+1}} \cdots u_k^{p_{lk}} p_{lj} \max\{u_j^{p_{lj}-1}, v_j^{p_{lj}-1}\} \leq C,$$

since all exponents are nonnegative. Finally, assume $p_{lj} \in (0, 1)$. Using (4.6.14), (4.6.15), for each fixed $l = 1, \dots, k$ and $j = 1, \dots, k$, we deduce that

$$\begin{aligned}
c_{lj}(x) &= v_1^{p_{l1}} \cdots v_{j-1}^{p_{l,j-1}} u_{j+1}^{p_{l,j+1}} \cdots u_k^{p_{lk}} p_{lj} \int_0^1 [tv_j + (1-t)u_j]^{p_{lj}-1} dt \\
&\le v_1^{p_{l1}} \cdots v_{j-1}^{p_{l,j-1}} u_{j+1}^{p_{l,j+1}} \cdots u_k^{p_{lk}} \cdot \min\left\{u_j^{p_{lj}-1}, v_j^{p_{lj}-1}\right\} \\
&= \left(\frac{v_1(x)}{v_j(x)}\right)^{p_{l1}} \cdots \left(\frac{v_{j-1}(x)}{v_j(x)}\right)^{p_{l,j-1}} \cdot \left(\frac{u_{j+1}(x)}{u_j(x)}\right)^{p_{l,j+1}} \cdots \left(\frac{u_k(x)}{u_j(x)}\right)^{p_{lk}} \\
&\quad \cdot v_j^{p_{l1}+\cdots+p_{l,j-1}}(x) \cdot u_j^{p_{l,j+1}+\cdots+p_{l,k}}(x) \cdot \min\left\{u_j^{p_{lj}-1}, v_j^{p_{lj}-1}\right\} \\
&\le m_{1j}^{p_{l1}}(x^\gamma) \cdots m_{j-1,j}^{p_{l,j-1}}(x^\gamma) \cdot m_{j+1,j}^{p_{l,j+1}}(x) \cdots m_{kj}^{p_{lk}}(x) \\
&\quad \cdot \max\{u_j^{|\mathbf{p}_l|-1}(x^\gamma), u_j^{|\mathbf{p}_l|-1}(x)\} \\
&\le C
\end{aligned}$$

since all exponents in the last line are nonnegative. Noting $c_{lj}(x) \ge 0$ in Ω, this immediately concludes the proof of the lemma. □

PROOF OF LEMMA 4.4.3. The proof is rather standard, by the well-known moving-plane method, developed for single equations in [4,6,25,47] (see also [54]). The maximum principle for small volume domains (cf. Lemma 4.6.1) plays a crucial role. We divide the proof into two steps.

Step 1. We first prove that there exists $\gamma_1 \in (\beta_z, \gamma_z)$ such that

$$\mathbf{u}(x) < \mathbf{u}(x^\gamma), \quad x \in \Omega_\gamma \tag{4.6.16}$$

holds for all $\gamma \in (\gamma_1, \gamma_z)$. For all $\gamma \in (\beta_z, \gamma_z)$, $\mathbf{w}$ satisfies (4.6.13) with $1/2 < |x| < 1$. Clearly (4.6.13) is cooperative since all entries $c_{lj}(x)$ are nonnegative by (4.6.14), and Lemma 4.6.1 applies if (4.6.2) is valid.

By Lemma 4.6.3 and the fact $1/2 < |x| < 1$, all entries $c_{lj}(x)$ are (uniformly, depending on $\mathbf{u}$ of course) bounded in Ω_γ for all $\gamma \in (\beta_z, \gamma_z)$. On the other hand, for $\gamma < \gamma_z$, there holds

$$\lim_{\gamma \to \gamma_z -} \lambda_1(\Omega_\gamma) = \infty$$

since the volume of Ω_γ shrinks to zero as $\gamma \to \gamma_z-$. It follows that there exists $\gamma_1 \in (\beta_z, \gamma_z)$ (sufficiently close to γ_z) such that (4.6.2) is satisfied for all $\gamma \in (\gamma_1, \gamma_z)$. In turn, Lemma 2.2.4 applies to (4.6.13) for all $\gamma \in (\gamma_1, \gamma_z)$. That is, $\mathbf{w} \ge 0$ for all $\gamma \in (\gamma_1, \gamma_z)$. Now (4.6.16) follows from Lemma 4.6.1 immediately since no components of $\mathbf{w}$ can vanish identically, by the fact that $\mathbf{u}$ is positive in Ω and zero on $\partial\Omega$.

Step 2. Define

$$\gamma_0 = \inf\{\gamma_1 < \gamma_z : \ (4.6.16) \text{ holds for all } \gamma \ge \gamma_1\}.$$

It follows from Step 1 that $\gamma_0 \ge \beta_z$ is well defined by construction. We want to show $\gamma_0 = \beta_z$.

Suppose for contradiction $\gamma_0 > \beta_z$. Then again by Lemma 4.6.1

$$\mathbf{w}(x) > 0 \quad \text{in } \Omega_{\gamma_0}, \qquad \partial_{x_1}\mathbf{w}(x) > 0 \quad \text{on } \{x_1 = \gamma_0\}, \tag{4.6.17}$$

since $\mathbf{w}$ cannot vanish identically in any connected component of Ω_{γ_0} due to the positivity of $\mathbf{u}$ in Ω and the zero-boundary conditions of $\mathbf{u}$ on $\partial\Omega$.

Since $\mathbf{w} > 0$ in Ω_{γ_0} by (4.6.17), there exists a compact subset $\Omega' \subset\subset \Omega_{\gamma_0}$ such that

$$2w_l^0 := \min_{\Omega'} w_l(x) > 0, \quad l = 1, \dots, k$$

and (2.4) holds on $(\Omega_{\gamma_0} \setminus \Omega')$ (i.e., the volume of $(\Omega_{\gamma_0} \setminus \Omega')$ is small).

By continuity and the fact $\gamma_0 > \beta_z$, there exists $\varepsilon > 0$ such that for all $\gamma \in [\gamma_0 - \varepsilon, \gamma_0]$ there hold

(1) $(\Omega_\gamma)^\gamma \subset \Omega$;
(2) (4.6.2) holds on $(\Omega_\gamma \setminus \Omega')$; and
(3) for $l = 1, \dots, k$

$$w_l \ge w_l^0 > 0, \qquad x \in \Omega' \subset \Omega_\gamma.$$

It follows that, for all $\gamma \in [\gamma_0 - \varepsilon, \gamma_0]$, Lemma 4.6.1 applies to (4.6.13) on $(\Omega_\gamma \setminus \Omega')$ since $\mathbf{w} \ge 0$ on $\partial(\Omega_\gamma \setminus \Omega')$, and consequently $\mathbf{w} \ge 0$ in $(\Omega_\gamma \setminus \Omega')$, and also in Ω' by construction. Hence, (4.6.16) holds for $\gamma \in (\gamma_0 - \varepsilon, \gamma_0]$ since $\mathbf{w} \not\equiv 0$, which obviously contradicts the definition of γ_0. □

4.7. *Proof of Lemma* 4.4.2

In this subsection, we prove Lemma 4.4.2 for solutions of (4.6.6) for arbitrary $k \ge 1$. We would like to point out that the lemma remains valid for general functions $\mathbf{f}$ satisfying for $\mathbf{u} \ge 0$

$$f^l(x, \mathbf{u}) \ge C\mathbf{u}^{\mathbf{p}_l} - M, \quad l = 1, \dots, k$$

for some constants $C, M > 0$ and $\mathbf{p}_l \in \Xi$ with $|\mathbf{p}_l| > 1$.

PROOF OF LEMMA 4.4.2. Let $\mathbf{u}$ be a nonnegative solution of (4.6.6). Fix a ball $B \subset\subset \Omega$ with a positive radius and put

$$p_0 := \min_l |\mathbf{p}_l| > 1.$$

In turn, by Green's formula, for each $l = 1, \dots, k$

$$u_l(x) = \int_\Omega (t + \mathbf{u}^{\mathbf{p}_l})G(x, y)dy \ge t \int_\Omega G(x, y)dy = C_1 t, \quad x \in B,$$

where G is Green's function of $(-\Delta, H_0(\Omega))$ (see for example [28,29,35]) and

$$C_1 = \min_{x \in B} \int_\Omega G(x, y)dy > 0.$$

In the sequel, we assume $t \geq C_0$ for some C_0 sufficiently large to be determined later, for otherwise there is nothing left to prove. Now we proceed as follows. For each fixed $l = 1, \ldots, k$, we have

$$F_l := \min_{y \in B} \mathbf{u}^{\mathbf{p}_l}(y) \geq \prod_{j=1}^{k} (\min_{y \in B} u_j^{p_{lj}}(y)) \geq \prod_{j=1}^{k} (C_1 t)^{p_{lj}} = (C_1 t)^{|\mathbf{p}_l|} \geq (C_1 t)^{p_0},$$

where we have taken $C_1 t \geq 1$, since $|\mathbf{p}_l| \geq p_0 > 1$. We iterate using Green's formula to deduce that for $l = 1, \ldots, k$,

$$\begin{aligned} u_l(x) &= \int_\Omega (t + \mathbf{u}^{\mathbf{p}_l}) G(x, y) dy \\ &\geq \int_B \mathbf{u}^{\mathbf{p}_l}(y) G(x, y) dy \geq F_l \int_B G(x, y) dy \\ &\geq (C_1 t)^{p_0} \int_B G(x, y) dy = C_2 (C_1 t)^{p_0}, \quad x \in B, \end{aligned}$$

where

$$C_2 = \min_{x \in B} \int_B G(x, y) dy > 0.$$

Substituting into F_l, we obtain for $l = 1, \ldots, k$

$$\begin{aligned} F_l &\geq \prod_{j=1}^{k} (\min_{y \in B} u_j^{p_{lj}}(y)) \geq \prod_{j=1}^{k} [C_2 (C_1 t)^{p_0}]^{p_{lj}} = [C_2 (C_1 t)^{p_0}]^{|\mathbf{p}_l|} \\ &\geq [C_2 (C_1 t)^{p_0}]^{p_0}, \end{aligned}$$

where we have taken $C_2 (C_1 t)^{p_0} \geq 1$. Repeating the process yields for $l = 1, \ldots, k$,

$$\begin{aligned} u_l(x) &\geq \int_\Omega (t + \mathbf{u}^{\mathbf{p}_l}) G(x, y) dy \\ &\geq \int_B \mathbf{u}^{\mathbf{p}_l}(y) G(x, y) dy \geq C_2 \cdot F_l \\ &\geq C_2 \cdot [C_2 (C_1 t)^{p_0}]^{p_0} = C_2^{1+p_0} (C_1 t)^{p_0^2}, \quad x \in B, \end{aligned}$$

and

$$\begin{aligned} F_l &\geq \prod_{j=1}^{k} (\min_{y \in B} u_j^{p_{lj}}(y)) \geq \prod_{j=1}^{k} [C_2^{1+p_0} (C_1 t)^{p_0^2}]^{p_{ij}} \\ &= [C_2^{1+p_0} (C_1 t)^{p_0^2}]^{|\mathbf{p}_l|} \geq [C_2^{1+p_0} (C_1 t)^{p_0^2}]^{p_0} = C_2^{p_0 + p_0^2} (C_1 t)^{p_0^3}, \end{aligned}$$

where we have again taken $C_2^{1+p_0} (C_1 t)^{p_0^2} \geq 1$. A direct iteration yields for $l = 1, \ldots, k$ and $j = 1, 2, \ldots,$

$$\begin{aligned} u_l(x) &\geq \int_B \mathbf{u}^{\mathbf{p}_l}(y) G(x, y) dy \geq C_2 \cdot F_l \\ &\geq C_2 \cdot C_2^{p_0 + \cdots + p_0^{j-1}} (C_1 t)^{p_0^j} = C_2^{1 + p_0 + \cdots + p_0^{j-1}} (C_1 t)^{p_0^j}, \quad x \in B \end{aligned}$$

and

$$F_l \geq \prod_{j=1}^{k} (\min_{y \in B} u_j^{p_{lj}}(y)) \geq \prod_{j=1}^{k} [C_2^{1+p_0+\cdots+p_0^{j-1}} (C_1 t)^{p_0^j}]^{|\mathbf{p}_i|}$$
$$\geq [C_2^{1+p_0+\cdots+p_0^{j-1}} (C_1 t)^{p_0^j}]^{p_0} = C_2^{p_0+\cdots+p_0^j} (C_1 t)^{p_0^{j+1}},$$

provided

$$t \geq C_0(j) \Longrightarrow C_2^{1+p_0+\cdots+p_0^{j-1}} (C_1 t)^{p_0^j} \geq 1, \quad j = 1, 2, \ldots, \tag{4.7.1}$$

where $C_3 = C_2^{1/(p_0-1)}$ and

$$C_0(j) := C_1^{-1} \cdot C_3^{p_0^{-j}-1} \leq C_0 := C_1^{-1} \max\{1, C_3^{-1}\}$$

since $p_0 > 1$ and $j \geq 1$. In particular, for $l = 1, \ldots, k$ and $j = 1, 2, \ldots,$

$$u_l(x) \geq C_2^{1+p_0+\cdots+p_0^{j-1}} (C_1 t)^{p_0^j}$$
$$= C_2^{(p_0^j-1)/(p_0-1)} (C_1 t)^{p_0^j} = C_3^{-1} (C_1 C_3 t)^{p_0^j}, \quad x \in B,$$

provided that (4.7.1) holds. Hence the conclusion follows immediately by taking

$$T_0 = 2 \max\{(C_1 C_3)^{-1}, C_0\}$$

and the proof is complete. □

4.8. *Proof of Lemma* 4.4.1

We now turn to prove Lemma 4.4.1, which is crucial to the proofs of Theorems 4.4.1 and 4.5.1. The monotonicity Lemma 4.4.3 comes right into play.

Since Ω is bounded with a C^2-boundary, for each $x \in \partial\Omega$, there exists a pair of balls $B = B(x)$ and an 1-1 mapping ψ of B onto $E = \psi(B)$ such that

$$\psi(B \cap \Omega) \subset \mathbb{R}^n_+; \qquad \psi(B \cap \partial\Omega) \subset \partial\mathbb{R}^n_+; \qquad \psi \in C^2(B), \ \ \psi^{-1} \in C^2(E),$$

see for example [28]. Using the compactness of $\partial\Omega$, there exist an integer $I > 0$ and I C^2-pairs (B_i, ψ_i), $i = 1, \ldots, I$, and a constant $C > 0$ such that

$$\|\psi_i\|_{1,\beta_0,\overline{B_i}} \leq C, \qquad \|\psi_i^{-1}\|_{1,\beta_0,\overline{\psi_i(B_i)}} \leq C, \quad i = 1, \ldots, I$$

and that for each $x \in \partial\Omega$, there holds $x \in B_i$ for some $i \in \{1, \ldots, I\}$. In particular, note that Ω satisfies a uniform interior sphere condition.

PROOF OF LEMMA 4.4.1. First, by Lemma 4.6.2 and the fact $|\mathbf{p}_l| \geq 1$, without loss of generality one may assume that there exists $\zeta_{l,j} \in \Omega$ such that

$$n_{l,j}(\zeta_{l,j}) = N_{l,j}, \quad l = 1, 2,\ j = 1, 2, \ldots.$$

Denote for $j = 1, 2, \ldots$

$$N_j = \max\{N_{1,j}, N_{2,j}\}, \qquad \zeta_j = \zeta_{l,j} \quad \text{if } N_j = N_{l,j}.$$

We shall argue by contradiction, that is, suppose that Lemma 4.4.1 fails. Then there exists $C > 0$ such that

$$\sup_j Q_j \operatorname{dist}(\zeta_j, \partial\Omega) = \sup_j Q_j \operatorname{dist}(\zeta_j, \zeta_j') \leq C, \tag{4.8.1}$$

where $\zeta_j' \in \partial\Omega$ satisfies

$$\operatorname{dist}(\zeta_j, \partial\Omega) = \operatorname{dist}(\zeta_j, \zeta_j').$$

It follows that, by the facts that Ω satisfies a uniform interior sphere condition and $N_j \to \infty$ as $j \to \infty$, one may choose a point $\zeta_j'' \in \Omega$ on the line through ζ_j and ζ_j' such that

$$\tau_j := Q_j' \operatorname{dist}(\zeta_j'', \partial\Omega) = Q_j' \operatorname{dist}(\zeta_j'', \zeta_j') \in [\varepsilon_0, 1/\varepsilon_0], \quad j = 1, 2, \ldots,$$

for some positive constant $\varepsilon_0 < \min\{1/C, \operatorname{diam}(\Omega)\}$, where

$$Q'^2_j := \max\{N_j, t_j u_{1,j}^{-1}(\zeta_j''), t_j u_{2,j}^{-1}(\zeta_j'')\} \to \infty.$$

In (4.4.2), take

$$z^j = \zeta_j'', \qquad Q_j = Q_j'.$$

Then $\mathbf{v}_j$ satisfies (4.4.4) and (4.4.6).

Similarly as in Case II) of Theorem 4.4.1, by the choice of Q_j, there exist $\kappa_1, \kappa_2, \delta_1, \delta_2 \in [0, 1]$ such that

$$\lim_{j\to\infty} Q_j^{-2} \bar{n}_{l,j}(y) = \kappa_l, \qquad \lim_{j\to\infty} Q_j^{-2} t_j u_{l,j}^{-1}(\zeta_j'') = \delta_l; \quad l = 1, 2.$$

By the choices of Q_j, one has

$$\sup_j |Q_j^{-2} t_j u_{l,j}^{-1}(\zeta_j'')| \leq 1, \qquad \sup_j \sup_{y\in\Omega_j} |Q_j^{-2} \bar{n}_{l,j}(y)| \leq \sup_j Q_j^{-2} N_{l,j} \leq 1 \tag{4.8.2}$$

and

$$\mathbf{v}_j(0) = (1, 1), \qquad \operatorname{dist}(0, \partial\Omega_j) \in [\varepsilon_0, 1/\varepsilon_0]. \tag{4.8.3}$$

Note, however, $\kappa_1 + \kappa_2 + \delta_1 + \delta_2 = 0$ may occur due to the modification of the choice of $z^j = \zeta_j''$.

With an appropriate rotation, we may assume $\Omega_j \to \{y_n > -\varepsilon\}$ for some $\varepsilon \geq \varepsilon_0$ as $j \to \infty$. For each $r > 0$, let $B = B_r(0)$ be the ball centered at the origin with radius r.

Applying the Harnack inequality Lemma 4.2.6 to the equations $(4.4.6)_{l,j}$, with the aid of the uniform estimates (4.8.2) and (4.8.3), we infer that each $\mathbf{v}_j$ is uniformly bounded on the set $\Gamma_j := B \cap \Omega_j \cap \{y_n \geq -\varepsilon_0/2\}$ for all $j > 0$ sufficiently large (so $\Gamma_j \subset\subset \Omega_j$ with $dist(\Gamma_j, \partial\Omega_j) \geq \varepsilon_0/4$ for j large), with a uniform bound $C > 0$ (also depending on r)

$$\|\mathbf{v}_j\|_{0,\overline{\Gamma_j}} \leq C.$$

Since $\mathbf{u}_j$ is monotonic (Lemma 4.4.3), every $\mathbf{v}_j$ is also monotonic in Ω_j by $\mathbf{v}_j$'s construction. Hence there holds for all j large

$$v_{l,j}(y) \leq v_{l,j}(y^\gamma) \quad \text{in } (\Omega_j \cap B_r)_\gamma$$

for all $\gamma \in [-\text{dist}(0, \partial\Omega_j), 0]$. Therefore the above uniform estimates actually hold on $B \cap \Omega_j$ (j large)

$$\|\mathbf{v}_j\|_{0,\overline{B\cap\Omega_j}} \leq \|\mathbf{v}_j\|_{0,\overline{\Gamma_j}} \leq C. \tag{4.8.4}$$

Assume $\zeta_j' \to \zeta_0 \in (\partial\Omega \cap B_{i_0})$ for some $i_0 > 0$, where $\{\psi_i, B_i\}_1^I$ are given in the beginning of this section. Clearly, $\zeta_j' \in (\partial\Omega \cap B_{i_0})$ and $\{x \mid y = (x - \zeta_j'')Q_j \in B\} \subset B_{i_0}$ for j large since $\zeta_j' \to \zeta_0 \in (\partial\Omega \cap B_{i_0})$ and $Q_j \to \infty$. Then the 1-1 mapping

$$\phi_{i_0}(y) := [\psi_{i_0}(x) - \psi_{i_0}(\zeta_j')]Q_j$$

straightens the boundary $\partial\Omega_j \cap B$ (mapping the point $(\zeta_j' - \zeta_j'')Q_j$ to the origin) with

$$\|\phi_{i_0}\|_{1,\beta_0,\overline{B}} \leq C, \qquad \|\phi_{i_0}^{-1}\|_{1,\beta_0,\overline{\phi_{i_0}(B)}} \leq C.$$

It follows that, by applying Lemma 2.2.3 to the equations $(4.4.6)_{l,j}$, there exists $\beta \in (0, 1)$ such that every $\mathbf{v}_j$ belongs to the Banach space $C^{1,\beta}(\overline{B_{r/2}(0) \cap \Omega_j})$ for $j > 0$ large, again, with a uniform bound $C > 0$ (also depending on the $C^{1,\beta}$-norm of ψ_i on B_i but independent of j)

$$\|\mathbf{v}_j\|_{1,\beta,\overline{B_{r/2}(0)\cap\Omega_j}} \leq C, \tag{4.8.5}$$

in view of (4.8.2)–(4.8.4). Note $r > 0$ is arbitrary.

With a proper rotation and translation, one may choose $y^j = (\zeta_j' - \zeta_j'')Q_j$ to be the new origin and the direction of $(\zeta_j'' - \zeta_j')$ the new (positive) y_n-direction. Consequently, the sequence of domains Ω_j converges to the half-space $\mathbb{R}^n_+ = \{y \in \mathbb{R}^n \mid y_n > 0\}$. It follows, by the estimate (4.8.5) and the Ascoli-Arzela theorem, that there exists $\mathbf{v} \in C^{1,\beta/2}_{\text{loc}}(\overline{\mathbb{R}^n_+})$ such that

$$\mathbf{v} = 0 \text{ on } \partial\mathbb{R}^n_+; \qquad \mathbf{v}(e) = (1, 1),$$

where e is the old origin $(\zeta_j'' - \zeta_j'') = 0$ which has the new coordinates $(0', 1)$,[6] and

$$\lim_{j\to\infty} \mathbf{v}_j(y) = \mathbf{v}(y),$$

and

$$\lim_{j\to\infty} Q_j^{-2}\bar{n}_{l,j}(y)v_{l,j}(y) = \lim_{j\to\infty} Q_j^{-2}n_{l,j}(\zeta_j'')\mathbf{v}_j^{\mathbf{p}_l}(y) = \kappa_l \mathbf{v}^{\mathbf{p}_l}(y)$$

on any compact set $\Gamma_1 \subset \mathbb{R}^n_+$ in $C^{1,\beta/2}$-topology. Moreover, the limiting function $\mathbf{v} \geq 0$ satisfies the limiting equations

$$\begin{aligned} \Delta v_1 + \kappa_1 \mathbf{v}^{\mathbf{p}_1}(y) + \delta_1 = 0, \\ \Delta v_2 + \kappa_2 \mathbf{v}^{\mathbf{p}_2}(y) + \delta_2 = 0, \end{aligned} \quad \text{in } \mathbb{R}^n_+$$

with $\delta_1, \delta_2, \kappa_1, \kappa_2 \geq 0$.

Clearly, under our assumptions, Lemma 4.2.5 applies. It follows that

$$\delta_1 = \delta_2 = \kappa_1 = \kappa_2 = 0, \qquad \mathbf{v}(y) = \mathbf{v}(y_n), \qquad \mathbf{v}'(y_n) \geq 0.$$

In turn, we have for $l = 1, 2$

$$v_l'' = 0 \quad \text{for } y_n > 0, \quad v_l(0) = 0, \quad v_l(1) = 1.$$

Plainly

$$v_1(y_n) = v_2(y_n) = y_n.$$

Evidently, the line segment $\Gamma' = \{(0', t) \,|\, t \in [0, 1/\varepsilon_0]\}$ is contained in the set $\overline{\mathbb{R}^n_+ \cap \Omega_j}$ for all $j > 0$. Thus the uniform estimates (4.8.5) of $\mathbf{v}_j$ in $C^{1,\beta}_{\text{loc}}(\overline{B_{r/2}(0) \cap \Omega_j})$ and the Ascoli-Arzela theorem imply

$$\|\mathbf{v}_j - \mathbf{v}\|_{1,\beta/2,\Gamma'} \to 0 \quad \text{as } j \to \infty.$$

In particular, for $l = 1, 2$,

$$\begin{aligned} &\sup_{0\leq y_n\leq 1/\varepsilon_0} |\partial_{y_n} v_{l,j}(0', y_n) - \partial_{y_n} v_l(0', y_n)| \\ &\quad = \sup_{0\leq y_n\leq 1/\varepsilon_0} |\partial_{y_n} v_{l,j}(0', y_n) - 1| \to 0 \quad \text{as } j \to \infty. \end{aligned}$$

Therefore

$$\frac{y_n}{2} \leq v_{l,j}(0', y_n) = \int_0^{y_n} \partial_{y_n} v_{l,j}(0', t)dt \leq \frac{3y_n}{2}$$

for $y_n \in [0, 1/\varepsilon_0]$ and j large. It follows that, by definition (see after (4.4.2)) and the fact $z^j = \zeta_j''$, for $y_n \in [0, 1/\varepsilon_0]$ and j large

$$\bar{n}_{l,j}(0', y_n) = n_{l,j}(\zeta_j'')\mathbf{v}_j^{\mathbf{p}_l}(0', y_n)v_{l,j}^{-1}(0', y_n) \leq c n_{l,j}(\zeta_j'') y_n^{|\mathbf{p}_l|-1}.$$

In turn,

$$n_{l,j}(\zeta_j) = \bar{n}_{l,j}(0', \eta_j) \leq c n_{l,j}(\zeta_j'')\eta_j^{|\mathbf{p}_l|-1}, \tag{4.8.6}$$

where $(0', \eta_j)$ is ζ_j's image on the y_n-axis with $\eta_j \in (0, 1/\varepsilon_0]$.

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[6] Its nth coordinate is bounded from both above and zero in view of (4.8.3) and 1 is taken merely for convenience.

On the other hand, one has

$$Q_j^2 = N_j = \max_l N_{l,j} = \max_l n_{l,j}(\zeta_j) \to \infty, \tag{4.8.7}$$

since $\delta_1 = \delta_2 = 0$.

Combining (4.8.6) and (4.8.7) yields

$$\max_l \kappa_l = \lim_{j\to\infty} Q_j^{-2} \max_l n_{l,j}(\zeta_j'') = \lim_{j\to\infty} \frac{\max_l n_{l,j}(\zeta_j'')}{\max_l n_{l,j}(\zeta_j)} \geq c\varepsilon_0^{1-|\mathbf{p}_2|} > 0.$$

This is a contradiction and completes the proof of the lemma. □

5. The m-Laplacians as principal parts

In this section, we consider the nonlinear functions

$$\mathbf{A}^l(x, u, \mathbf{p}) = |\mathbf{p}|^{m_l-2}\mathbf{p}, \quad l = 1, \ldots, k,$$

where $m_l > 1$ are positive numbers. Then the principal parts of (2.1.5) are given by the so-called m-Laplace operators:

$$\operatorname{div} \mathbf{A}^l(x, u, \nabla u) = \Delta_{m_l} u = \operatorname{div}(|\nabla u|^{m_l-2}\nabla u), \quad l = 1, \ldots, k.$$

One then readily verifies that (A1)–(A3) are valid (with $\delta = 0$). We assume that the function $\mathbf{f}$ satisfies a positivity condition of the form

$$f^l(x, \mathbf{u}, \nabla\mathbf{u}) + L|u_l|^{m_l-2}u_l \geq 0,$$

where $L > 0$ is a (large) constant. We shall also assume that the boundary condition (2.1.7) is homogeneous, i.e., $\mathbf{u}_0 = 0$ whenever Ω is bounded. We are seeking a positive vector-valued function $\mathbf{u} \in C^{1,\beta}(\overline{\Omega})$ ($\beta > 0$) satisfying the BVP (2.1.5) and (2.1.7) on bounded Ω's. As before, we shall exclusively focus ourselves on systems *without* variational structure.

5.1. *Existence*

We first consider scalar equations, i.e., $k = 1$. Then the BVP (2.1.5) and (2.1.7) becomes

$$\begin{aligned} \Delta_m u + f(x, u, \nabla u) = 0, &\quad \text{in } \Omega, \\ u = 0 &\quad \text{on } \partial\Omega, \end{aligned} \tag{5.1.1}$$

where $m > 1$. For $p > m - 1$, denote

$$\alpha = \frac{p - (m-1)}{m} > 0.$$

We say that $\mathbf{f} = f(x, u, \mathbf{p})$ satisfies a growth-limit condition (GL1), provided

(GL1) There exist positive numbers K, L and $p \in (m-1, m_*)$ ($p > m-1$ if $m \geq n$) such that the following conclusions hold.

(1) There exists a bounded function $F : \mathbb{R}_+ \to [0, 1]$ such that

$$|f(x, u, \mathbf{p})| \leq K[1 + u^p + F(|\mathbf{p}|)|\mathbf{p}|^{mp/(p+1)}], \quad (x, u, \mathbf{p}) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}^n,$$

and $F(|\mathbf{p}|) \to 0$ as $|\mathbf{p}| \to \infty$.

(2) There exists a continuous function $b : \overline{\Omega} \to \mathbb{R}_+$ such that for any sequences $\{M_j \to \infty\}$ and $\{\mathbf{p}_j = O(M_j^{1+\alpha})\}$, there hold

$$\lim_{j \to \infty} M_j^{-p} f(x, M_j, \mathbf{p}_j) = b(x)$$

uniformly on Ω.

(3) There exists $\varepsilon > 0$ such that

$$f(x, u, \mathbf{p}) \leq (\lambda_1 - \varepsilon)u^{m-1} + o(|\mathbf{p}|^{m-1})$$

as $(u, \mathbf{p}) \to 0$ uniformly on Ω for $u \geq 0$.

(4) There exist constants $L > 0$ and $\delta \in (0, (m-1)/2)$ such that

$$f(x, u, \mathbf{p}) \geq \delta(2u^p - |\mathbf{p}|^{mp/(p+1)}) - L, \quad (x, u, \mathbf{p}) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}^n.$$

We then have the following existence result.

THEOREM 5.1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Assume that f satisfies a growth-limit condition* (GL1). *Then the BVP* (5.1.1) *has a positive solution u in the space $C^{1,\beta}(\overline{\Omega}) \cap C_0(\Omega)$ for some $\beta > 0$.*

PROOF. As before, it is straightforward to verify that all conditions of Theorem 2.5.2 are satisfied, except the (AP2). This proves Theorem 5.1.1 immediately when one can show the a priori estimate (AP2), which will be done in Section 5.3. □

A canonical prototype of f satisfying the conditions of Theorem 5.1.1 is given by

$$f(x, u, \mathbf{p}) = -D|u|^{m-2}u + E|u|^{p-1}u + F|\mathbf{p}|^q, \quad (x, u, \mathbf{p}) \in \Omega \times \mathbb{R} \times \mathbb{R}^n, \tag{5.1.2}$$

where $D, F \geq 0$ are nonnegative numbers, and E, p and q are positive numbers satisfying

$$p > m - 1, \qquad q < \frac{mp}{p+1}$$

(the "sublinear" growth $p < m - 1$ being trivial). A refinement of (5.1.2) is given by

$$f(x, u, \mathbf{p}) = -D|u|^{m-2}u + E|u|^{p-1}u + \frac{F|\mathbf{p}|^{mp/(p+1)}}{1 + \ln(1 + |\mathbf{p}|)}, \qquad (x, u, \mathbf{p}) \in \Omega \times \mathbb{R} \times \mathbb{R}^n.$$

Of course, the coefficients D, E and F can be nonnegative continuous functions on $\overline{\Omega}$ (either strictly positive or identically zero, but with $E > 0$).

It is worth pointing out that Theorem 5.1.1 is optimum for the canonical prototype

$$f(z, u, \mathbf{p}) = |u|^{p-1}u, \quad p > m-1.$$

In this case, generically speaking, the BVP (5.1.1) admits a positive solution if and only if $p < m_*$. Indeed, Theorem 5.1.1 fails on star-shaped domains for $p \geq m_*$ since (5.1.1) possesses no positive solutions on star-shaped domains by the Pohozaev identity when $p \geq m_*$. However, when f depends also on the gradient ∇u, the situation is far from clear.

Next we consider a canonical prototype 2-system in which

$$f^1(x, \mathbf{u}, Q) = u_1^a u_2^b, \qquad f^2(x, \mathbf{u}, Q) = u_1^c u_2^d, \tag{5.1.3}$$

where a, b, c and d are nonnegative numbers. Moreover, we assume $m_1 = m_2 = m > 1$. Then the BVP (2.1.5) and (2.1.7) becomes

$$\begin{aligned} \Delta_m u_1 + u_1^a u_2^b &= 0, \quad \text{in } \Omega, \\ \Delta_m u_2 + u_1^c u_2^d &= 0, \quad \text{in } \Omega, \\ \mathbf{u} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{5.1.4}$$

We shall assume

$$bc > 0, \qquad a + d > 0$$

so that (5.1.4) is strongly coupled.

As observed in Section 4, a monotone property of the solutions is needed to derive the a priori estimate (AP2) for (5.1.4), being strongly coupled. It is unclear if (5.1.4) possesses such a monotone property on general smooth domains like (4.1.1)–(4.1.2). However, an elementary yet useful observation here is that one only needs estimates (AP2) for a (certain) *subclass* of solutions to prove existence. Indeed, we can introduce a cone $\mathcal{C}$ of so-called monotone functions on an uniformly normal domain and show that the m-Laplace operator preserves such a monotonicity. We then are able to establish the desired a priori estimates for elements in the class of monotone functions and consequently obtain existence.

Write

$$\chi := bc - \alpha\delta \quad \text{where } \alpha := m - 1 - a, \delta := m - 1 - d.$$

We say that an (ALT) condition holds for (5.1.4), provided that the exponents m, a, b, c and d satisfy one of the following conditions:

(A) $n \leq m$;

(B) $n > m$, $\min(\alpha, \delta) > 0$ and

$$\max\{b + \delta, c + \alpha\} > \frac{n\chi}{mm_*};$$

(C) $n > m$, $\delta \leq 0 < \alpha$ and

$$\max\left\{\frac{\chi}{c - \delta}, c + \alpha\right\} > \frac{n\chi}{mm_*};$$

(D) $n > m, \alpha \leq 0 < \delta$ and

$$\max\left\{b + \delta, \frac{\chi}{b - \alpha}\right\} > \frac{n\chi}{mm_*};$$

(E) $n > m, \max(\alpha, \delta) \leq 0$ and

$$\min(a + b, c + d) < m_*.$$

Using the notion that Liouville theorems imply a priori estimates, we can state our first existence result for (5.1.4) as follows.

THEOREM 5.1.2. *Let $\Omega \subset \mathbb{R}^n$ be uniformly normal* (*see the definition in Section* 5.4). *Suppose that* (5.1.4) *(with no boundary conditions!) has no positive solutions on $\Omega = \mathbb{R}^n$ and there holds*

$$\chi > 0, \qquad \max(a, d) < m_*. \tag{5.1.5}$$

Then the BVP (5.1.4) *has a positive solution* $\mathbf{u}$ *in the space $C^{1,\beta}(\overline{\Omega}) \cap C_0(\Omega)$ for some $\beta > 0$.*

PROOF. The proof is similar to that of those earlier ones, i.e., being applications of Theorem 2.5.2. However, as mentioned above, it is not clear whether (AP2) holds for (5.1.4). That is, it is unknown if the estimate (2.5.2) holds for *all* positive solutions of (5.4.1). But as shown in Theorem 5.4.1, (2.5.2) does hold for *all monotone* (see the definition in Section 5.4) solutions of (5.4.1). This leads us to consider the following cone in X

$$\mathcal{C} = \{\mathbf{u} \in C_0(\Omega) \cap X \mid \mathbf{u} \in MO(\Omega)\}.$$

Now one readily checks that all steps of the proof of Theorem 2.5.2 carry over (see [60] for details) and it remains to verify (AP2) (but only for monotone solutions of (5.4.1)!) and Step 2. We shall show that $\chi > 0$ implies Step 2 of the proof of Theorem 2.5.2, replacing the superlinearity condition in Lemma 5.1.1 below and give the verification of (AP2) in Section 5.4. □

LEMMA 5.1.1. *Assume $\chi > 0$. Then there exists a positive number r such that $\mathbf{u} \neq t\Lambda(\mathbf{u})$ for all $t \in [0, 1]$ and $\mathbf{u} \in \mathcal{C}$ with $\|\mathbf{u}\| = r$.*

The proof of Lemma 5.1.1 is postponed to Section 5.6.

The following result is an immediate corollary of Theorem 5.1.2.

THEOREM 5.1.3. *Let $\Omega \subset \mathbb{R}^n$ be uniformly normal. Assume that* (5.1.5) *and* (ALT) *hold for* (5.1.4). *Then the BVP* (5.1.4) *has a positive solution* $\mathbf{u}$ *in the space $C^{1,\beta}(\overline{\Omega}) \cap C_0(\Omega)$ for some $\beta > 0$.*

When $m = 2$, Theorem 5.1.3 can be improved slightly.

THEOREM 5.1.4. *Let $\Omega \subset \mathbb{R}^n$ be uniformly normal. Assume $m = 2$, $\chi > 0$ and there holds*

$$\min(a+b, c+d) < m_*, \qquad \max(a+b, c+d) \le m_*. \tag{5.1.6}$$

Then the BVP (5.1.4) has a positive solution $\mathbf{u}$ in the space $C^{1,\beta}(\overline{\Omega}) \cap C_0(\Omega)$ for some $\beta > 0$.

PROOF OF THEOREMS 5.1.3 AND 5.1.4. Assuming Theorem 5.1.2, we readily derive Theorems 5.1.3 and 5.1.4 once we show that the conditions of Theorems 5.1.3 and 5.1.4 imply that (5.1.4) (with no boundary conditions!) has no positive solutions on $\Omega = \mathbb{R}^n$. But this is simply Lemma 5.2.2.

When $\Omega = B$ is an Euclidean ball, the above 2-system was treated in [12] for radially symmetric solutions and a priori estimate and existence results have been obtained, with different exponents m_1 and m_2. In [5,13], a different system was studied in which $f^1 = f(v)$ and $f^2 = f(u)$ (cf., a Hamiltonian structure).

5.2. *Liouville theorems*

In this subsection, we gather several Liouville theorems for (2.1.5) when the principal parts are the m-Laplace operators, which shall be used in the blow-up process later. Again, we omit all proofs.

Consider the scalar equation

$$\Delta_m u + \kappa u^p = 0, \qquad \text{in } \Omega, \tag{5.2.1}$$

where $p > m-1$ and $\kappa \ge 0$ are positive numbers and the 2-system

$$\begin{aligned} &\Delta_m u_1 + \kappa_1 u_1^a u_2^b = 0, \qquad \text{in } \Omega, \\ &\Delta_m u_2 + \kappa_2 u_1^c u_2^d = 0, \qquad \text{in } \Omega, \end{aligned} \tag{5.2.2}$$

where a, b, c, d and κ_1, κ_2 are nonnegative numbers.

We first have the following theorem for (5.2.1).

LEMMA 5.2.1. *Assume $p \in (m-1, m_*)$. Then the equation (5.2.1) does not admit any nonnegative nontrivial solution on the entire space $\Omega = \mathbb{R}^n$, or on the half-space $\Omega = \mathbb{R}^n_+$ vanishing on $\partial\mathbb{R}^n_+$.*

When $\Omega = \mathbb{R}^n$, Lemma 5.2.1 is a special case of Part (b) of Theorem IV of [50] (see also the references therein). For $\Omega = \mathbb{R}^n_+$, Lemma 5.2.1 is again a special case of [63]. The conclusion for $\Omega = \mathbb{R}^n$ is optimum, in view of the existence of radial solutions for $f = u^p$ with $p \ge m_*$.

The second Liouville theorem is for (5.2.2) on exterior domains Ω. We say that a domain $\Omega \subset \mathbb{R}^n$ is exterior if $\Omega \supset \{x \in \mathbb{R}^n \mid |x| > R\}$ for some $R > 0$.

LEMMA 5.2.2. *Let $k = 2$ and $\Omega \subset \mathbb{R}^n$ be exterior. Then the system of inequalities*

$$\Delta_m u_1 + u_1^a u_2^b \le 0, \qquad \Delta_m u_2 + u_1^c u_2^d \le 0$$

does not admit any positive solutions, provided $\chi > 0$ and the (ALT) *condition holds.*

When $m < n$ and $\min(\alpha, \delta) > 0$, Lemma 5.2.2 was proved in [7], see Theorem 5.3, p. 41, [7] (with different exponents $m_1 = p$ and $m_2 = q$). The rest of Lemma 5.2.2 was proved in [60].

5.3. *A priori estimates I: The scalar case*

We want to establish the estimate (AP2) for (5.1.1) with (2.5.3) becoming[7]

$$\begin{aligned} \Delta_m u + f(x, u, \nabla u) + t = 0 \quad & \text{in } \Omega, \\ u = 0 \quad & \text{on } \partial\Omega \end{aligned} \tag{5.3.1}$$

where $t \geq 0$ is a parameter. Below is the precise statement of (AP2).

THEOREM 5.3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Assume that the conditions of Theorem 5.1.1 are satisfied. Then the a priori estimate* (2.5.2) *holds for all nonnegative solutions u of* (5.3.1).

We first have the following upper bound for the parameter $t \geq 0$.

LEMMA 5.3.1. *Let (t, u) be a nonnegative solution of* (5.3.1). *Then there exists a constant $t_0 > 0$ independent of u or t such that*

$$t \leq t_0.$$

The proof of Lemma 5.3.1 is deferred.

Proof of Theorem 5.3.1. Suppose that Theorem 5.3.1 is false. Then there exists a sequence of solutions $\{(t_j, u_j(x)\}$ of (5.3.1) such that

$$\lim_{j\to\infty} \|u_j\|_{L^\infty(\Omega)} = \infty$$

since $t \leq t_0$ by Lemma 5.3.1. Put

$$M_j = \max_{x\in\Omega} u_j(x) = u_j(\xi^j) \to \infty, \quad \xi^j \in \Omega.$$

Denote

$$v(y) = M_j^{-1} u_j(x), \qquad y = (x - \xi^j) M_j^\alpha,$$

and

$$\Omega_j = \{y \in \mathbb{R}^n \mid x = \xi^j + M_j^{-\alpha} y \in \Omega\}.$$

By direct calculations, v_j satisfies

$$\begin{aligned} \Delta_m v_j + M_j^{-p} B(\xi^j + M_j^{-\alpha} y, M_j v_j, M_j^{1+\alpha} \nabla v_j) = 0, \quad & y \in \Omega_j \\ v_j = 0 \quad & \text{on } \partial\Omega_j, \end{aligned} \tag{5.3.2}$$

[7] The parameter t below should be of form t^{m-1}, but it makes no essential difference in view of the fact $m > 1$. See also (5.4.1).

and

$$0 < v_j(y) \le 1, \quad y \in \Omega_j; \qquad v_j(0) = 1. \tag{5.3.3}$$

It follows that, with the aid of the growth-limit condition (GL1), part 1 (assuming $M_j > 1$)

$$\begin{aligned} &M_j^{-p}|B(\xi^j + M_j^{-\alpha} y, M_j v_j, M_j^{1+\alpha}\nabla v_j)| \\ &\quad \le K M_j^{-p}[1 + M_j^p v_j^p + (M_j^{1+\alpha}|\nabla v_j|)^{mp/(p+1)}] \\ &\quad \le K[1 + v_j^p + |\nabla v_j|^{mp/(p+1)}] \le K(3 + |\nabla v_j|^m). \end{aligned}$$

Invoking Lemma 2.2.3, one then infers that there exist positive constants $\beta = \beta(K, n, m) \in (0, 1)$ and $C = C(\|\partial\Omega\|_{1,1}, K, n, m) > 0$ such that

$$\|v_j\|_{C^{1,\beta}(\overline{\Omega}_j)} \le C. \tag{5.3.4}$$

In particular, by combining $(5.3.2)_2$, (5.3.3) and (5.3.4), one infers that there exists a constant $C = C(\|\partial\Omega\|_{1,1}, K, n, m) > 0$ such that

$$\mathrm{dist}(0, \partial\Omega_j) = d_j M_j^\alpha \ge C, \qquad d_j = \mathrm{dist}(\xi^j, \partial\Omega). \tag{5.3.5}$$

Next we consider two cases.

Case I: The sequence $\{d_j M_j^\alpha\}$ is unbounded, say (without loss of generality), $d_j M_j^\alpha \to \infty$ as $j \to \infty$. Plainly $\Omega_j \to \mathbb{R}^n$ as $j \to \infty$. With the aid of (5.3.4), one can apply the Ascoli-Arzela theorem to infer that there exists a nonnegative function $v \in C^{1,\beta/2}(\mathbb{R}^n)$ such that

$$\lim_{j\to\infty} v_j(y) = v(y) \ge 0, \qquad v(0) = 1, \tag{5.3.6}$$

uniformly on any compact subset of $\mathbb{R}^n$ in $C^{1,\beta/2}$-topology. Combining (5.3.4), the uniform convergence (5.3.5) and the growth-limit condition (GL1), one readily deduces that

$$\lim_{j\to\infty} M_j^{-p} f(\xi^j + M_j^{-\alpha} y, M_j v_j, M_j^{1+\alpha}\nabla v_j) = b(\xi^0) v^p(y)$$

uniformly on any compact subset of $\mathbb{R}^n$, where

$$\xi^0 = \lim_{j\to\infty} \xi^j \in \overline{\Omega}.$$

Now fix any function $\phi \in C_0^\infty(\mathbb{R}^n)$. Taking ϕ as a test function in (5.3.2) (for j large so Ω_j contains the support of ϕ) and letting $j \to \infty$, one readily verifies that the limiting function $v \ge 0$ satisfies the following limiting equation

$$\Delta_m v + b(\xi^0) v^p = 0 \quad \text{in } \mathbb{R}^n$$

with $b(\xi^0) > 0$ and $p \in (m - 1, m_*)$. Thus $v \equiv 0$ by Lemma 5.2.1. This contradicts the fact $v(0) = 1$.

Case II: The sequence $\{d_j M_j^\alpha\}$ is bounded as $j \to \infty$. In this case, with a proper rotation, the sequence of domains Ω_j converges (up to a subsequence) to the half-space

$\mathbb{R}^n_\varepsilon = \{y \in \mathbb{R}^n \mid y_n > -\varepsilon\}$ for some $\varepsilon > 0$, in view of (5.3.6). Similarly as in Case I), one deduces that there exists a nonnegative function $v \in C^{1,\beta/2}(\mathbb{R}^n_\varepsilon)$ such that

$$\lim_{j\to\infty} v_j(y) = v(y) \geq 0, \qquad v(0) = 1,$$

uniformly on any compact subset of $\mathbb{R}^n_\varepsilon$ in $C^{1,\beta/2}$-topology and

$$\lim_{j\to\infty} M_j^{-p} B(\xi^j + M_j^{-\alpha} y, M_j v_j, M_j^{1+\alpha} \nabla v_j) = b(\xi^0) v^p(y)$$

uniformly on any compact subset of $\mathbb{R}^n_\varepsilon$ for some $\xi^0 \in \overline{\Omega}$. Moreover, v vanishes on $\partial\mathbb{R}^n_\varepsilon$ since v_j vanishes on $\partial\Omega_j$ for every j. It follows that $v \geq 0$ satisfies the

$$\begin{aligned} \Delta_m v + b(\xi^0) v^p &= 0 \qquad \text{in } \mathbb{R}^n_\varepsilon \\ v &= 0 \qquad \text{on } \partial\mathbb{R}^n_\varepsilon \end{aligned}$$

with $b(\xi^0) > 0$ and $p \in (m-1, m_*)$. Thus $v \equiv 0$ by Lemma 5.2.1 again, a contradiction to the fact $v(0) = 1$. □

5.4. *A priori estimates II: The system case*

In this subsection, we want to establish the (AP2) property for the BVP (5.1.4). Then (2.5.3) becomes

$$\begin{aligned} \Delta_m u_1 + u_1^a u_2^b + t &= 0 \quad \text{in } \Omega, \\ \Delta_m u_2 + u_1^c u_2^d + t &= 0 \quad \text{in } \Omega, \\ \mathbf{u} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{5.4.1}$$

where $t \geq 0$ is a parameter.

We begin with the notion of uniformly normal domains and the class of monotone functions.

Using the notations from Section 4, recall

$$\beta_z = \beta(z) := \inf\{\sigma \in \mathbb{R} \mid (\Omega_{t,\nu_z})^{t,\nu_z} \subset \Omega \text{ for all } t \in (\sigma, \sigma_z)\} \in (-\infty, z_\nu), \tag{5.4.2}$$

provided that Ω is normal at $z \in \partial\Omega$.

A domain Ω is called uniformly normal if Ω is normal at every $z \in \partial\Omega$, and there holds

$$\delta_0 = \delta_0(\Omega, \partial\Omega) := \inf_{z\in\partial\Omega} \{z_\nu - \beta(z)\} > 0, \tag{5.4.3}$$

where $z_\nu = z \cdot \nu_z$ and ν_z is the (unit) outer-normal at z and $\beta(z)$ is given by (5.4.2).

We next define the class of monotone functions on a uniformly normal Ω. Let Ω be uniformly normal and let $g(x)$ be a nonnegative continuous function on Ω. We say that g is monotone in Ω if for all $z \in \partial\Omega$ and for all $\sigma \in (\beta_z, \sigma_z)$

$$g(x^{\sigma,\nu_z}) \geq g(x), \quad x \in \Omega_{\sigma,\nu_z}.$$

We denote by $MO(\Omega)$ the set of all monotone functions on Ω.

Now we are ready to state the a priori estimates' result for (5.4.1).

THEOREM 5.4.1. *Let Ω be a uniformly normal domain. Suppose that the conditions of Theorem 5.1.2 hold. Then the a priori estimate (2.5.2) holds for all nonnegative solutions* $\mathbf{u}$ *of* (5.4.1) *which are monotone in* Ω.

The proof of Theorem 5.4.1 is similar to that of Theorem 4.4.1. We shall only consider *positive* solutions $\mathbf{u}$ and use exactly the same notations as in Theorem 4.4.1.

We need two technical lemmas as before. The first is an upper-bound for the parameter $t \geq 0$, whose proof is given in next section.

LEMMA 5.4.1. *There exists a constant $t_0 > 0$ independent of* $\mathbf{u}$ *or t such that*

$$t \leq t_0.$$

The second is on bounds of the ratios of the components of $\mathbf{u}$.

LEMMA 5.4.2. *Let $\Gamma \subset \Omega$ be bounded and smooth, and* $\mathbf{u}$ *be a positive solution of* (5.4.1). *Then there exists a positive constant $C = C(\mathbf{u}, \partial\Omega) > 0$ such that*

$$\frac{1}{C} \leq \inf_{x\in\Gamma} \frac{u_1(x)}{u_2(x)} \leq \sup_{x\in\Gamma} \frac{u_1(x)}{u_2(x)} \leq C.$$

PROOF. When $m = 2$, this is essentially Lemma 4.6.2. For general $m > 1$, the proof of Lemma 4.6.2 carries over by using the strong maximum principle for the m-Laplace operator (Lemma 2.2.4). □

PROOF OF THEOREM 5.4.1. Suppose for contradiction that (2.5.2) is false. Then there exist a sequence of positive *monotone* solutions $\mathbf{u}_j = (u_{1,j}, u_{2,j})$ of (5.4.1) such that

$$\lim_{j\to\infty} \max\{U_{1,j}, U_{2,j}\} = \infty \tag{5.4.4}$$

since $t_j \leq t_0$ by Lemma 5.4.1. We divide the proof into two cases.

Case I). $\min\{a, d\} \geq m - 1$. Apply the transform (4.4.3) to the solutions $\mathbf{u}_j$ and denote the resulted functions by $\{\mathbf{v}_j\}$. Then, by direct calculations, $\mathbf{v}_j$ satisfies

$$\mathbf{v}_j(0) = (v_{1,j}(0), v_{2,j}(0)) \equiv (1, 1), \quad j = 1, 2, \ldots. \tag{5.4.5}$$

and

$$\begin{aligned} \Delta_m v_{l,j} + Q_j^{-m} n_{l,j}(y) v_{l,j}^{m-1} + Q_l^{-m} t_l u_{l,j}^{-(m-1)}(z^j) &= 0 \quad \text{in } \Omega_j, \\ \mathbf{v}_j &= 0 \quad \text{on } \partial\Omega_j, \end{aligned} \tag{5.4.6}$$

where

$$n_{l,j}(y) = n_{l,j}(x) = \mathbf{u}_j^{\mathbf{p}_l}(x) u_{l,j}^{-(m-1)}(x) = n_{l,j}(z^j)\mathbf{v}_j^{\mathbf{p}_l}(y) v_{l,j}^{-(m-1)}(y).$$

Since Ω is uniformly normal and $\{\mathbf{u}_j\}$ are monotone in Ω, there exists $\delta_0 > 0$ (given by (5.4.3)) such that for all $z \in \partial\Omega$ and for all $\gamma \in (\gamma_z - \delta_0, \gamma_z)$

$$\mathbf{u}_j(x^{\gamma,\nu_z}) \geq \mathbf{u}_j(x), \quad x \in \Omega_{\gamma,\nu_z}.$$

In turn,

$$\max_{x\in\Omega} u_{l,j}(x) = \max_{x\in\Omega_0} u_{l,j}(x), \quad l = 1, 2;\ j = 1, 2, \ldots,$$

where

$$\Omega_0 = (\Omega\backslash\Omega_{\delta_0}) := \{x \in \Omega \,|\, dist(x, \partial\Omega) \geq \delta_0\}.$$

With the help of $\min\{a, d\} \geq m - 1$ and the monotonicity of $\mathbf{u}_j$, it follows that there exist $\xi_{l,j} \in \Omega_0$ and $\zeta_{l,j} \in \Omega_0$ such that

$$U_{i,l} = u_{l,j}(\xi_{l,j}) = \max_{x\in\Omega} u_{l,j}(x), \qquad N_{i,l} = n_{l,j}(\zeta_{l,j}) = \max_{x\in\Omega} n_{l,j}(x)$$

for $l = 1, 2;\ \ j = 1, 2, \ldots$. In particular,

$$\min_l\{\text{dist}(\xi_{l,j}, \partial\Omega), \text{dist}(\zeta_{l,j}, \partial\Omega)\} \geq \delta_0. \tag{5.4.7}$$

We further divide the proof into three subcases and, without loss of generality, assume

$$U_{1,j} = \max\{U_{1,j}, U_{2,j}\} \to \infty.$$

Subcase (i). $\bar{N}_1 = 0$. One argues in exactly the same way as in Case I) of Theorem 4.4.1 to derive a contradiction since $U_{1,j} \to \infty$.

Subcase (ii). $\bar{N}_1 = \infty$. In (4.4.3), take

$$z^j := \zeta_j, \qquad Q_j^m := \max\{N_j, t_j u_{1,j}^{-(m-1)}(\zeta_j), t_j u_{2,j}^{-(m-1)}(\zeta_j)\} \to \infty,$$

where

$$N_j := \max\{N_{1,j}, N_{2,j}\} \to \infty, \qquad \zeta_j := \zeta_{l,j} \text{ if } N_j = N_{l,j}.$$

We claim that all of the following hold simultaneously.

(1) The system (5.1.4) (with no boundary conditions) admits no positive solutions on $\Omega = \mathbb{R}^n$,

(2) The single equation $\Delta_m u + u^\sigma = 0$, where either $\sigma = a$ or $\sigma = d$, has no positive solutions on $\Omega = \mathbb{R}^n$, and

(3) There holds

$$\liminf_{j\to\infty} \min_l Q_j \text{dist}(z^j, \partial\Omega) = \infty.$$

Indeed, (1) is our assumption. (2) follows from Lemma 5.2.1 since $\max(a, d) < m_*$. (3) follows from (5.4.7) and the facts $z^j = \zeta_j$ and $Q_j \to \infty$ as $j \to \infty$. Now the rest of the proof proceeds essentially in the same manner as in the proving of Theorem 4.4.1.

Subcase (iii). $\bar{N}_1 \in (0, \infty)$. Then as in the proofs of Theorem 4.4.1, one infers $\bar{N}_2 = \infty$ since $\chi, b, c > 0$. Take

$$z^j = \zeta_{2,j}, \qquad Q_j^m = \max\{N_{2,j}, t_j u_{1,j}^{-(m-1)}(\zeta_{2,j}), t_j u_{2,j}^{-(m-1)}(\zeta_{2,j})\} \to \infty$$

and this becomes an analogue of Subcase (ii) and one deduces a contradiction similarly.

Case (II). $\min(a, c) < m - 1$. Without loss of generality, assume

$$a = \min(a, d) < m - 1 \Longrightarrow \alpha = m - 1 - a > 0.$$

We next consider two subcases.

Subcase (i). $U_{2,j} \le C < \infty$. Then $U_{1,j} \to \infty$ by (5.4.4). By Lemma 5.4.1,

$$\max\{U_{1,j}^{-1} t_j, U_{1,j}^{-\alpha} U_{2,j}^{b}\} \to 0$$

since $\alpha, b > 0$. Rewrite $(5.4.6)_1$ with $Q_j = 1$ into

$$\Delta_m w_{1,j}(x) + U_{1,j}^{-\alpha} u_{2,j}^{b}(x) w_{1,j}^{a}(x) + U_{1,j}^{-(m-1)} t_j = 0, \tag{5.4.8}$$

where

$$w_{1,j}(x) = u_{1,j}(x) U_{1,j}^{-1} \in (0, 1], \qquad \max_{x \in \Omega} w_{1,j}(x) = 1,$$
$$w_{1,j}(x)|_{x \in \partial\Omega} = 0.$$

Letting $j \to \infty$ in (5.4.8) immediately yields a contradiction since

$$\max\{U_{1,j}^{-(m-1)} t_j, \sup_{x \in \Omega} U_{1,j}^{-\alpha} u_{2,j}^{b}(x)\} \le \max\{U_{1,j}^{-(m-1)} t_j, U_{1,j}^{-\alpha} U_{2,j}^{b}\} \to 0.$$

Subcase (ii). There holds $U_{2,j} \to \infty$. We claim

$$\max_{l} n_{l,j}(\xi_{2,j}) \to \infty.$$

To this end, suppose, say, $n_{1,j}(\xi_{2,j}) \le C$ for some $C > 0$, $j = 1, 2, \ldots$. Then

$$u_{1,j}(\xi_{2,j}) = n_{1,j}^{-\alpha}(\xi_{2,j}) U_{2,j}^{b/\alpha} \ge C U_{2,j}^{b/\alpha}$$

since $\alpha > 0$. In turn,

$$n_{2,j}(\xi_{2,j}) = u_{1,j}^{c}(\xi_{2,j}) U_{2,j}^{-\delta} \ge C U_{2,j}^{bc/\alpha - \delta} = C U_{2,j}^{\beta/\alpha} \to \infty,$$

since $U_{2,j} \to \infty$ and $\alpha, \beta > 0$, which yields the claim.

Now, in (4.4.3), take

$$z^j := \xi_{2,j},$$
$$Q_j^m := \max\{n_{1,j}(z^j), n_{2,j}(z^j), t_j u_{1,j}^{-(m-1)}(z^j), t_j u_{2,j}^{-(m-1)}(z^j)\} \to \infty.$$

We claim all (1)–(3) in Case (II) of Theorem 4.4.1 remain valid under our assumptions. Indeed, (3) follows from (5.4.7) and the facts $z^j = \xi_{2,j}$ and $Q_j \to \infty$ as $j \to \infty$. By (5.1.5), (2) follows from Lemma 5.2.1 since $\max(a, d) < m_*$. (1) is simply our assumption.

However, by our choice of z^j, one need to show that the functions $\{v_{1,j}\}$ is uniformly bounded on any compact set $\Gamma \subset \mathbb{R}^n$ for j large (so $\Gamma \subset \Omega_j$. Note $0 \le v_{2,j} \le 1$ by construction). To this end, one notices that by the choice of Q_j

$$0 \le Q_j^{-m} n_{1,j}(\xi_{2,j}) v_{1,j}^a(y) v_{2,j}^b(y) \le v_{1,j}^a(y) \le 1 + v_{1,j}^{m-1}(y),$$

since $b > 0$ and $a \in [0, m-1)$, and

$$0 \le Q_j^{-m} t_j u_{1,j}^{-(m-1)}(\xi_{2,j}) \le 1.$$

Moreover, $v_{1,j}$ satisfies the equation

$$\Delta_m v_{1,j}(y) + Q_j^{-m} n_{1,j}(\xi_{2,j}) v_{1,j}^a(y) v_{2,j}^b(y) + Q_j^{-2} t_j u_{1,j}^{-1}(\xi_{2,j}) = 0.$$

Now the uniform boundedness of $\{v_{1,j}\}$ follows from the Harnack inequality Lemma 4.2.6 since $v_{1,j}(0,0) = 1$ for $j = 1, 2, \ldots$. The rest of the proof becomes an analogue of Case (II) of Theorem 4.4.1 and the proof is complete. □

5.5. *Proofs of Lemmas 5.3.1 and 5.4.1*

In this subsection, we prove Lemmas 5.3.1 and 5.4.1.

PROOF OF LEMMA 5.3.1. Let (t, u) be a nonnegative solution of (5.3.1) and $\phi_1 > 0$ a first eigenfunction associated to the first eigenvalue λ_1 of $-\Delta_m$ (with homogeneous Dirichlet boundary data). Then

$$2^{m-1}\lambda_1 \int_\Omega \phi_1^m - 2^{m-1} \int_\Omega |\nabla \phi_1|^m = 0. \tag{5.5.1}$$

For $\varepsilon > 0$, write $u_\varepsilon = u + \varepsilon$. Multiply (2.3.2) by the test function $\phi = \phi_1^m / u_\varepsilon^{m-1} = \psi^m / u_\varepsilon \in C^{1,\beta}(\Omega) \cap C_0(\Omega)$ and integrate over Ω to obtain

$$\int_\Omega (t + f(x, u, \nabla u))\phi = \int_\Omega |\nabla u|^{m-2} \nabla u \nabla \phi. \tag{5.5.2}$$

Applying the Young inequality, we infer that

$$\begin{aligned} \int_\Omega |\nabla u|^{mp/(p+1)} \phi &= \int_\Omega \phi_1^m (|\nabla u|/u_\varepsilon)^{mp/(p+1)} u_\varepsilon^{(p-m+1)/(p+1)} \\ &\le \frac{p}{p+1} \int_\Omega \psi^m |\nabla u|^m + \frac{1}{p+1} \int_\Omega \phi_1^m u_\varepsilon^{p-m+1}, \end{aligned} \tag{5.5.3}$$

and

$$m \left| \int_\Omega \psi^{m-1} |\nabla u|^{m-2} \nabla u \nabla \phi_1 \right| \le \frac{m-1}{2} \int_\Omega \psi^m |\nabla u|^m + 2^{m-1} \int_\Omega |\nabla \phi_1|^m.$$

It follows that (noting $\nabla u_\varepsilon = \nabla u$)

$$\begin{aligned}
&\int_\Omega |\nabla u|^{m-2}\nabla u \nabla \phi + \delta \int_\Omega \psi^m |\nabla u|^m - 2^{m-1}\int_\Omega |\nabla \phi_1|^m \\
&\quad = m\int_\Omega \psi^{m-1}|\nabla u|^{m-2}\nabla u\nabla\phi_1 - (m-1-\delta)\int_\Omega \psi^m|\nabla u|^m - 2^{m-1}\int_\Omega |\nabla\phi_1|^m \\
&\quad \le m\int_\Omega \psi^{m-1}|\nabla u|^{m-2}\nabla u\nabla\phi_1 - \frac{m-1}{2}\int_\Omega \psi^m|\nabla u|^m - 2^{m-1}\int_\Omega |\nabla\phi_1|^m \le 0,
\end{aligned} \tag{5.5.4}$$

since $\delta \le (m-1)/2$. Now combining (5.5.1)–(5.5.4) and using the growth-limit condition (GL1), part 4, we obtain for all $\varepsilon > 0$

$$\begin{aligned}
\int_\Omega (t - L + 2\delta u^p)\phi &\le \int_\Omega \left[t + f(x,u,\nabla u) + \delta|\nabla u|^{mp/(p+1)}\right]\phi \\
&\quad + 2^{m-1}\lambda_1 \int_\Omega \phi_1^m - 2^{m-1}\int_\Omega |\nabla\phi_1|^m \\
&\le \int_\Omega |\nabla u|^{m-2}\nabla u\nabla\phi + \delta\int_\Omega \psi^m|\nabla u|^m - 2^{m-1}\int_\Omega |\nabla\phi_1|^m \\
&\quad + 2^{m-1}\lambda_1\int_\Omega \phi_1^m + \delta\int_\Omega \phi_1^m u_\varepsilon^{p-m+1} \\
&\le 2^{m-1}\lambda_1\int_\Omega \phi_1^m + \delta\int_\Omega \phi_1^m u_\varepsilon^{p-m+1}
\end{aligned} \tag{5.5.5}$$

(the coefficients $p/(p+1)$ and $1/(p+1)$ being dropped from (5.5.3) for simplicity).

We claim that there exists $C = C(p,m,\delta) > 0$ such that

$$t \le t_0 := C(2^{m-1}\lambda_1)^{p/(p+1-m)} + L. \tag{5.5.6}$$

If $t \le L$, there is nothing left to prove. So assume $t > L$. Then (5.5.5) implies

$$u > 0, \quad \text{a.e.}$$

In turn,

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+} u_\varepsilon = u, \\
&\lim_{\varepsilon\to 0^+} (t - L + 2\delta u^p)\phi = [(t-L)u^{1-m} + 2\delta u^{p+1-m}]\phi_1^m, \quad \text{a.e.}
\end{aligned}$$

Plainly $(t - L + 2\delta u^p)\phi > 0$ is monotonically increasing as $\varepsilon \to 0^+$. Hence by the monotone convergence theorem, with the help of (5.5.5), we have

$$\lim_{\varepsilon\to 0^+}\int_\Omega (t - L + 2\delta u^p)\phi = \int_\Omega [(t-L)u^{1-m} + 2\delta u^{p+1-m}]\phi_1^m < \infty$$

and (obviously)

$$\lim_{\varepsilon\to 0^+}\int_\Omega \phi_1^m u_\varepsilon^{p+1-m} = \int_\Omega u^{p+1-m}\phi_1^m.$$

It follows that, by letting $\varepsilon \to 0^+$ in (5.5.5),

$$\int_\Omega [(t-L)u^{1-m} + \delta u^{p+1-m}]\phi_1^m \le 2^{m-1}\lambda_1 \int_\Omega \phi_1^m. \tag{5.5.7}$$

Using Young's inequality once more, there exists $C = C(p, m, \delta) > 0$ such that for $t - L > 0$

$$\inf_{u>0}[(t-L)u^{1-m} + \delta u^{p+1-m}] \ge C(t-L)^{(p+1-m)/p}. \tag{5.5.8}$$

Combining (5.5.7) and (5.5.8) immediately yields the desired upper bound (5.5.6) for t. □

PROOF OF LEMMA 5.4.1. The proof is similar to that of Lemma 5.3.1. We shall consider strictly positive solutions $\mathbf{u} = (u, v) > 0$ of (5.4.1) ($\mathbf{u} \equiv 0 \Rightarrow t = 0$).

For $\varepsilon > 0$ and $m > 1$, it has been observed in [22], as a consequence of the Young inequality, that there holds for $u_1 = u + \varepsilon$ (note $\nabla u_1 = \nabla u$)

$$|\nabla\phi_1|^m - \nabla\left(\frac{\phi_1^m}{u_1^{m-1}}\right)|\nabla u|^{m-2}\nabla u \ge 0, \quad x \in \Omega.$$

It follows that, by using the equations of ϕ_1 and u respectively

$$\begin{aligned}\lambda_1 \int_\Omega \phi_1^m - \int_\Omega (t + u^a v^b)\frac{\phi_1^m}{u_1^{m-1}} \\ = \int_\Omega \left(|\nabla\phi_1|^m - \nabla\left(\frac{\phi_1^m}{u_1^{m-1}}\right)|\nabla u|^{m-2}\nabla u\right) \ge 0.\end{aligned}$$

Letting $\varepsilon \to 0$ immediately yields (note $\{\phi_1^m/u_1^{m-1}\}$ are uniformly bounded in Ω for $\varepsilon > 0$)

$$\int_\Omega (t + u^a v^b)u^{1-m}\phi_1^m \le \lambda_1 \int_\Omega \phi_1^m.$$

Similarly, one has

$$\int_\Omega (t + u^c v^d)v^{1-m}\phi_1^m \le \lambda_1 \int_\Omega \phi_1^m.$$

Combining the two together, we have

$$\int_\Omega \{(t + u^a v^b)u^{1-m} + (t + \kappa_2 u^c v^d)v^{1-m}\}\phi_1^m \le 2\lambda_1 \int_\Omega \phi_1^m. \tag{5.5.9}$$

Since $\chi, b, c > 0$, one readily infers that there exists $\lambda, \mu \ge 0$ ($\lambda, \mu > 0$ if $\alpha, \delta > 0$) such that $\lambda + \mu = 1$, and

$$r, s \ge 0, \quad r + s > 0, \quad \text{where } r = -\lambda\alpha + \mu c,\ s = \lambda b - \mu\delta.$$

For example, one may take (with $\mu = 1 - \lambda$) $\lambda = 1$ if $\alpha \le 0$, $\lambda = 0$ if $\delta \le 0$ and $\delta/(b + \delta) < \lambda < c/(\alpha + c)$ if $\alpha, \delta > 0$. By Young's inequality, we have for $u, v > 0$

$$(u^a v^b)u^{1-m} + (u^c v^d)v^{1-m} = u^{-\alpha}v^b + u^c v^{-\delta} \ge C u^r v^s$$

for some $C = C(r, s, \lambda, \mu) > 0$. It follows that

$$(t + u^a v^b)u^{1-m} + (t + u^c v^d)v^{1-m} \geq t(u^{1-m} + v^{1-m}) + Cu^r v^s.$$

But, a direct computation shows for $t \geq 0$ (noting $m > 1$)

$$\inf_{u,v>0}\{t(u^{1-m} + v^{1-m}) + Cu^r v^s\} = Ct^{(r+s)/(r+s+m-1)}$$

for some positive constant C depending only on r, s, λ, μ and m. Now the conclusion follows from (3.2) by taking $t_0 = (2\lambda_1 C^{-1})^{(r+s+m-1)/(r+s)}$. □

5.6. *Proofs of Lemma 5.1.1*

In this subsection, we prove Lemma 5.1.1.

PROOF OF LEMMA 5.1.1. Consider $\mathbf{u} = t\Lambda(\mathbf{u})$ with $t \in [0, 1]$ and $\mathbf{u} \in \mathcal{C}$ with $\|\mathbf{u}\| = r > 0$, that is,

$$\begin{aligned} \Delta_m u + tu^a v^b &= 0, \qquad \text{in } \Omega,\\ \Delta_m v + tu^c v^d &= 0, \qquad \text{in } \Omega,\\ u = v &= 0, \qquad \text{on } \partial\Omega. \end{aligned} \tag{5.6.1}$$

For $e, f \geq 1$, multiply the first equation by u^e and the second equation by v^f and integrate over Ω to obtain

$$\int_\Omega |\nabla u|^m u^{e-1} \leq C\int_\Omega u^{a+e}v^b, \qquad \int_\Omega |\nabla v|^m v^{f-1} \leq C\int_\Omega u^c v^{d+f} \tag{5.6.2}$$

since $t \in [0, 1]$.

We want to show that there exists $r_0 > 0$ such that the equation $\mathbf{u} = t\Lambda(\mathbf{u})$, i.e., (5.6.1), actually has no solution in the punctuated ball $B_{r_0}(0) - \{0\}$ for all $t \in [0, 1]$. By the strong maximum principle, using (5.6.1), $\mathbf{u}$ is strictly positive in Ω since $\|\mathbf{u}\| = r > 0$ and (5.6.1) is fully coupled. In the sequel, $\mathbf{u}$ is always taken as a positive solution of (5.6.1) with $r = \|\mathbf{u}\| > 0$.

We consider four cases.

Case (I). $\max\{a, d\} \geq m - 1$, say, $d \geq m - 1$. Taking $f = 1$ in $(5.6.2)_2$, we deduce

$$\int_\Omega |\nabla v|^m = O(r^{c+d+1-m})\int_\Omega v^m = O(r^{c+d+1-m})\int_\Omega |\nabla v|^m,$$

where $r = \|\mathbf{u}\|$, $c > 0$ and $d + 1 \geq m$, and we have used the Poincare inequality. This is impossible if $r = \|\mathbf{u}\|$ is small since $v > 0$. It follows that there exists $r_0 > 0$ such that the equation $\mathbf{u} = t\Lambda(\mathbf{u})$ has no solution in $B_{r_0}(0) - \{0\}$ for all $t \in [0, 1]$.

Case (II). $\max\{a, d\} < m-1$, $m \in (1, n)$ and $\max\{a+b, c+d\} \le m_*$. Plainly,

$$\max\{a, d\} < m-1 \Longrightarrow \alpha, \delta > 0. \tag{5.6.3}$$

We take $e = f = 1$ in (5.6.2), and apply the Hölder and Sobolev inequalities to deduce

$$\begin{aligned}\int_\Omega |\nabla u|^m \le \int_\Omega u^{a+1} v^b &\le \left(\int_\Omega u^{m_*+1}\right)^{\frac{a+1}{m_*+1}} \left(\int_\Omega v^{\frac{b(m_*+1)}{m_*-a}}\right)^{\frac{m_*-a}{m_*+1}} \\ &\le C\left(\int_\Omega |\nabla u|^m\right)^{(a+1)/m} \left(\int_\Omega v^{\frac{b(m_*+1)}{m_*-a}}\right)^{\frac{m_*-a}{m_*+1}} \\ &\le C\left(\int_\Omega |\nabla u|^m\right)^{(a+1)/m} \left(\int_\Omega v^{m_*+1}\right)^{b/(m_*+1)} \\ &\le C\left(\int_\Omega |\nabla u|^m\right)^{(a+1)/m} \left(\int_\Omega |\nabla v|^m\right)^{b/m},\end{aligned}$$

where we have used the fact $b(m_*+1)/(m_*-a) \le m_*+1$ (by assumption $a+b \le m_*$). In turn,

$$\int_\Omega |\nabla u|^m \le C\left(\int_\Omega |\nabla v|^m\right)^{b/\alpha}.$$

Similarly, one has

$$\int_\Omega |\nabla v|^m \le C\left(\int_\Omega |\nabla u|^m\right)^{c/\delta}.$$

Combining the two inequalities together yields

$$1 \le C\left(\int_\Omega |\nabla u|^m\right)^{\chi/\alpha\delta} \le C\|\mathbf{u}\|_X^{m\chi/\alpha\delta},$$

since $\alpha, \chi, \delta > 0$ by (5.6.3). In turn, again, there exists $r_0 > 0$ such that the equation $\mathbf{u} = t\Lambda(\mathbf{u})$ has no solution in $B_{r_0}(0) - \{0\}$ for all $t \in [0, 1]$.

Case (III). $\max\{a, d\} < m-1$, $m \in (1, n)$ and $\max\{a+b, c+d\} > m_*$. Rewrite (5.6.2) into

$$\int_\Omega |\nabla w|^m \le C\int_\Omega w^{a'+1} z^{b'}, \qquad \int_\Omega |\nabla z|^m \le C\int_\Omega w^{c'} z^{d'+1},$$

where $u = w^{m/(e+m-1)}$, $v = z^{m/(f+m-1)}$ and

$$a' = \frac{m(a+e)}{e+m-1} - 1 > 0, \qquad b' = \frac{mb}{f+m-1} > 0,$$

and

$$c' = \frac{mc}{e+m-1} > 0, \qquad d' = \frac{m(d+f)}{f+m-1} - 1 > 0$$

since $e, f \ge 1$.

We claim that one can choose suitable $e, f \geq 1$ so that

$$a' = \frac{m(a+e)}{e+m-1} - 1 < m-1, \qquad d' = \frac{m(d+f)}{f+m-1} - 1 < m-1, \tag{5.6.4}$$

$$\chi' := b'c' - \alpha'\delta' = b'c' - (m-1-a')(m-1-d') > 0 \tag{5.6.5}$$

and

$$\max\{a'+b', c'+d'\} \leq m_*. \tag{5.6.6}$$

Indeed, direct computations show that (5.6.4) and (5.6.5) are equivalent to $\max\{a,d\} < m-1$ and to $\chi > 0$, respectively, for any $e, f > 0$. To see (5.6.6), first fix $f_0 \geq 1$ such that

$$m + \frac{bm}{f_0+m-1} < m_* + 1, \qquad \frac{(d+f_0)m}{f_0+m-1} < m_* + 1, \tag{5.6.7}$$

which is equivalent to

$$\frac{b}{f_0+m-1} < \frac{m}{n-m}, \qquad \frac{d+f_0}{f_0+m-1} < \frac{n}{n-m}$$

which is obviously possible by taking $f_0 \geq 1$ large. Next one simply chooses $e_0 \geq 1$ so that

$$\frac{(a+e_0)m}{e_0+m-1} + \frac{bm}{f_0+m-1} \leq m_* + 1,$$
$$\frac{(d+f_0)m}{f_0+m-1} + \frac{cm}{e_0+m-1} \leq m_* + 1$$

which is (5.6.6) with e_0 and f_0, and is again possible by taking e_0 large since

$$\frac{(a+e)m}{e+m-1} \to m, \qquad \frac{cm}{e+m-1} \to 0$$

as $e \to \infty$, in view of (5.6.7). Now one readily applies the arguments of Case (II) to the pair of $(w, z) > 0$, with the positive exponents a', b', c', d' satisfying (5.6.4)–(5.6.6), to draw the same conclusion as in Case (II) (note $\|(w,z)\| \to 0$ as $r = \|\mathbf{u}\| \to 0$).

Case (IV). $\max\{a,d\} < m-1$ and $m \geq n$. Plainly, the arguments of Case (II) apply (with slight modifications) since $m_* = \infty$. □

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